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Differential Subordination Results for Holomorphic Functions Associated with Wanas Operator

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Abstract

In this investigation, we define a certain class of holomorphic functions defined by Wanas operator in the open unit disk U. We establish some important geometric properties for this class such as inclusion relationship, integral representation and argument estimate.

1. Introduction

Let $\mathcal{A}$ indicate the family of functions f which are holomorphic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and have the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $m, \lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$, we consider the differential operator $W_{\alpha,\beta}^{k,\lambda} : \mathcal{A} \to \mathcal{A}$, introduced by Wanas [11], where

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$$W_{\alpha,\beta}^{k,\lambda} f(z) = z + \sum_{n=2}^{\infty}\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m}\right)\right]^{\lambda} a_n z^n. \tag{1.2}$$

It is easily verified from (1.2) that

$$z(W_{\alpha,\beta}^{k,\lambda} f(z))' = \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\right] W_{\alpha,\beta}^{k,\lambda+1} f(z)$$

$$-\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right] W_{\alpha,\beta}^{k,\lambda} f(z). \tag{1.3}$$

Some of the special cases of the operator defined by (1.2) can be found in [1, 2, 3, 8, 10].

Let T stand for the family of mapping h of the form:

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n,$$

which are holomorphic and convex univalent in U and satisfy the condition:

$$\operatorname{Re}\{h(z)\} > 0, \ (z \in U).$$

Given two functions f and g which are holomorphic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function w which is holomorphic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)), (z \in U)$. In particular, if the function g is univalent in U, then we have the following equivalent (see [7]), $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

To establish our main results, we require the following lemmas.

Lemma 1.1 [5]. *Let $u, v \in \mathbb{C}$ and suppose that ψ is convex and univalent in U with $\psi(0) = 1$ and $\operatorname{Re}\{u\psi(z) + v\} > 0, (z \in U)$. If q is holomorphic in U with $q(0) = 1$, then the subordination*

$$q(z) + \frac{zq'(z)}{uq(z) + v} \prec \psi(z)$$

implies that $q(z) \prec \psi(z)$.

Lemma 1.2 [6]. *Let h be convex univalent in U and $\mathcal{T}$ be holomorphic in U with* $\operatorname{Re}\{\mathcal{T}(z)\} \geq 0,\ (z \in U)$. *If q is holomorphic in U and $q(0) = h(0)$, then the subordination*

$$q(z) + \mathcal{T}(z) z q'(z) \prec h(z)$$

implies that $q(z) \prec h(z)$.

Lemma 1.3 [4]. *Let q be holomorphic in U with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$. If there exists two points $z_1, z_2 \in U$ such that*

$$-\frac{\pi}{2} b_1 = \arg(q(z_1)) < \arg(q(z)) < \arg(q(z_2)) = \frac{\pi}{2} b_2,$$

for some b_1 and b_2 $(b_1 > 0, b_2 > 0)$ and for all z $(|z| < |z_1| = |z_2|)$, then

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{b_1 + b_2}{2}\right)\eta \quad \text{and} \quad \frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{b_1 + b_2}{2}\right)\eta,$$

where

$$\eta \geq \frac{1 - |\varepsilon|}{1 + |\varepsilon|} \quad \text{and} \quad \varepsilon = i \tan\frac{\pi}{4}\left(\frac{b_2 - b_1}{b_1 + b_2}\right).$$

2. Main Results

We begin this section by defining the function class $\mathcal{W}(\delta, \alpha, \beta, k, \lambda; h)$ as follows:

Definition 2.1. A function $f \in \mathcal{A}$ is called in the family $\mathcal{W}(\delta, \alpha, \beta, k, \lambda; h)$ if it satisfies the following differential subordination:

$$\frac{1}{1-\delta}\left(\frac{z(W_{\alpha,\beta}^{k,\lambda} f(z))'}{W_{\alpha,\beta}^{k,\lambda} f(z)} - \delta\right) \prec h(z), \tag{2.1}$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $m, \lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 \leq \delta < 1$ and $h \in T$.

In the following theorem, we establish the inclusion relationship for the class $\mathcal{W}(\delta, \alpha, \beta, k, \lambda; h)$.

Theorem 2.1. *Let* $\operatorname{Re}\left\{\delta + (1-\delta)h(z) + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right\} > 0.$ *Then*

$$\mathcal{W}(\delta, \alpha, \beta, k, \lambda+1; h) \subset \mathcal{W}(\delta, \alpha, \beta, k, \lambda; h).$$

Proof. Assume that $f \in \mathcal{W}(\delta, \alpha, \beta, k, \lambda+1; h)$ and put

$$q(z) = \frac{1}{1-\delta}\left(\frac{z(W_{\alpha,\beta}^{k,\lambda}f(z))'}{W_{\alpha,\beta}^{k,\lambda}f(z)} - \delta\right). \tag{2.2}$$

Then q is holomorphic in U with $q(0) = 1.$ Making use of the identity (1.3), we find from (2.2) that

$$\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m} + 1\right)\right]\frac{W_{\alpha,\beta}^{k,\lambda+1}f(z)}{W_{\alpha,\beta}^{k,\lambda}f(z)}$$

$$= \delta + (1-\delta)q(z) + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}. \tag{2.3}$$

Differentiating both sides of (2.3) with respect to z and multiplying by z, we have

$$q(z) + \frac{zq'(z)}{\delta + (1-\delta)q(z) + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}}$$

$$= \frac{1}{1-\delta}\left(\frac{z(W_{\alpha,\beta}^{k,\lambda+1}f(z))'}{W_{\alpha,\beta}^{k,\lambda+1}f(z)} - \delta\right) \prec h(z). \tag{2.4}$$

Since $\operatorname{Re}\left\{\delta + (1-\delta)h(z) + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right\} > 0,$ applying Lemma 1.1 to the subordination (2.4), yields $q(z) \prec h(z),$ which implies $f \in \mathcal{W}(\delta, \alpha, \beta, k, \lambda; h).$

Next, we find integral representation of the class $\mathcal{W}(\delta, \alpha, \beta, k, \lambda; h).$

Theorem 2.2. *Let* $f \in \mathcal{W}(\delta, \alpha, \beta, k, \lambda; h).$ *Then*

$$W_{\alpha,\beta}^{k,\lambda} f(z) = z \cdot \exp\left[(1-\delta)\int_0^z \frac{h(w(s))-1}{s} ds\right],$$

where w is holomorphic in U with $w(0) = 0$ *and* $|w(z)| < 1$, $(z \in U)$.

Proof. Suppose that $f \in \mathcal{W}(\delta, \alpha, \beta, k, \lambda; h)$. It is easy to see that subordination condition (2.1) can be written as follows

$$\frac{z(W_{\alpha,\beta}^{k,\lambda} f(z))'}{W_{\alpha,\beta}^{k,\lambda} f(z)} = (1-\delta)h(w(z)) + \delta, \tag{2.5}$$

where w is holomorphic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$.

From (2.5), we find that

$$\frac{(W_{\alpha,\beta}^{k,\lambda} f(z))'}{W_{\alpha,\beta}^{k,\lambda} f(z)} - \frac{1}{z} = (1-\delta)\frac{h(w(z))-1}{z}. \tag{2.6}$$

After integrating both sides of (2.6), we have

$$\log\left(\frac{W_{\alpha,\beta}^{k,\lambda} f(z)}{z}\right) = (1-\delta)\int_0^z \frac{h(w(s))-1}{s} ds. \tag{2.7}$$

Therefore, from (2.7), we obtain the required result.

Theorem 2.3. *Let* $f \in \mathcal{A}$, $0 < a_1, a_2 \le 1$ *and* $0 \le \delta < 1$. *If*

$$-\frac{\pi}{2}a_1 < \arg\left(\frac{z(W_{\alpha,\beta}^{k,\lambda+1} f(z))'}{W_{\alpha,\beta}^{k,\lambda+1} g(z)} - \delta\right) < \frac{\pi}{2}a_2,$$

for some $g \in \mathcal{W}\left(\delta, \alpha, \beta, k, \lambda+1; \frac{1+AZ}{1+Bz}\right)$, $(-1 \le B < A \le 1)$, *then*

$$-\frac{\pi}{2}b_1 < \arg\left(\frac{z(W_{\alpha,\beta}^{k,\lambda} f(z))'}{W_{\alpha,\beta}^{k,\lambda} g(z)} - \delta\right) < \frac{\pi}{2}b_2,$$

where b_1 *and* b_2 $(0 < b_1, b_2 \le 1)$ *are the solutions of the equations*:

$$a_1 = \begin{cases} b_1 + \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\delta + \frac{(1+A)(1-\delta)}{1+B} + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) + (1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right), & B \neq -1, \\ b_1, & B = -1 \end{cases} \tag{2.8}$$

and

$$a_2 = \begin{cases} b_2 + \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\delta + \frac{(1+A)(1-\delta)}{1+B} + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) + (1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right), & B \neq -1, \\ b_2, & B = -1 \end{cases} \tag{2.9}$$

with

$$\varepsilon = i\tan\frac{\pi}{2}\left(\frac{b_2-b_1}{b_1+b_2}\right)$$

and

$$t = \frac{2}{\pi}\sin^{-1}\left(\frac{(A-B)(1-\delta)}{\left(\delta + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) + (1-B^2) + (1-\delta)(1-AB)}\right). \tag{2.10}$$

Proof. Define the function G by

$$G(z) = \frac{1}{1-\tau}\left(\frac{z(W_{\alpha,\beta}^{k,\lambda} f(z))'}{W_{\alpha,\beta}^{k,\lambda} g(z)} - \tau\right), \tag{2.11}$$

where $g \in \mathcal{W}\left(\delta, \alpha, \beta, k, \lambda+1; \frac{1+AZ}{1+Bz}\right)$, $(-1 \le B < A \le 1)$ and $0 \le \tau < 1$.

Then G is holomorphic in U with $G(0) = 1$. Thus in view of (1.3) and (2.11), we observe that

$$((1-\tau)G(z)+\tau)W_{\alpha,\beta}^{k,\lambda} g(z) = \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\right] W_{\alpha,\beta}^{k,\lambda+1} f(z)$$

$$- \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right] W_{\alpha,\beta}^{k,\lambda} f(z).$$

Differentiating above relation with respect to z and multiplying by z, we get

$$((1-\tau)G(z)+\tau)z(W_{\alpha,\beta}^{k,\lambda} g(z))' + (1-\tau)zG'(z)W_{\alpha,\beta}^{k,\lambda} g(z)$$

$$= \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\right] z(W_{\alpha,\beta}^{k,\lambda+1} f(z))'$$

$$- \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right] z(W_{\alpha,\beta}^{k,\lambda} f(z))'. \tag{2.12}$$

Suppose that

$$H(z) = \frac{1}{1-\delta}\left(\frac{z(W_{\alpha,\beta}^{k,\lambda} g(z))'}{W_{\alpha,\beta}^{k,\lambda} g(z)} - \delta\right).$$

Using (1.3) again, we have

$$\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\right] \frac{W_{\alpha,\beta}^{k,\lambda+1} g(z)}{W_{\alpha,\beta}^{k,\lambda} g(z)}$$

$$= \delta + (1-\delta)H(z) + \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right]. \tag{2.13}$$

From (2.12) and (2.13), we easily get

$$G(z) + \frac{zG'(z)}{\delta + (1-\delta)H(z) + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}} = \frac{1}{1-\tau}\left(\frac{z(W_{\alpha,\beta}^{k,\lambda+1}f(z))'}{W_{\alpha,\beta}^{k,\lambda+1}g(z)} - \tau\right). \tag{2.14}$$

Notice that from Theorem 2.1, $g \in \mathcal{W}\left(\delta, \alpha, \beta, k, \lambda+1; \frac{1+AZ}{1+Bz}\right)$ implies $g \in \mathcal{W}\left(\delta, \alpha, \beta, k, \lambda; \frac{1+AZ}{1+Bz}\right)$. Thus,

$$H(z) \prec \frac{1+AZ}{1+Bz} \quad (-1 \le B < A \le 1).$$

By applying the result of Silverman and Silvia [9], we have

$$\left|H(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (B \ne -1, z \in U) \tag{2.15}$$

and

$$\operatorname{Re}\{H(z)\} > \frac{1-A}{2} \quad (B = -1, z \in U). \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$\left|\delta + (1-\delta)H(z) + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m} - \frac{\left(\delta + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)(1-B^2) + (1-\delta)(1-AB)}{1-B^2}\right|$$

$$< \frac{(A-B)(1-\delta)}{1-B^2}, \ (B \neq -1, \ z \in U)$$

and

$$\operatorname{Re}\left\{\delta + (1-\delta)H(z) + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right\}$$

$$> \delta + \frac{(1-A)(1-\delta)}{2} + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m, \ (B = -1, \ z \in U).$$

Putting

$$\delta + (1-\delta)H(z) + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m = \rho e^{i\frac{\pi}{2}\phi},$$

where

$$-\frac{(A-B)(1-\delta)}{\left(\delta + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)(1-B^2) + (1-\delta)(1-AB)} < \phi$$

$$< \frac{(A-B)(1-\delta)}{\left(\delta + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)(1-B^2) + (1-\delta)(1-AB)}, \ (B \neq -1)$$

and $-1 < \phi < 1, \ (B = -1),$ then

$$\delta + \frac{(1-A)(1-\delta)}{1-B} + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m < \rho$$

$$< \delta + \frac{(1+A)(1-\delta)}{1+B} + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m, (B \neq -1)$$

and

$$\delta + \frac{(1-A)(1-\delta)}{1-B} + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m < \rho < \infty, \ (B = -1).$$

An application of Lemma 1.2 with

$$\mathcal{T}(z)=\frac{1}{\delta+(1-\delta)H(z)+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}},$$

yields $G(z)\prec h(z)$.

If there exist two points $z_1,\ z_2\in U$ such that

$$-\frac{\pi}{2}b_1=\arg(G(z_1))<\arg(G(z))<\arg(G(z_2))=\frac{\pi}{2}b_2,$$

then by Lemma 1.3, we get

$$\frac{z_1G'(z_1)}{G(z_1)}=-\frac{\eta i}{2}(b_1+b_2)\ \text{ and }\ \frac{z_2G'(z_2)}{G(z_2)}=\frac{\eta i}{2}(b_1+b_2),$$

where

$$\eta\geq\frac{1-|\varepsilon|}{1+|\varepsilon|}\ \text{ and }\ \varepsilon=i\tan\frac{\pi}{4}\left(\frac{b_2-b_1}{b_1+b_2}\right).$$

Now, for the case $B\neq-1$, we obtain

$$\arg\left(\frac{1}{1-\tau}\left(\frac{z_1(W_{\alpha,\beta}^{k,\lambda+1}f(z_1))'}{W_{\alpha,\beta}^{k,\lambda+1}g(z_1)}-\tau\right)\right)$$

$$=\arg\left(G(z_1)+\frac{z_1G'(z_1)}{\delta+(1-\delta)H(z_1)+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}}\right)$$

$$=\arg(G(z_1))+\arg\left(1+\frac{z_1G'(z_1)}{\left[\delta+(1-\delta)H(z_1)+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right]G(z_1)}\right)$$

$$= -\frac{\pi}{2}b_1 + \arg\left(1 - \frac{\eta i}{2\rho}(b_1 + b_2)e^{-i\frac{\pi}{2}\phi}\right)$$

$$= -\frac{\pi}{2}b_1 + \arg\left(1 - \frac{\eta}{2\rho}(b_1 + b_2)\cos\frac{\pi}{2}(1-\phi) + \frac{\eta i}{2\rho}(b_1 + b_2)\sin\frac{\pi}{2}(1-\phi)\right)$$

$$\le -\frac{\pi}{2}b_1 - \tan^{-1}\left(\frac{\eta(b_1 + b_2)\sin\frac{\pi}{2}(1-\phi)}{2\rho + \eta(b_1 + b_2)\cos\frac{\pi}{2}(1-\phi)}\right)$$

$$\le -\frac{\pi}{2}b_1 - \tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1 + b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\delta + \frac{(1+A)(1-\delta)}{1+B} + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) + (1-|\varepsilon|)(b_1 + b_2)\sin\frac{\pi}{2}t}\right)$$

$$= -\frac{\pi}{2}a_1,$$

where a_1 and t are given by (2.8) and (2.10), respectively.

Also,

$$\arg\left(\frac{1}{1-\tau}\left(\frac{z_2(W_{\alpha,\beta}^{k,\lambda+1}f(z_2))'}{W_{\alpha,\beta}^{k,\lambda+1}g(z_2)} - \tau\right)\right)$$

$$\ge \frac{\pi}{2}b_2 + \tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1 + b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\delta + \frac{(1+A)(1-\delta)}{1+B} + \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) + (1-|\varepsilon|)(b_1 + b_2)\sin\frac{\pi}{2}t}\right)$$

$$= \frac{\pi}{2} a_2,$$

where a_2 and t are given by (2.9) and (2.10), respectively.

Similarly, for the case $B = -1,$ we have

$$\arg\left(\frac{1}{1-\tau}\left(\frac{z_1(W_{\alpha,\beta}^{k,\lambda+1}f(z_1))'}{W_{\alpha,\beta}^{k,\lambda+1}g(z_1)} - \tau\right)\right) \leq -\frac{\pi}{2} b_1$$

and

$$\arg\left(\frac{1}{1-\tau}\left(\frac{z_2(W_{\alpha,\beta}^{k,\lambda+1}f(z_2))'}{W_{\alpha,\beta}^{k,\lambda+1}g(z_2)} - \tau\right)\right) \geq \frac{\pi}{2} b_2.$$

The above two cases disagree with the assumptions. Therefore, the proof of the theorem is complete.

References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, *Int. J. Math. Math. Sci.* 2004, Article ID 172525, 8 pp. https://doi.org/10.1155/S0161171204108090

[2] N. E. Cho and T. H. Kim, Multiplier transformations and strongly close-to-convex functions, *Bull. Korean Math. Soc.* 40(3) (2003), 399-410. https://doi.org/10.4134/BKMS.2003.40.3.399

[3] N. E. Cho and H. M. Srivastava, Argument estimates of certain analytic functions defined by a class of multiplier transformations, *Math. Comput. Modelling* 37(1-2) (2003), 39-49. https://doi.org/10.1016/S0895-7177(03)80004-3

[4] A. Ebadian, S. Shams, Z. G. Wang and Y. Sun, A class of multivalent analytic functions involving the generalized Jung-Kim-Srivastava operator, *Acta Univ. Apulensis* 18 (2009), 265-277.

[5] P. Eenigenburg, P. T. Mocanu, S. S. Miller and M. O. Reade, On a Briot-Bouquet differential subordination, *General Inequalities*, Birkhäuser, Basel 3 (1983), 339-348. https://doi.org/10.1007/978-3-0348-6290-5_26

[6] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.* 28 (1981), 157-171. https://doi.org/10.1307/mmj/1029002507

[7] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[8] G. S. Salagean, Subclasses of univalent functions, Lecture Notes in Math., Springer Verlag, Berlin, 1013 (1983), 362-372. https://doi.org/10.1007/BFb0066543

[9] H. Silverman and E. M. Silvia, Subclasses of starlike functions subordinate to convex functions, *Canad. J. Math.* 37 (1985), 48-61. https://doi.org/10.4153/CJM-1985-004-7

[10] S. R. Swamy, Inclusion properties of certain subclasses of analytic functions, *Int. Math. Forum* 7(36) (2012), 1751-1760.

[11] A. K. Wanas, New differential operator for holomorphic functions, *Earthline J. Math. Sci.* 2(2) (2019), 527-537. https://doi.org/10.34198/ejms.2219.527537