Coupled Anti Fuzzy Subrings Using t-Conorms

Clement Boateng Ampadu

31 Carrolton Road, Boston, MA 02132-6303, USA

e-mail: drampadu@hotmail.com

Abstract

The aim of this paper is to introduce the concept of coupled anti fuzzy subrings by using t-conorm C. By using t-conorm C; we consider the relationship between coupled subrings and coupled anti fuzzy subrings and we prove that the intersection of two coupled anti fuzzy subrings are also coupled anti fuzzy subring. Also we obtain some results for coupled anti fuzzy subrings under the ring homomorphisms. Finally, we show that the quotient of coupled anti fuzzy subring is also a coupled anti fuzzy subring with respect to t-conorm C. Our work is inspired by [1].

1 Some Old and New Notions and Notations

Definition 1.1. Let X be a set. A fuzzy subset of $X \times X$ will be a function from $X \times X$ into $[0, 1]$. The set of all fuzzy subsets of $X \times X$ will be called the $[0, 1]$-power set of $X \times X$, and will be denoted by $[0, 1]^{X \times X}$.

Definition 1.2. [2] A t-conorm C is a function $C : [0, 1] \times [0, 1] \mapsto [0, 1]$ having the following four properties:

(a) $C(x, 0) = x$,

(b) $C(x, y) \leq C(x, z)$, if $y \leq z$

(c) $C(x, y) = C(y, x)$,

(d) $C(x, C(y, z)) = C(C(x, y), z)$

for all $x, y, z \in [0, 1]$.

Example 1.3. The basic t-conorms are standard union, $C_m(x, y) = \max\{x, y\}$. Bounded sum, $C_b(x, y) = \min\{1, x + y\}$, and algebraic sum, $C_p(x, y) = x + y - xy$ for all $x, y \in [0, 1]$.

Definition 1.4. The t-conorm C is called idempotent if for all $x \in [0, 1]$, $C(x, x) = x$.

Lemma 1.5. [3] *Let C be a t-conorm. Then*

$$C(C(x, y), C(w, z)) = C(C(x, w), C(y, z))$$

for all $x, y, z, w \in [0, 1]$.

Definition 1.6. (Compare with [4]) Let f be a mapping from ring R into S, $\mu \in [0, 1]^{R \times R}$, $\beta \in [0, 1]^{S \times S}$. Define $f(\mu) \in [0, 1]^{S \times S}$ and $f^{-1}(\beta) \in [0, 1]^{R \times R}$ such that for all $(y, v) \in S \times S$, if $f^{-1}(y), f^{-1}(v) \neq \emptyset$, then

$$f(\mu)(y, v) = \sup\{\mu(x, m) | (x, m) \in R \times R, f(x) = y, f(m) = v\}.$$

If $f^{-1}(y), f^{-1}(v) = \emptyset$, then

$$f(\mu)(y, v) = 0.$$

Theorem 1.7. (Compare with [5]) *Let R be a ring. A nonempty subset S of $R \times R$ is a coupled subring of $R \times R$ iff $(x - y, m - v) \in S$ and $(xy, mv) \in S$ for all $(x, m), (y, v) \in S$.*

Definition 1.8. (Compare with [5]) Let R be a ring, and I be a nonempty subset of $R \times R$. We say I is a left(right) ideal of $R \times R$ if for all $(x, m), (y, v) \in I$ and for all $r \in R$, $(x - y, m - v) \in I, (rx, rm) \in I[(x - y, m - v) \in I, (xr, mr) \in I]$.

Definition 1.9. Let R be a ring, and μ be a fuzzy subset of $R \times R$. We say μ is an anti fuzzy subring of $R \times R$ under a t-conorm C iff for all $(x, m), (y, v) \in R \times R$, we have the following

(a) $\mu(x - y, m - v) \leq C[\mu(x, m), \mu(y, v)]$;

(b) $\mu(xy, mv) \leq C[\mu(x, m), \mu(y, v)]$.

Notation 1.10. By $AF(R \times R)$ we mean the set of all anti fuzzy subrings of $R \times R$ under t-conorm C.

Definition 1.11. Let $\mu_1, \mu_2 \in AF(R \times R)$ and $(x, m) \in R \times R$. We define the following

(a) $\mu_1 \subseteq \mu_2$ iff $\mu_1(x, m) \leq \mu_2(x, m)$;

(b) $\mu_1 = \mu_2$ iff $\mu_1(x, m) = \mu_2(x, m)$;

(c) $(\mu_1 \cap \mu_2)(x, m) = C[\mu_1(x, m), \mu_2(x, m)]$.

Remark 1.12. By (c) and (d) of Definition 1.2, we also have

(a) $\mu_1 \cap \mu_2 = \mu_2 \cap \mu_1$;

(b) $\mu_1 \cap \mu_2 \cap \mu_3 = (\mu_1 \cap \mu_2) \cap \mu_3 = \mu_1 \cap (\mu_2 \cap \mu_3)$.

Definition 1.13. Let R be a ring, I be an ideal of R and $\mu_I \in AF(R \times R)$. If $x \neq i, m \neq j$, define $\mu : \frac{R \times R}{I} \mapsto [0, 1]$ by $\mu(x + I, m + I) = C[\mu_I(x, m), \mu_I(i, j)]$, and if $x = i, m = j$, define $\mu : \frac{R \times R}{I} \mapsto [0, 1]$ by $\mu(x + I, m + I) = 1$, for all $(x, m) \in R \times R$, $i, j \in I$.

2 Some Properties

Proposition 2.1. *Let $\mu_1, \mu_2 \in AF(R \times R)$. Then $\mu_1 \cap \mu_2 \in AF(R \times R)$.*

Proof. Let $(x, m), (y, v) \in R \times R$. Observe

$$
\begin{aligned}
(\mu_1 \cap \mu_2)(x - y, m - v) &= C[\mu_1(x - y, m - v), \mu_2(x - y, m - v)] \\
&\leq C[C[\mu_1(x, m), \mu_1(y, v)], C[\mu_2(x, m), \mu_2(y, v)]] \\
&= C[C[\mu_1(x, m), \mu_2(x, m)], C[\mu_1(y, v), \mu_1(y, v)]] \\
&= C[(\mu_1 \cap \mu_2)(x, m), (\mu_1 \cap \mu_2)(y, v)].
\end{aligned}
$$

Also

$$(\mu_1 \cap \mu_2)(xy, mv) = C[\mu_1(xy, mv), \mu_2(xy, mv)]$$
$$\leq C[C[\mu_1(x, m), \mu_1(y, v)], C[\mu_2(x, m), \mu_2(y, v)]]$$
$$= C[C[\mu_1(x, m), \mu_2(x, m)], C[\mu_1(y, v), \mu_1(y, v)]]$$
$$= C[(\mu_1 \cap \mu_2)(x, m), (\mu_1 \cap \mu_2)(y, v)].$$

It follows that $\mu_1 \cap \mu_2 \in AF(R \times R)$. $\qquad\square$

Proposition 2.2. *Let $\mu \in AF(R \times R)$ and C be idempotent. Then for all $t \in [0, 1]$, $R_t = \{(x, m) \in R \times R | \mu(x, m) \leq t\}$ is a coupled subring of $R \times R$.*

Proof. Let $(x, m), (y, v) \in R_t$. Then

$$\mu(x - y, m - v) \leq C[\mu(x, m), \mu(y, v)]$$
$$\leq C(t, t)$$
$$= t$$

which implies $(x - y, m - v) \in R_t$. Also

$$\mu(xy, mv) \leq C[\mu(x, m), \mu(y, v)]$$
$$\leq C(t, t)$$
$$= t$$

which implies $(xy, mv) \in R_t$. Thus R_t is a coupled subring of $R \times R$. $\qquad\square$

Proposition 2.3. *Let R be a ring, and μ be a fuzzy subset of $R \times R$, and C be idempotent, such that for all $(x, m), (y, v) \in R \times R$, and $r \in R$, $\mu(x - y, m - v) \leq C[\mu(x, m), \mu(y, v)]$ and $\mu(rx, rm) \leq \mu(x, m)$. Then*

(a) $\mu(0, 0) \leq \mu(x, m)$, for all $(x, m) \in R \times R$;

(b) $\mu(x, m) = \mu(-x, -m)$, for all $(x, m) \in R \times R$;

(c) For all $t \in [0, 1]$, $R_t = \{(x, m) \in R \times R | \mu(x, m) \leq t\}$ is a left ideal of $R \times R$;

(d) $R_0 = \{(x, m) \in R \times R | \mu(x, m) = \mu(0, 0)\}$ *is a left ideal of* $R \times R$.

Proof. Let $(x, m) \in R \times R$.

(a): $\mu(0, 0) = \mu(x - x, m - m) \leq C[\mu(x, m), \mu(x, m)] = \mu(x, m)$.

(b):

$$
\begin{aligned}
\mu(x, m) &= \mu(0 - (-x), 0 - (-m)) \\
&\leq C[\mu(0, 0), \mu(-x, -m)] \\
&\leq C[\mu(-x, -m), \mu(-x, -m)] \\
&= \mu(-x, -m) \\
&= \mu(0 - x, 0 - m) \\
&\leq C[\mu(0, 0), \mu(x, m)] \\
&\leq C[\mu(x, m), \mu(x, m)] \\
&= \mu(x, m).
\end{aligned}
$$

So $\mu(x, m) = \mu(-x, -m)$.

(c): If $(x, m), (y, v) \in R_t$, $r \in R$, then, $\mu(x - y, m - v) \leq C[\mu(x, m), (y, v)] \leq C[t, t] = t$ and $\mu(rx, rm) \leq C[\mu(rx, rm), \mu(rx, rm)] \leq C[\mu(x, m), \mu(x, m)] \leq C[t, t] = t$. So, $(x - y, m - v), (rx, rm) \in R_t$, thus R_t is a left ideal of $R \times R$.

(d): If $(x, m), (y, v) \in R_0$, $r \in R$, then, $\mu(x - y, m - v) \leq C[\mu(x, m), (y, v)] \leq C[\mu(0, 0), \mu(0, 0)] = \mu(0, 0)$ and $\mu(rx, rm) \leq C[\mu(rx, rm), \mu(rx, rm)] \leq C[\mu(x, m), \mu(x, m)] \leq C[\mu(0, 0), \mu(0, 0)] = \mu(0, 0)$. So, $(x - y, m - v), (rx, rm) \in R_0$, thus R_0 is a left ideal of $R \times R$. $\qquad\square$

Proposition 2.4. *Let* $\mu \in AF(R \times R)$ *and* C *be idempotent. Then* $\mu(x - y, m - v) = \mu(-y, -v)$ *iff* $\mu(x, m) = \mu(0, 0)$ *for all* $(x, m), (y, v) \in R \times R$.

Proof. Let $(x, m), (y, v) \in R \times R$ and $\mu(x - y, m - v) = \mu(-y, -v)$. By letting $y = v = 0$, we get $\mu(x, m) = \mu(0, 0)$. Conversely, suppose that $\mu(x, m) = \mu(0, 0)$.

Then from the previous Proposition $\mu(x, m) \leq \mu(x - y, m - v), \mu(-y, -v)$. Now

$$
\begin{aligned}
\mu(x - y, m - v) &\leq C[\mu(x, m), \mu(y, v)] \\
&\leq C[\mu(-y, -v), \mu(y, v)] \\
&= C[\mu(-y, -v), \mu(-y, -v)] \\
&= \mu(-y, -v) \\
&= \mu(x - y - x, m - v - m) \\
&\leq C[\mu(x - y, m - v), \mu(x, m)] \\
&\leq C[\mu(x - y, m - v), \mu(x - y, m - v)] \\
&= \mu(x - y, m - v).
\end{aligned}
$$

So, $\mu(x - y, m - v) = \mu(-y, -v)$. □

Proposition 2.5. *Let $\mu \in AF(R \times R)$ and S be a ring. Suppose that f is onto homomorphism of R into S. Then $f(\mu) \in AF(S \times S)$.*

Proof. Let $s_1, s_2, s_3, s_4 \in S$, then there exists $(x, m), (y, v) \in R \times R$ such that $s_1 = f(x), s_2 = f(y), s_3 = f(m), s_4 = f(v)$. Now

$$
\begin{aligned}
&f(\mu)(s_1 - s_2, s_3 - s_4) \\
&= \sup\{\mu(x - y, m - v) | s_1 = f(x), s_2 = f(y), s_3 = f(m), s_4 = f(v)\} \\
&\leq \sup\{C[\mu(x, m), \mu(y, v)] | s_1 = f(x), s_2 = f(y), s_3 = f(m), s_4 = f(v)\} \\
&= C[\sup\{\mu(x, m) | s_1 = f(x), s_3 = f(m)\}, \sup\{\mu(y, v) | s_2 = f(y), s_4 = f(v)\}] \\
&= C[f(\mu)(s_1, s_3), f(\mu)(s_2, s_4)].
\end{aligned}
$$

Also

$$
\begin{aligned}
&f(\mu)(s_1 s_2, s_3 s_4) \\
&= \sup\{\mu(xy, mv) | s_1 = f(x), s_2 = f(y), s_3 = f(m), s_4 = f(v)\} \\
&\leq \sup\{C[\mu(x, m), \mu(y, v)] | s_1 = f(x), s_2 = f(y), s_3 = f(m), s_4 = f(v)\} \\
&= C[\sup\{\mu(x, m) | s_1 = f(x), s_3 = f(m)\}, \sup\{\mu(y, v) | s_2 = f(y), s_4 = f(v)\}] \\
&= C[f(\mu)(s_1, s_3), f(\mu)(s_2, s_4)].
\end{aligned}
$$

Therefore, $f(\mu) \in AF(S \times S)$. □

Proposition 2.6. *Let S be a ring, and $\beta \in AF(S \times S)$. If f is a homomorphism of R into S, then $f^{-1}(\beta) \in AF(R \times R)$.*

Proof. Let $(x, m), (y, v) \in R \times R$. Observe

$$
\begin{aligned}
f^{-1}(\beta)(x - y, m - v) &= \beta(f(x - y), f(m - v)) \\
&= \beta(f(x) - f(y), f(m) - f(v)) \\
&\leq C[\beta(f(x), f(m)), \beta(f(y), f(v))] \\
&= C[f^{-1}(\beta)(x, m), f^{-1}(\beta)(y, v)].
\end{aligned}
$$

Also

$$
\begin{aligned}
f^{-1}(\beta)(xy, mv) &= \beta(f(xy), f(mv)) \\
&= \beta(f(x)f(y), f(m)f(v)) \\
&\leq C[\beta(f(x), f(m)), \beta(f(y), f(v))] \\
&= C[f^{-1}(\beta)(x, m), f^{-1}(\beta)(y, v)].
\end{aligned}
$$

So, $f^{-1}(\beta) \in AF(R \times R)$. □

Proposition 2.7. *Let R be a ring, and I be an ideal of R. If C is idempotent, then $\mu \in AF\left(\frac{R \times R}{I}\right)$.*

Proof. Let $(x + I, m + I), (y + I, v + I) \in \frac{R \times R}{I}$, and $i, j \in I$, and $\mu_I \in AF(R \times R)$ such that $x \neq i \neq y$, $m \neq j \neq v$. Observe

$$
\begin{aligned}
&\mu[(x + I) - (y + I), (m + I) - (v + I)] \\
&= \mu[(x - y) + I, (m - v) + I] \\
&= C[\mu_I(x - y, m - v), \mu_I(i, j)] \\
&\leq C[C[\mu_I(x, m), \mu_I(y, v)], \mu_I(i, j)] \\
&= C[C[\mu_I(x, m), \mu_I(y, v)], C[\mu_I(i, j), \mu_I(i, j)]] \\
&= C[C[\mu_I(x, m), \mu_I(i, j)], C[\mu_I(y, v), \mu_I(i, j)]] \\
&= C[\mu(x + I, m + I), \mu(y + I, v + I)].
\end{aligned}
$$

Also

$$\mu[(x+I)(y+I),(m+I)(v+I)] = \mu[xy+I, mv+I]$$
$$= C[\mu_I(xy,mv), \mu_I(i,j)]$$
$$\leq C[C[\mu_I(x,m), \mu_I(y,v)], \mu_I(i,j)]$$
$$= C[C[\mu_I(x,m), \mu_I(y,v)], C[\mu_I(i,j), \mu_I(i,j)]]$$
$$= C[C[\mu_I(x,m), \mu_I(i,j)], C[\mu_I(y,v), \mu_I(i,j)]]$$
$$= C[\mu(x+I, m+I), \mu(y+I, v+I)].$$

Finishing the proof.   □

3 An Open Problem

We begin by introducing the following

Definition 3.1. Let R and S be two rings such that $\mu \in AF(R \times R)$ and $\beta \in AF(S \times S)$. The direct product of μ and β, denoted by $\mu \times \beta$, is the fuzzy subset of ring $R^2 \times S^2$ such that for all $(x,m) \in R \times R$ and $(y,v) \in S \times S$, we have

$$(\mu \times \beta)[(x,m),(y,v)] = C[\mu(x,m), \beta(y,v)].$$

Conjecture 3.2. If $\mu_I \in AF(R_i \times R_i)$ for $i = 1, 2$, then $\mu_1 \times \mu_2 \in AF(R_1^2 \times R_2^2)$.

Conjecture 3.3. Let $\mu_i \in AF(R_i \times R_i)$ for $i = 1, 2, \cdots, n$. Then

$$\mu_1 \times \mu_2 \times \cdots \times \mu_n \in AF(R_1^2 \times R_2^2 \times \cdots \times R_n^2).$$

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