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Anticommutativity and n-schemes

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Abstract

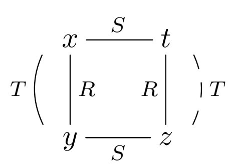
The purpose of this paper is two-fold. A first and more concrete aim is to give new characterizations of equivalence distributive Goursat categories (which extend 3-permutable varieties) through variations of the little Pappian Theorem involving reflexive and positive relations. A second and more abstract aim is to show that every finitely complete category $\mathcal{E}$ satisfying the n-scheme is locally anticommutative.

1 Introduction and Preliminaries

In this section we recall some basic definitions and results from the literature, needed throughout the article.

1.1 n-schemes

For a sublattice L of an equivalence lattice EqA, Gumm's Shifting Lemma [11] is stated as follows. Given congruences R, S and T on the same algebra X in V such that $R \wedge S \leqslant T$, whenever x, y, z, t are elements in X with $(x, y) \in R \wedge T$, $(x, t) \in S$, $(y, z) \in S$ and $(t, z) \in R$, it then follows that $(t, z) \in T$. We display this condition as



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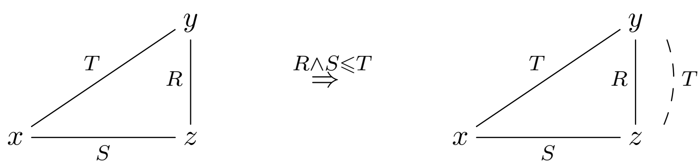
A variety $\mathcal{V}$ of universal algebras satisfies the Shifting Lemma precisely when it is congruence modular, this meaning that the lattice of congruences on any algebra in $\mathcal{V}$ is modular. In particular, since any 3-permutable variety is congruence modular, it always satisfies the Shifting Lemma.

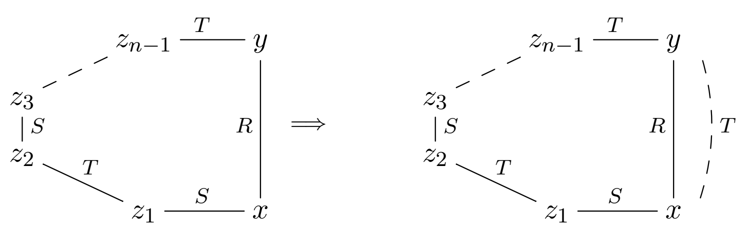
Recall from [11] that a sublattice L of an equivalence lattice EqA satisfies the Triangular scheme if for each $R, S, T \in L$ with $R \wedge S \leqslant T$ and for $x, y, z \in A$ such that $\langle x, y\rangle \in T$, $\langle x, z\rangle \in S$, $\langle z, y\rangle \in R$ we have $\langle z, y\rangle \in T$.
This can be visualized as follows



A sublattice L of EqA satisfies the n-scheme if for each $R, S, T \in L$ with $R \wedge S \leqslant T$ and for $x, y, z_1, \cdots, z_n \in A$ such that

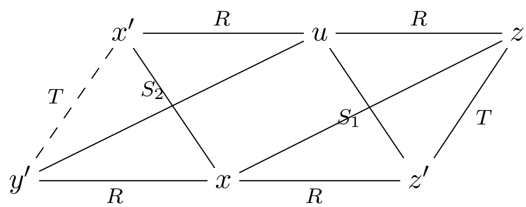
$$\langle x, y\rangle \in R, \langle x, z_1\rangle \in S, \langle z_1, z_2\rangle \in T, \langle z_2, z_3\rangle \in S, \cdots, \langle z_{n-1}, y\rangle \in S$$

for n odd and $\langle z_{n-1}, y\rangle \in T$ for n even we have $\langle x, y\rangle \in T$. These schemes can be also visualized but, contrary to the previous cases, classes of the same congruence fail to be parallel:



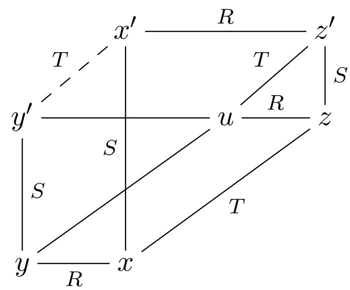
A sublattice L of EqA satisfies the little Pappian Theorem, [21] if given congruences R, S_i and T on the same algebra X in L such that $R \wedge S_i \leqslant T$, whenever x, y, u, z, x', y', z' are element in X with $(u, y'), (x, z) \in S_1$, $(x', x), (u, z') \in$

$S_2, (x', u), (u, z), (y', x), (x, z') \in R$ and $(z, z') \in T$, then $(x', y') \in T$:



(1.1)

Similarly, on identifying S_2 with T and u with z' we obtain A sublattice L of EqA satisfies the scheme-1 if given congruences R, S and T on the same algebra X in L such that $R \wedge S \leqslant T$, one has



(1.2)

1.2 Anticommutative categories

Our categories will always be regular, in the sense of Barr [2]; we recall that a category is regular if it has finite limits, each arrow factors as a regular epi followed by a mono, and regular epis are pull-back stable. (It turns out that in a regular category the kernel pair of an arrow always has a coequalizer, given by the regular epi part of the factorization of the arrow) In a regular category, it is possible to compose relations. If (R, r_1, r_2) is a relation from X to Y and (S, s_1, s_2) a relation from Y to Z, their composite SR is a relation from X to Z obtained as the regular image of the arrow

$$(r_1\pi_1, s_2\pi_2) : R \times_Y S \longrightarrow X \times Z,$$

where $(R \times_Y S, \pi_1, \pi_2)$ is the pullback of r_2 along s_1. The composition of relations is then associative, thanks to the fact that regular epimorphisms are assumed to be pullback stable. A relation E on X is called positive when it is of the form $E = R^{\circ}R$ for some relation $R \rightarrowtail X \times Y$. Recall that a category is said to be pointed if it admits a zero object 0, i.e., an object which is both initial and terminal. A point in a category $\mathcal{E}$ is a split epimorphism $p : A \to X$ together with a fixed splitting $s : X \to A$, usually depicted as

$$A \underset{s}{\overset{p}{\rightleftarrows}} X \,.$$

Let $\mathcal{E}$ be an arbitrary category. The category $Pt_{\mathcal{E}}(X)$ [3] of points of $\mathcal{E}$ over X is the category of pointed objects of the comma category $\mathcal{E} \downarrow X$, that is,

$$Pt_{\mathcal{E}}(X) = (X, 1_X) \downarrow (\mathcal{E} \downarrow X).$$

Explicitly, objects of this category are triples (A, p, s) where A is an object of $\mathcal{E}$ and $p : A \to X$ and $s : X \to A$ are morphisms in $\mathcal{E}$ with $p \circ s = 1_X$. A morphism $f : (A, p, s) \to (B, q, t)$ in $Pt_{\mathcal{E}}(X)$ is a morphism $f : A \to B$ in $\mathcal{E}$ such that $q \circ f = p$ and $f \circ s = t$. The category $Pt_{\mathcal{E}}(X)$ is always pointed, where the zero-object is $(X, 1_X, 1_X)$, and if $\mathcal{E}$ is finitely complete, then so is $Pt_{\mathcal{E}}(X)$. Recall that two morphisms $f : A \to C$ and $g : B \to C$ in a pointed category $\mathcal{E}$ with binary products are said to commute [15] if there exists a morphism $\rho : A \times B \to C$ such that $\rho \circ \iota_1 = f$ and $\rho \circ \iota_2 = g$, where $\iota_1 : A \to A \times B$ and $\iota_2 : B \to A \times B$ are the canonical product inclusions.

$$\begin{array}{ccccc} A & \xrightarrow{\iota_1} & A \times B & \xleftarrow{\iota_2} & B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \rho} & \swarrow{\scriptstyle g} & \\ & & C & & \end{array}$$

Two morphisms $f : X \to Z$ and $g : Y \to Z$ in a pointed category $\mathcal{E}$ are said to be disjoint if for any commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

we have $g \circ a = 0 = f \circ b$. This brings us to the main definition of this paper: A pointed category $\mathcal{E}$ with binary products is a called anticommutative if every pair of commuting morphisms are disjoint.

2 Majority Categories and Goursat Categories

For a regular category $\mathcal{E}$ the property of being a majority category can be equivalently defined as follows (see [16]): for any reflexive relations R, S and T on the same object X in $\mathcal{E}$, the inequality

$$R \wedge (ST) \leqslant (R \wedge S)(R \wedge T)$$

holds. We then observe that any regular majority category satisfies the 3-scheme and, consequently, also the 2-scheme and Shifting Lemma):

Lemma 2.1. *The n-scheme holds true in any regular majority category $\mathcal{E}$.*

Proof. Given equivalence relations R, S and T on the same object such that $R \wedge S \leqslant T$, then

$$R \wedge (S,T)_n \leqslant (R \wedge S)(R \wedge T) \cdots (R \wedge S) \leqslant T$$

for n odd and

$$R \wedge (S,T)_n \leqslant (R \wedge S)(R \wedge T) \cdots (R \wedge T) \leqslant T$$

for n even. Here $(S,T)_n$ denotes the composite $STST \cdots$ of S and T, n times. □

Corollary 2.2. *Let $\mathcal{E}$ be a regular majority category.*

(1) *The little Pappian Theorem holds true in $\mathcal{E}$.*

(2) *The scheme-*1 *holds true in $\mathcal{E}$.*

A variety $\mathcal{V}$ of universal algebras is called 3-permutable when the strictly weaker equality $RSR = SRS$ holds. Such varieties are characterized by the existence of two quaternary operations p and q satisfying the identities $p(x,y,y,z) = x,\ p(u,u,v,v) = q(u,u,v,v)$, $q(x,y,y,z) = z$ (see [10]). The notions of 3-permutability can be extended from varieties to regular categories by replacing congruences with (internal) equivalence relations, allowing one to explore some interesting new (non-varietal) examples. Regular categories that are 3-permutable are usually called Goursat categories. As examples of Goursat categories we have :compact groups, topological groups, torsion-free abelian groups, reduced commutative rings. It is well-known that any 3-permutable variety is congruence modular, thus the Shifting Lemma and 3-scheme hold. This result also extends to the regular categorical context.

Theorem 2.3. *[10] Let $\mathcal{E}$ be a regular category. The following statements are equivalent:*

(i) *$\mathcal{E}$ is a Goursat category;*

(ii) $\forall R, S \in Equiv(X), RSR = SRS \in Equiv(X),$ *for any X;*

(iii) *every relation $P \to X \times Y$ in $\mathcal{E}$, $PP^{\circ}PP^{\circ} = PP^{\circ}$;*

(iv) *every reflexive relation F in $\mathcal{E}$, $F^{\circ}F = FF^{\circ} \in Equiv(X)$;*

(v) *every reflexive and positive relation in $\mathcal{E}$ is an equivalence relation.*

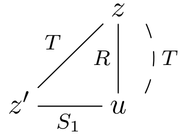
Let us begin with the following observation:

Proposition 2.4. *Let $\mathcal{E}$ be an equivalence distributive Goursat categories.*

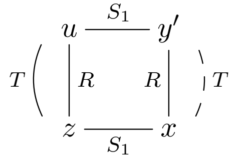
(1) *The Little Pappian Theorem holds true in $\mathcal{E}$ when S_i is a reflexive relation and R and T are equivalence relations.*

(2) *The scheme-1 holds true in $\mathcal{E}$ when S is a reflexive relation and R and T are equivalence relations.*

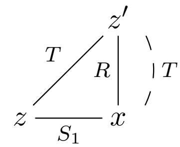
Proof. The proof of this result is based on that of Proposition 5.3 in [12] which claims that a Goursat category satisfies the Shifting Lemma, 2-scheme and 3-scheme when S is a reflexive relation and R and T are equivalence relations.
We prove (1). Let R and T be equivalence relations and let S_i be a reflexive relation on an object X such that $R \wedge S_i \leqslant T$. Suppose that x, y, z, u, x', y', z' are elements in X related as in (1.1). We are going to show that $(x', y') \in T$.
We apply 2-scheme to



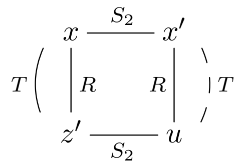
We now apply the Shifting Lemma to



Next we apply 2-scheme to



We now apply the Shifting Lemma to



It follows that,$(x', u), (u, z) \in T$ and $(z, z'), (z', x) \in T$, $(x, y') \in T$. We conclude that $x'Ty'$ (T is transitive), as desired. □

We are now ready to prove the main result in this section:

Theorem 2.5. *Let $\mathcal{E}$ be a regular category. The following conditions are equivalent:*

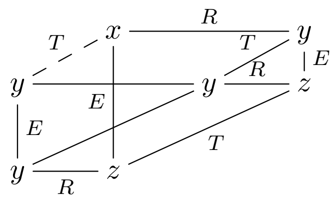
(1) *$\mathcal{E}$ is an equivalence distributive Goursat category;*

(2) *the Little Pappian Theorem holds true in $\mathcal{E}$ when S_i is a reflexive relation and R and T are reflexive and positive relations;*

(3) *the scheme-1 holds true in $\mathcal{E}$ when S is a reflexive relation and R and T are reflexive and positive relations.*

Proof. $(1) \Rightarrow (2)$ This implication follows from the fact that reflexive and positive relations are necessarily equivalence relations in the Goursat context (Theorem 2.3) and from Proposition 2.4.
$(2) \Rightarrow (3)$ Obvious. $(3) \Rightarrow (1)$ We shall prove that for any reflexive relation E on X in $\mathcal{E}$, $EE^{\circ} = E^{\circ}E$ (see Theorem 2.3 (iv)). Suppose that $(x, y) \in EE^{\circ}$. Then, for some z in X, one has that $(z, x) \in E$ and $(z, y) \in E$. Consider the reflexive and positive relations

$R = EE^{\circ}$ and $T = E^{\circ}E$, and the reflexive relation E on X. Then we have:



to conclude that $(x, y) \in E^{\circ}E$. Having proved that $EE^{\circ} \leqslant E^{\circ}E$ for every reflexive relation E, the equality $EE^{\circ} = E^{\circ}E$ follows immediately.

□

3 Locally Anticommutative Categories

The fibration of points $\pi : Pt(\mathcal{E}) \to \mathcal{E}$ classifies many central notions in categorical algebra, such as, Mal'tsev categories:a finitely complete category $\mathcal{E}$ is Mal'tsev if and only if every fibre $Pt_{\mathcal{E}}(X)$ of the fibration of points is unital, strongly unital or subtractives [3].

Definition 3.1. [15] A category $\mathcal{E}$ is locally anticommutative if for any object X in $\mathcal{E}$, the category $Pt_{\mathcal{E}}(X)$ is anticommutative.

Proposition 3.2. *If $\mathbb{D}$ is any finitely complete category which satisfies the n-scheme and $U : \mathcal{E} \to \mathbb{D}$ is any conservative functor (i.e., reflects isomorphisms) which preserves pullbacks and equalizers, then $\mathcal{E}$ satisfies the n-scheme.*

Note that the assumptions on the functor U imply that it preserves monomorphisms, and that if R is an equivalence relation in $\mathcal{E}$, then $U(R)$ the relation obtained by applying U to the representative of R is an equivalence relation in $\mathbb{D}$.

Proof. Let R, S, T are equivalence relations on an object X in $\mathcal{E}$ such that $R \wedge S \leqslant T$ and for $x, y, z_1, \cdots, z_n$ are related such that

$$\langle x, y\rangle \in R, \langle x, z_1\rangle \in S, \langle z_1, z_2\rangle \in T, \langle z_2, z_3\rangle \in S, \cdots, \langle z_{n-1}, y\rangle \in S$$

for n odd and $\langle z_{n-1}, y\rangle \in T$ for n even. Then we are required to show that xTy, which is equivalent to showing that in the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{p_1} & T \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle t} \\ R & \xrightarrow[\langle x,y\rangle]{} & X \times X \end{array}$$

p_2 is an isomorphism. Applying U to the diagram above, we obtain a pullback diagram in $\mathbb{D}$. The assumptions on U easily imply that the canonical morphism $U(X \times X) \to U(X) \times U(X)$ is a monomorphism, which implies that $(U(P), U(p_1), U(p_2))$ form a pullback of $U(t)$ along $(U(x), U(y))$. Since

$$\langle U(x), U(y)\rangle \in U(R), \langle U(x), U(z_1)\rangle \in U(S), \langle U(z_1), U(z_2)\rangle \in U(T),$$

$$\langle U(z_2), U(z_3)\rangle \in U(S), \cdots, \langle U(z_{n-1}), U(y)\rangle \in U(S)$$

for n odd and $\langle U(z_{n-1}), U(y)\rangle \in U(T)$ for n even. Since $\mathbb{D}$ satisfies the n-scheme $(U(x), U(y))$ factors through T , which implies that $U(p_2)$ is an isomorphism, so that p_2 is an isomorphism since U reflects isomorphisms. □

Corollary 3.3.

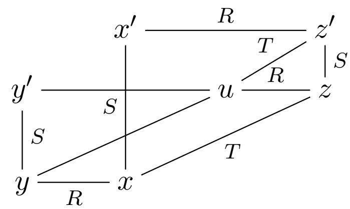
(i) *If $\mathcal{E}$ is a finitely complete category which satisfies the n-scheme, then so does $\mathcal{E} \downarrow X$ and $X \downarrow \mathcal{E}$ for any object X. In particular, it follows that $Pt_{\mathcal{E}}(X)$ satisfies the n-scheme if $\mathcal{E}$ does.*

(ii) *Every finitely complete category $\mathcal{E}$ satisfying the n-scheme is locally anticommutative.*

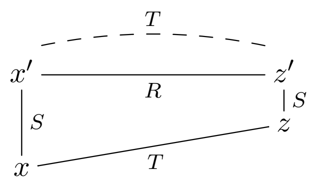
Proof. The proof follows from the fact that the codomain-assigning functor $X \downarrow \mathcal{E} \to \mathcal{E}$ and the domain-assigning functors $\mathcal{E} \downarrow X \to \mathcal{E}$ and $Pt_{\mathcal{C}}(X) \to \mathcal{E}$ satisfy the conditions of Proposition 3.2. □

Proposition 3.4. *If $\mathbb{D}$ is any finitely complete category which satisfies the the scheme-*1 *and $U : \mathcal{E} \to \mathbb{D}$ is any conservative functor (i.e., reflects isomorphisms) which preserves pullbacks and equalizers, then $\mathcal{E}$ satisfies the scheme-*1.

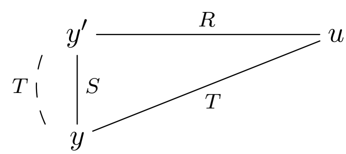
Proof. Let R, S, T are equivalence relations on an object X in $\mathcal{E}$ such that $R \wedge S \leqslant T$ and for x, y, z, x', u, y', z' are related as follows



we show that $x'Ty'$.
We apply Proposition 3.2 (3-scheme) to



Next, We apply Proposition 3.2 (2-scheme) to



It follows that,$(x', z') \in T$, $(z', y) \in T$ and $(y, y') \in T$, we conclude that $x'Ty'$ (T is transitive), as desired. □

Corollary 3.5.

(i) *If $\mathcal{E}$ is a finitely complete category which satisfies the scheme-*1*, then so does $\mathcal{E} \downarrow X$ and $X \downarrow \mathcal{E}$ for any object X. In particular, it follows that $Pt_{\mathcal{E}}(X)$ satisfies the the scheme-*1 *if $\mathcal{E}$ does.*

(ii) *Every finitely complete category $\mathcal{E}$ satisfying the the scheme-*1 *is locally anticommutative.*

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