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Explicit Euclidean Norm, Eigenvalues, Spectral Norm and Determinant of Circulant Matrix with the Generalized Tribonacci Numbers

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Abstract

In this paper, we obtain explicit Euclidean norm, eigenvalues, spectral norm and determinant of circulant matrix with the generalized Tribonacci (generalized (r, s, t)) numbers. We also present the sum of entries, the maximum column sum matrix norm and the maximum row sum matrix norm of this circulant matrix. Moreover, we give some bounds for the spectral norms of Kronecker and Hadamard products of circulant matrices of (r, s, t) and Lucas (r, s, t) numbers.

1 Introduction

The generalized (r, s, t) sequence (or generalized Tribonacci sequence or generalized 3-step Fibonacci sequence)

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.

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This sequence has been studied by many authors, see for example [2, 3, 4, 10, 11, 18, 22, 28, 30, 44, 41, 45, 51, 52].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a third-order recurrence sequence (difference equation), its characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.2}$$

whose roots α, β, γ satisfy the following identities:

$$\begin{aligned} \alpha + \beta + \gamma &= r, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -s, \\ \alpha\beta\gamma &= t. \end{aligned}$$

It is well known that the generalized (r, s, t) numbers (the generalized Tribonacci numbers) can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \tag{1.3}$$

where

$$p_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0,\ p_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0,$$

$$p_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0.$$

We need the special cases of the generalized (r, s, t) sequence $\{W_n\}$ which are called (r, s, t) and Lucas (r, s, t) sequences. (r, s, t) sequence $\{G_n\}_{n\geq 0}$ and Lucas (r, s, t) sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} G_{n+3} &= rG_{n+2} + sG_{n+1} + tG_n, \qquad G_0 = 0, G_1 = 1, G_2 = r, & (1.4) \\ H_{n+3} &= rH_{n+2} + sH_{n+1} + tH_n, \qquad H_0 = 3, H_1 = r, H_2 = 2s + r^2. & (1.5) \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{s}{t}G_{-(n-1)} - \frac{r}{t}G_{-(n-2)} + \frac{1}{t}G_{-(n-3)}, \\ H_{-n} &= -\frac{s}{t}H_{-(n-1)} - \frac{r}{t}H_{-(n-2)} + \frac{1}{t}H_{-(n-3)} \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1.4) and (1.5) hold for all integers n.

Some special cases of (r, s, t) sequence $\{G_n(0, 1, r; r, s, t)\}_{n\geq 0}$ and Lucas (r, s, t) sequence $\{H_n(3, r, 2s + r^2; r, s, t)\}_{n\geq 0}$ are as follows:

1. $G_n(0, 1, 1; 1, 1, 1) = T_n$, Tribonacci sequence,
2. $H_n(3, 1, 3; 1, 1, 1) = K_n$, Tribonacci-Lucas sequence,
3. $G_n(0, 1, 2; 2, 1, 1) = P_n$, third order Pell sequence,
4. $H_n(3, 2, 6; 2, 1, 1) = Q_n$, third order Pell-Lucas sequence,
5. $G_n(0, 1, 0; 0, 1, 1) = U_n$, adjusted Padovan sequence,
6. $H_n(3, 0, 2; 0, 1, 1) = E_n$, Perrin (Padovan-Lucas) sequence.

Note that we use the Lucas (r, s, t) numbers in Lemma 2.13 for calculating the determinants.

The following Theorem presents a summing formula of generalized Tribonacci numbers with positive subscripts.

Theorem 1.1. *Let x be a real or complex number. For $n \geq 0$, we have the following formula: If $tx^3 + sx^2 + rx - 1 \neq 0$, then*

$$\sum_{k=0}^{n} x^k W_k = \frac{\Theta_1(x)}{\Theta(x)},$$

where

$$\begin{aligned} \Theta_1(x) &= x^{n+3}W_{n+3} - (rx-1)\,x^{n+2}W_{n+2} - (sx^2+rx-1)x^{n+1}W_{n+1} \\ &\quad -x^2W_2 + x\,(rx-1)\,W_1 + (sx^2+rx-1)W_0, \\ \Theta(x) &= tx^3 + sx^2 + rx - 1. \end{aligned}$$

Proof. It is given in [40]. □

Now, we give an alternative proof. By using Binet's formula of generalized Tribonacci numbers and the following identities

$$\begin{aligned}\alpha+\beta+\gamma &= r,\\ \alpha\beta+\alpha\gamma+\beta\gamma &= -s,\\ \alpha\beta\gamma &= t,\end{aligned}$$

we obtain

$$\begin{aligned}\sum_{k=0}^{n-1}x^kW_k &= \sum_{k=0}^{n-1}x^k\left(\frac{p_1\alpha^k}{(\alpha-\beta)(\alpha-\gamma)}+\frac{p_2\beta^k}{(\beta-\alpha)(\beta-\gamma)}+\frac{p_3\gamma^k}{(\gamma-\alpha)(\gamma-\beta)}\right)\\ &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)}\left(\frac{(\alpha x)^n-1}{\alpha x-1}\right)+\frac{p_2}{(\beta-\alpha)(\beta-\gamma)}\left(\frac{(\beta x)^n-1}{\beta x-1}\right)\\ &\quad +\frac{p_3}{(\gamma-\alpha)(\gamma-\beta)}\left(\frac{(\gamma x)^n-1}{\gamma x-1}\right)\\ &= \frac{\begin{array}{c}p_1((\alpha x)^n-1)(\beta x-1)(\gamma x-1)(\beta-\gamma)\\ +p_2((\beta x)^n-1)(\alpha x-1)(\gamma x-1)(\gamma-\alpha)\\ +p_3((\gamma x)^n-1)(\beta x-1)(\alpha x-1)(\alpha-\beta)\end{array}}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)(\alpha x-1)(\beta x-1)(\gamma x-1)}\\ &= \frac{\begin{array}{c}x^{n+2}W_{n+2}-x^{n+1}(rx-1)W_{n+1}-x^n(sx^2+rx-1)W_n\\ -x^2W_2+x(rx-1)W_1+(sx^2+rx-1)W_0\end{array}}{tx^3+sx^2+rx-1}.\end{aligned}$$

Now, from the equality

$$\sum_{k=0}^{n}x^kW_k = x^nW_n+\sum_{k=0}^{n-1}x^kW_k,$$

the claim of the theorem follows. □

Note that

$$\sum_{k=0}^{n-1}x^kW_k = \frac{\partial}{tx^3+sx^2+rx-1}, \tag{1.6}$$

where

$$\partial = x^{n+2}W_{n+2} - x^{n+1}(rx-1)W_{n+1} - x^{n}(sx^{2}+rx-1)W_{n} - x^{2}W_{2} + x(rx-1)W_{1} + (sx^{2}+rx-1)W_{0}.$$

The case $x=1$ of the above theorem can be given as follows.

Theorem 1.2. *For $n \geq 0$, we have the following formula: If $t+s+r-1 \neq 0$, then*

$$\sum_{k=0}^{n} W_{k} = \frac{\Theta_{1}}{\Theta},$$

where

$$\Theta_{1} = W_{n+3} - (r-1)W_{n+2} - (s+r-1)W_{n+1} - W_{2} + (r-1)W_{1} + (s+r-1)W_{0},$$
$$\Theta = t+s+r-1.$$

Let

$$\Delta = (s+rt-t^{2}+1)(r+s+t-1)(r-s+t+1).$$

Theorem 1.3. *If $\Delta \neq 0$, then*

$$\sum_{k=1}^{n} W_{k}^{2} = \frac{\Delta_{1}}{\Delta},$$

where

$$\begin{aligned}
\Delta_{1} = {} & -(t^{2}+rt+s-1)W_{n+3}^{2} - (r^{3}t+r^{2}t^{2}+r^{2}s+r^{2}+t^{2}+2rst+rt+s-1)W_{n+2}^{2} \\
& -(r^{3}t+r^{2}t^{2}+s^{2}t^{2}-rs^{2}t-s^{3}+r^{2}s+4rst+r^{2}+s^{2}+t^{2}+rt+s-1)W_{n+1}^{2} \\
& +2(r+t)(s+rt)W_{n+3}W_{n+2} + 2t(r+st)W_{n+3}W_{n+1} - 2t(s-1)(s+rt)W_{n+2}W_{n+1} \\
& +(2rst+2r^{2}+t^{2}+rt+s-1)W_{3}^{2} + (r^{3}t+r^{2}t^{2}+r^{2}s+2rst+r^{2}+t^{2}+rt+s-1)W_{2}^{2} \\
& +(r^{3}t+r^{2}t^{2}+s^{2}t^{2}-rs^{2}t-s^{3}+r^{2}s+4rst+r^{2}+s^{2}+t^{2}+rt+s-1)W_{1}^{2} \\
& -2(r+st)W_{4}W_{3} - 2t(r^{2}-s^{2}+rt+s)W_{3}W_{2} + 2t(s-1)(s+rt)W_{2}W_{1}.
\end{aligned}$$

Proof. It is given in Soykan [42, Theorem 2.1]. See also Soykan [43, Theorem 3.1]. □

Remark 1.4. Using Theorems 1.1, 1.2 and 1.3, we give relatively short proofs for our results which are given in the next section.

2 Main Results

In this section, we obtain explicit Euclidean norm, eigenvalues, spectral norm and determinant of circulant matrix with the generalized Tribonacci (generalized (r,s,t)) numbers. We also present the sum of entries, the maximum column sum matrix norm and the maximum row sum matrix norm of this circulant matrix. Moreover, we give some bounds for the spectral norms of Kronecker and Hadamard products of circulant matrices of (r,s,t) and Lucas (r,s,t) numbers. For our work, we need to recall a $n \times n$ circulant matrix and various norm on matrices and their properties.

Let $n \geq 2$ be an integer. An $n \times n$ matrix $C = (c_{ij}) \in M_{n\times n}(\mathbb{C})$ is called a circulant matrix if it is of the form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix}_{n\times n}$$

i.e.

$$c_{ij} = \begin{cases} c_{j-i} & , \quad j \geq i \\ c_{n+j-i} & , \quad j < i \end{cases}$$

and the circulant matrix C is denoted by $C = Circ(c_0, c_1, ..., c_{n-1})$. Circulant matrix was first proposed by Davis in [7]. This matrix has many interesting properties, and it is one of the most important research subject in the field of the computational and pure mathematics (see for example references given in Table 1).

For a $m \times n$ matrix $A = (a_{ij}) \in M_{m\times n}(\mathbb{C})$, the spectral norm of A is given by

$$\|A\|_2 = \left(\max_{1\leq i\leq n} |\lambda_i(A^*A)| \right)^{1/2}$$

where $\lambda_i(A^*A)$'s are the eigenvalues of the matrix A^*A and A^* is the conjugate of transpose of the matrix A. The Frobenius (or Euclidean) norm a $m \times n$ matrix

$A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ is as follows:

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

The following inequality holds for any matrix $A = (a_{ij})_{m\times n} \in M_{n\times n}(\mathbb{C})$ (see [53, Theorem 1 and Table 1]):

$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F . \tag{2.1}$$

It follows that

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\, \|A\|_2 .$$

In literature there are other types of norms of matrices. The maximum column sum matrix norm of $n \times n$ matrix $A = (a_{ij})$ is $\|A\|_1 = \max\limits_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$ and the maximum row sum matrix norm is $\|A\|_\infty = \max\limits_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$.

The maximum column length norm $c_1(.)$ and the maximum row length norm $r_1(.)$ of an matrix of order $m \times n$ are defined as follows:

$$c_1(A) = \max_{1\leq j\leq n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad r_1(A) = \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

There is a relation between $\|.\|_2$, $c_1(.)$ and $r_1(.)$ norms:

Lemma 2.1. *[15] For any matrices $A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ and $B = (b_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$, we have*

$$\|A \circ B\|_2 \leq r_1(A) c_1(B)$$

and

$$\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2$$

and

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$$

where $A \circ B$ *is the Hadamard product which is defined by*

$$A \circ B = (a_{ij} b_{ij}),$$

$A \otimes B$ *is the Kronecker product which is defined by*

$$A \otimes B = (a_{ij} B),$$

and $r_1(A) = \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$, $c_1(B) = \max_{1 \leq j \leq n} \left(\sum_{i=1}^{n} |b_{ij}|^2 \right)^{1/2}$.

The Kronecker product is also sometimes called matrix direct product.

For more details on norm of matrices, see for example [14]. In the following Table 1, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences which require sum formulas of second powers of numbers in m-step Fibonacci sequences ($m = 2, 3, 4$). For m-step Fibonacci sequences, see, for example, [39].

Table 1. Papers on the norms.

Name of sequence	Papers
second order↓	second order↓
Fibonacci, Lucas	[8, 9, 13, 17, 19, 27, 31, 32, 33, 34, 36, 37, 38, 46]
Pell, Pell-Lucas	[1, 47]
Jacobsthal, Jacobsthal-Lucas	[24, 48, 49, 50]
third order↓	third order↓
Tribonacci, Tribonacci-Lucas	[16, 25, 26]
Padovan, Perrin	[6, 21, 29]
fourth order↓	fourth order↓
Tetranacci, Tetranacci-Lucas	[20]

See also Polatlı [23] for the spectral norms of k-circulant matrices with a type of Catalan triangle numbers.

We need the following three lemmas for our calculations.

Lemma 2.2. *[7, 5] Let $C = Circ(c_0, c_1, ..., c_{n-1})$ be a $n \times n$ circulant matrix. Then the eigenvalues of C are*

$$\lambda_j(C) = \sum_{k=0}^{n-1} \omega^{-jk} c_k \tag{2.2}$$

and the corresponding eigenvectors are

$$v_j = (1, \omega^{-j}, \omega^{-2j}, \omega^{-3j}, ..., \omega^{-(n-1)j})^T$$

where $\omega = \exp(2\pi i/n)$, $i = \sqrt{-1}$, $j = 0, 1, 2, ..., n-1$.

Note that (2.2) is equivalent to

$$\lambda_{n-j}(C) = \sum_{p=0}^{n-1} \omega^{jp} c_p, \qquad j = 0, 1, 2, ..., n-1$$

or

$$\lambda_m(C) = \sum_{p=0}^{n-1} \omega^{mp} c_p, \qquad m = n, n-1, ..., 2, 1$$

or

$$\mu_j(C) = \sum_{p=0}^{n-1} \omega^{jp} c_p, \qquad j = 0, 1, ..., n-1 \quad (\mu_j = \lambda_{n-j}).$$

We have the determinants and inverses of nonsingular circulant matrices [35, 12, 7, 5]:

Lemma 2.3. *Determinant of a circulant matrix C is*

$$\det(C) = \prod_{j=0}^{n-1} \lambda_j(C) = \prod_{j=0}^{n-1} \left(\sum_{p=0}^{n-1} \omega^{-jp} c_p \right) = \prod_{j=0}^{n-1} \left(\sum_{p=0}^{n-1} \omega^{jp} c_p \right)$$

and if C is nonsingular circulant matrix then its inverse is

$$C^{-1} = Circ(a_0, a_1, ..., a_{n-1})$$

where

$$a_{n-p} = \frac{1}{n} \sum_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} \omega^{-jk} W_k \right)^{-1} \omega^{-pj} = \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j^{-1} \omega^{-pj}, \qquad p = 1, 2, 3, ..., n$$

or

$$a_p = \frac{1}{n}\sum_{j=0}^{n-1}\left(\sum_{k=0}^{n-1}\omega^{jk}W_k\right)^{-1}\omega^{-pj}, \quad p = 0, 1, 2, ..., n-1.$$

Lemma 2.4. *[14] Let A be a $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of AA^* are $|\lambda_1|^2, |\lambda_2|^2, |\lambda_3|^2, ..., |\lambda_n|^2$ where A^* is the conjugate of transpose of the matrix A.*

Next, we define circulant matrix with generalized (r, s, t) numbers entries.

Definition 2.5. A $n \times n$ circulant matrix with generalized (r, s, t) (generalized Tribonacci) numbers entries is defined by

$$C_n(W) = \begin{pmatrix} W_0 & W_1 & W_2 & \cdots & W_{n-2} & W_{n-1} \\ W_{n-1} & W_0 & W_1 & \cdots & W_{n-3} & W_{n-2} \\ W_{n-2} & W_{n-1} & W_0 & \cdots & W_{n-4} & W_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ W_1 & W_2 & W_3 & \cdots & W_{n-1} & W_0 \end{pmatrix}_{n\times n} = Circ(W_0, W_1, ..., W_{n-1}). \tag{2.3}$$

We call this matrix as generalized Tribonacci (generalized (r, s, t)) circulant matrix.

We consider two special cases of generalized Tribonacci (generalized (r, s, t)) circulant matrix, namely (r, s, t)) circulant matrix: $C_n(G) = Circ(G_0, G_1, ..., G_{n-1})$ and Lucas (r, s, t) circulant matrix: $C_n(H) = Circ(H_0, H_1, ..., H_{n-1})$.

We denote the sum of entries of $C_n(W)$ as $S(C_n(W))$.

Lemma 2.6. *The sum of entries of $C_n(W)$ is*

$$S(C_n(W)) = n\left(-W_n + \frac{\Theta_1}{\Theta}\right)$$

where Θ_1 and Θ are as in Theorem 1.2.

Proof. From the definition of $C_n(W)$, using Theorem 1.2, we obtain

$$S(C_n(W)) = n\sum_{i=0}^{n-1} W_i = n\left(-W_n + \sum_{i=0}^{n} W_i\right) = n\left(-W_n + \frac{\Theta_1}{\Theta}\right).$$

□

Next, we present the maximum column sum matrix norm $\|C_n(W)\|_1$ and the maximum row sum matrix norm $\|C_n(W)\|_\infty$ of matrix $C_n(W) = (a_{ij})$ under certain condition on the generalized Tribonaci sequence W_n.

Theorem 2.7. *Suppose that $W_p \geq 0$ for all the nonnegative integers p. Then we have the following formula:*

$$\|C_n(W)\|_1 = \|C_n(W)\|_\infty = -W_n + \frac{\Theta_1}{\Theta}.$$

Proof. From the definition of the matrix $C_n(W) = (a_{ij})$ we can write

$$\begin{aligned}
\|C_n(W)\|_1 &= \max_{1\leq j\leq n}\sum_{i=1}^{n} |a_{ij}| = \max_{1\leq j\leq n}\{|a_{1j}| + |a_{2j}| + |a_{3j}| + ... + |a_{nj}|\} \\
&= |a_{1n}| + |a_{2n}| + |a_{3n}| + ... + |a_{nn}| \\
&= W_{n-1} + W_{n-2} + ... + W_3 + W_2 + W_1 + W_0 \\
&= -W_n + \sum_{i=0}^{n} W_i \\
&= -W_n + \frac{\Theta_1}{\Theta}
\end{aligned}$$

Similarly, we have

$$\|C_n(W)_k\|_\infty = -W_n + \frac{\Theta_1}{\Theta}.$$

□

The following theorem gives the Euclidean (Frobenius) norm of circulant matrix $C_n(W)$.

Theorem 2.8. *The Euclidean norm of circulant matrix $C_n(W)$ is:*

$$\|C_n(W)\|_F = \sqrt{n\left(W_0^2 - W_n^2 + \frac{\Delta_1}{\Delta}\right)}$$

where Δ and Δ_1 are as in Theorem 1.3.

Proof. From the definition of the Euclidean norm of a matrix, using Theorem 1.3, we obtain

$$(\|C_n(W)\|_F)^2 = n\sum_{i=0}^{n-1} W_i^2 = n\left(W_0^2 - W_n^2 + \sum_{i=1}^{n} W_i^2\right) = n\left(W_0^2 - W_n^2 + \frac{\Delta_1}{\Delta}\right).$$

It follows that

$$\|C_n(W)\|_F = \sqrt{n\left(W_0^2 - W_n^2 + \frac{\Delta_1}{\Delta}\right)}.$$

□

The following theorem gives us the eigenvalues of the matrix in (2.3).

Theorem 2.9. *The eigenvalues of $C_n(W)$ are*

$$\lambda_j(C_n(W)) = \frac{\Phi_j}{r\omega^{-j} + s\omega^{-2j} + t\omega^{-3j} - 1}$$

where $\Phi_j = (W_n - W_0) - (-W_{n+1} + rW_n + W_1 - rW_0)\omega^{-j} + (W_{n+2} - rW_{n+1} - sW_n - W_2 + rW_1 + sW_0)\omega^{-2j}$, and

$$\begin{aligned} \omega &= \exp(2\pi i/n), \\ j &= 0, 1, 2, 3, ..., n-1. \end{aligned}$$

Proof. By using (1.6) or Theorem 1.1 (by putting $x = \omega^{-j}$), we obtain

$$\begin{aligned} &\lambda_j(C_n(W)) \\ =\; &\sum_{k=0}^{n-1} \omega^{-jk} W_k \\ =\; &-\omega^{-jn} W_n + \sum_{k=0}^{n} (\omega^{-j})^k W_k \\ =\; &\frac{\begin{array}{c} \omega^{-j(n+2)} W_{n+2} - \omega^{-j(n+1)}(r\omega^{-j} - 1)W_{n+1} - \omega^{-jn}(s\omega^{-2j} + r\omega^{-j} - 1)W_n \\ -\omega^{-2j} W_2 + \omega^{-j}\left(r\omega^{-j} - 1\right) W_1 + (s\omega^{-2j} + r\omega^{-j} - 1)W_0 \end{array}}{t\omega^{-3j} + s\omega^{-2j} + r\omega^{-j} - 1}. \end{aligned}$$

By simplifying the last equality, we get

$$\lambda_j(C_n(W)) = \frac{\Phi_j}{r\omega^{-j} + s\omega^{-2j} + t\omega^{-3j} - 1}.$$

□

The following theorem presents the spectral norm of $C_n(W)$.

Theorem 2.10. *Suppose that $W_p \geq 0$ for all the nonnegative integers p. The spectral norm of $C_n(W)$ is*

$$\|C_n(W)\|_2 = \frac{W_{n+2} - (r-1)W_{n+1} - (s+r-1)W_n - W_2 + (r-1)\,W_1 + (s+r-1)W_0}{t+s+r-1}.$$

Proof. For $j = 0$, we have

$$\lambda_0(C_n(W)) = \sum_{p=0}^{n-1} \omega^{-0\times k} W_k = \sum_{p=0}^{n-1} W_p.$$

Note that the matrix $C_n(W)$ is a normal matrix since

$$C_n(W)^* C_n(W) = C_n(W) C_n(W)^*.$$

Now, using Lemma 2.4, we see that

$$\|C_n(W)\|_2 = \left(\max_{1\leq j\leq n} |\lambda_j(C_n(W)|^2 \right)^{1/2}.$$

In the last equality, if we take $j = 0$, then λ_0 becomes the maximum eigenvalue because

$$\|C_n(W)\|_2 = \left(\max_{1\leq j\leq n} |\lambda_j(C_n(W))|^2 \right)^{1/2} = \left(|\lambda_0(C_n(W))|^2, \max_{1\leq j\leq n-1} |\lambda_j(C_n(W))|^2 \right)^{1/2}$$

and

$$\max_{1\leq j\leq n-1} |\lambda_j(C_n(W))| = \left| \sum_{p=0}^{n-1} \omega^{-jp} W_p \right| \leq \sum_{p=0}^{n-1} \left|\omega^{-jp}\right| |W_p| = \sum_{k=0}^{n-1} |W_p| = \sum_{k=0}^{n-1} W_p = \lambda_0(C_n(W)).$$

Hence

$$\|C_n(W)\|_2 = |\lambda_0(C_n(W))| = \sum_{p=0}^{n-1} W_p.$$

By applying Theorem 2.9 (or from Theorem 1.1), we obtain

$$\begin{aligned}&\|C_n(W)\|_2\\ = \ &\frac{W_{n+2}-(r-1)W_{n+1}-(s+r-1)W_n-W_2+(r-1)W_1+(s+r-1)W_0}{t+s+r-1}.\end{aligned}$$

□

The following corollary presents the spectral norms of $(r,s,t))$ circulant matrix: $C_n(G)=Circ(G_0,G_1,...,G_{n-1})$ and Lucas (r,s,t) circulant matrix: $C_n(H)=Circ(H_0,H_1,...,H_{n-1})$.

Corollary 2.11. *Suppose that $G_p\geq 0$ and $H_p\geq 0$ for all the nonnegative integers p. The spectral norm of $C_n(G)$ and $C_n(H)$ are*

$$\|C_n(G)\|_2=\frac{G_{n+2}-(r-1)G_{n+1}-(s+r-1)G_n-1}{t+s+r-1}$$

and

$$\|C_n(H)\|_2=\frac{H_{n+2}-(r-1)H_{n+1}-(s+r-1)H_n+2r+s-3}{t+s+r-1}$$

respectively.

Proof. Take $W_n=G_n$, $G_0=0, G_1=1, G_2=r$ and $W_n=H_n$, $H_0=3, H_1=r, H_2=2s+r^2$ in Theorem 2.10. □

The following corollary presents properties of the spectral norms of Hadamard product and Kronecker product of $C_n(G)=Circ(G_0,G_1,...,G_{n-1})$ and $C_n(H)=Circ(H_0,H_1,...,H_{n-1})$

Corollary 2.12. *Suppose that $G_p\geq 0$ and $H_p\geq 0$ for all the nonnegative integers p. The spectral norm of Hadamard product of $C_n(G)$ and $C_n(H)$ has the following property:*

$$\|C_n(G)\circ C_n(H)\|_2\leq\frac{\Upsilon}{(t+s+r-1)^2}$$

and the spectral norm of Kronecker product of $C_n(G)$ and $C_n(H)$ has the following property:

$$\|C_n(G)\otimes C_n(H)\|_2=\frac{\Upsilon}{(t+s+r-1)^2}$$

where $\Upsilon = (G_{n+2} - (r-1)G_{n+1} - (s+r-1)G_n - 1)(H_{n+2} - (r-1)H_{n+1} - (s+r-1)H_n + 2r + s - 3)$.

Proof. Since $\|C_n(G) \circ C_n(H)\|_2 \leq \|C_n(G)\|_2 \|C_n(H)\|_2$ and $\|C_n(G) \otimes C_n(H)\|_2 = \|C_n(G)\|_2 \|C_n(H)\|_2$, the proof is trivial from Corollary 2.11. $\square$

Next, we present the determinant of $C_n(W)$.

Theorem 2.13. *The determinant of $C_n(W)$ is given by*

$$\det(C_n(W)) = \frac{\Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(H_n + (1 - H_{-n})t^n - 1)}$$

where

$$\begin{aligned} \omega &= \exp(2\pi i/n), \\ \Lambda_1 &= W_n - W_0, \\ \Lambda_2 &= -W_{n+1} + rW_n + W_1 - rW_0, \\ \Lambda_3 &= W_{n+2} - rW_{n+1} - sW_n - W_2 + rW_1 + sW_0, \end{aligned}$$

Proof. By considering identities

$$\begin{aligned} &\prod_{j=0}^{n-1}(x - y\omega^{-j}) = x^n - y^n, \\ &\prod_{j=0}^{n-1}(x - y\omega^{-j} + z\omega^{-2j}) \\ = \;& x^n \left(1 - \left(\frac{y - \sqrt{y^2 - 4xz}}{2x}\right)^n - \left(\frac{y + \sqrt{y^2 - 4xz}}{2x}\right)^n + \left(\frac{z}{x}\right)^n\right) \end{aligned}$$

and

$$(r\omega^{-j} + s\omega^{-2j} + t\omega^{-3j} - 1) = (\alpha\omega^{-j} - 1)(\beta\omega^{-j} - 1)(\gamma\omega^{-j} - 1),$$

we see that

$$\prod_{j=0}^{n-1}(r\omega^{-j} + s\omega^{-2j} + t\omega^{-3j} - 1) = (-1)^{n+1}(H_n + (1 - H_{-n})t^n - 1).$$

and

$$\begin{aligned}
& \prod_{j=0}^{n-1}((W_n - W_0) - (-W_{n+1} + rW_n + W_1 - rW_0)\omega^{-j} \\
& +(W_{n+2} - rW_{n+1} - sW_n - W_2 + rW_1 + sW_0)\omega^{-2j}) \\
= & \Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)
\end{aligned}$$

$$\begin{aligned}
\omega &= \exp(2\pi i/n), \\
\Phi_j &= (W_n - W_0) - (-W_{n+1} + rW_n + W_1 - rW_0)\omega^{-j} \\
&\quad +(W_{n+2} - rW_{n+1} - sW_n - W_2 + rW_1 + sW_0)\omega^{-2j}
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_1 &= W_n - W_0, \\
\Lambda_2 &= -W_{n+1} + rW_n + W_1 - rW_0, \\
\Lambda_3 &= W_{n+2} - rW_{n+1} - sW_n - W_2 + rW_1 + sW_0,
\end{aligned}$$

From Theorem 2.10, we have

$$\begin{aligned}
\det(C_n(W)) &= \prod_{j=0}^{n-1} \lambda_j(C_n(W)) \\
& \prod_{j=0}^{n-1} \frac{\Phi_j}{(r\omega^{-j} + s\omega^{-2j} + t\omega^{-3j} - 1)} \\
&= \frac{\prod_{j=0}^{n-1} \Phi_j}{\prod_{j=0}^{n-1}(r\omega^{-j} + s\omega^{-2j} + t\omega^{-3j} - 1)} \\
&= \frac{\Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(H_n + (1 - H_{-n})t^n - 1)}
\end{aligned}$$

which completes the proof. □

Remark 2.14. Note that choosing suitable values on r, s, t and $W_0 = a, W_1 = b, W_2 = c$ (initial values) in Theorems 2.8, 2.9, 2.10, 2.13, lower and upper bounds of the Euclidean norm, eigenvalues, the spectral norm and determinant of circulant matrices for the special case of all third order sequences can be obtained.

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