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Triangular Scheme Revisited in the Light of n-permutable Categories

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Abstract

The first diagrammatic scheme was developed by H.P. Gumm under the name Shifting Lemma in case to characterize congruence modularity. A diagrammatic scheme is developed for the generalized semi distributive law in Mal'tsev categories. In this paper we study this diagrammatic scheme in the context of n-permutable, and of Mal'tsev categories in particular. Several remarks concerning the Triangular scheme case are included.

1 Introduction

We are going to establish a diagrammatic scheme for equivalence distributivity in equivalence n-permutable category. In this section we recall some basic definitions and results from the literature, needed throughout the article.

1.1 n-permutable varieties

A variety of universal algebras is called n-permutable, $n \geq 2$, when any pair of congruences R and S on a same algebra n-permutes: $(R, S)_n = (S, R)_n$. Where $(R, S)_n$ denotes the composite $RSRS\cdots$ of R and S, n times. This notion determines a sequence of families of varieties, whose first two instances are well known: for $n \geq 2$, we regain 2-permutable varieties [15], better known as Mal'tsev varieties; for $n \geq 3$,

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these are the 3-permutable varieties. The property of n-permutability makes sense in any regular category and was generalised to this categorical context in [5], where n-permutable categories were first studied. A variety is a Mal'tsev variety precisely when its theory contains a ternary operation p such that the identities $p(x,y,y)=x$ and $p(x,x,y)=y$ hold [14]. 3-permutable varieties are characterised by the existence of two ternary operations r and s satisfying the identities $r(x,y,y)=x$, $r(x,x,y)=s(x,y,y)$ and $s(x,x,y)=y$. Equivalently, they are characterised by the existence of quaternary operations p and q such that the identities $p(x,y,y,z)=x, p(x,x,y,y)=q(x,x,y,y)$ and $q(x,y,y,z)=z$ hold. Congruence distributive varieties were characterized by Jónsson [18] by means of the Maltsev condition. A lattice L is called distributive when

$$b\wedge(a\vee c)=(b\wedge a)\vee(b\wedge c). \qquad (d)$$

Equivalently, L is distributive if and only if it satisfies the Horn sentence

$$a\wedge b\leq c\Rightarrow b\wedge(a\vee b)\leq c. \qquad (s)$$

Hence, (s) is a Horn sentence and a lattice L satisfies (d) if and only if it satisfies (s). Thus (s) is another characterization of distributivity. A variety $\mathcal{V}$ of universal algebras is called congruence distributive when the lattice Cong(A) of congruences on any algebra A in $\mathcal{V}$ is distributive. Recall that a lattice L is $\wedge$-semidistributive if

$$y\wedge x=y\wedge z\ \Rightarrow\ y\wedge(x\vee z)=y\wedge z\ \text{ for all } x,y,z\in L. \qquad (SD_\wedge)$$

The $\wedge$-semidistributive law above is often denoted by $SD_\wedge$. More general (in fact, weaker) Horn sentences have been investigated in Geyer [12]. For $n\geq 2$ put $\underline{n}=\{0,1,\cdots,n-1\}$ and let $l_2(\underline{n})$ denote the set $\{I : I\subseteq \underline{n}$ and $|I|\geq 2\}$. For $\emptyset\neq K\subseteq l_2(\underline{n})$ we define the generalized meet semidistributive law $SD_\wedge(n,K)$ for lattices as follows: for all $x,y_0,\cdots,y_{n-1}$

$$x\wedge y_0=x\wedge y_1=\cdots=x\wedge y_{n-1}\Rightarrow x\wedge y_0=\bigwedge_{J\in K}\bigvee_{j\in J}y_j.$$

As a particular case, when $K=\{J : J\subseteq\underline{n}$ and $|J|=2\}$ is denoted by $SD_\wedge(n,2)$. Notice that $SD_\wedge(n,2)$ is the following lattice Horn sentence:

$$x\wedge y_0=x\wedge y_1=\cdots=x\wedge y_{n-1}\Rightarrow x\wedge y_0=x\wedge\bigwedge_{0\leq i<j<n}(y_i\vee y_j).$$

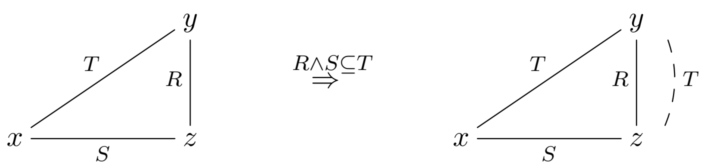
In particular, $SD_\wedge(2,2)$ is the $\wedge$-semidistributivity law defined in by:

$$x \wedge y = y \wedge z \ \Rightarrow \ y \wedge (x \vee z) = y \wedge z \ \text{ for all } x, y, z \in L. \qquad (SD_\wedge)$$

1.2 n-schemes and weak n-scheme

Recall from [16] that a sublattice L of an equivalence lattice EqA satisfies the Triangular scheme if for each $R, S, T \in L$ with $R \wedge S \subseteq T$ and for $x, y, z \in A$ such that $\langle x, y\rangle \in T$, $\langle x, z\rangle \in S$, $\langle z, y\rangle \in R$ we have $\langle z, y\rangle \in T$.

This can be visualized as follows:



The following assertions are proved in [17]:

(a) if $ConA$ is distributive, then it satisfies the Triangular scheme;

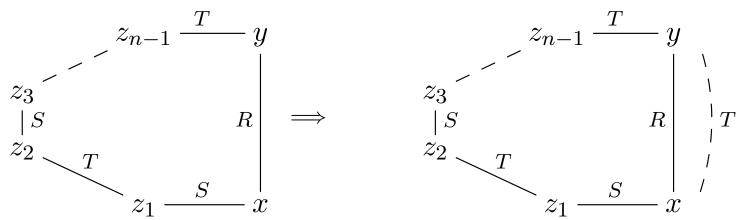
(b) if A is a congruence permutable algebra, then $ConA$ is distributive if and only if $ConA$ satisfies the Triangular scheme.

We are going to establish a diagrammatic scheme for equivalence distributivity in equivalence n-permutable category. The scheme is similar to that of Gumm [26] for congruence modularity.

A sublattice L of EqA satisfies the n-scheme (the weak n-scheme) if for each $R, S, T \in L$ with $R \wedge S \subseteq T$ (or $R \wedge S = R \wedge T$, respectively) and for $x, y, z_1, \cdots, z_n \in A$ such that

$$\langle x, y\rangle \in R, \langle x, z_1\rangle \in S, \langle z_1, z_2\rangle \in T, \langle z_2, z_3\rangle \in S, \cdots, \langle z_{n-1}, y\rangle \in S$$

for n odd and $\langle z_{n-1}, y\rangle \in T$ for n even we have $\langle x, y\rangle \in T$. These schemes can be also visualized but, contrary to the previous cases, classes of the same congruence fail to be parallel:



1.3 Regular categories and relations

Our categories will always be regular, in the sense of Barr [1]; we recall that a category is regular if it has finite limits, each arrow factors as a regular epi followed by a mono, and regular epis are pull-back stable. (It turns out that in a regular category the kernel pair of an arrow always has a coequalizer, given by the regular epi part of the factorization of the arrow). In a regular category, it is possible to compose relations. If (R, r_1, r_2) is a relation from X to Y and (S, s_1, s_2) a relation from Y to Z, their composite SR is a relation from X to Z obtained as the regular image of the arrow

$$(r_1\pi_1, s_2\pi_2) : R \times_Y S \longrightarrow X \times Z,$$

where $(R \times_Y S, \pi_1, \pi_2)$ is the pullback of r_2 along s_1. The composition of relations is then associative, thanks to the fact that regular epimorphisms are assumed to be pullback stable.

2 Regular Mal'tsev Categories and Triangular Scheme

A finitely complete category $\mathcal{E}$ is called a Mal'tsev category if every reflexive relation in $\mathcal{E}$ is an equivalence relation. These categories are also characterized by other properties on relations, as follows:

Theorem 2.1. [16] *Let $\mathcal{E}$ be a regular category. Then the following statements are equivalent:*

(i) *$\mathcal{E}$ is a Mal'tsev category;*

(ii) $\forall F, E \in Equiv(X), E \vee F = FE(= EF) \in Equiv(X)$, *for any object X;*

(iii) *every reflexive relation E in $\mathcal{E}$ is symmetric: $E^\circ = E$.*

Theorem 2.2. *Let $\mathcal{E}$ be a regular Mal'tsev category. Then the following conditions are equivalent:*

(1) *the Triangular scheme holds in $\mathcal{E}$;*

(2) $\forall E, F, P \in Equiv(X)$ *we have*

$$F \wedge (EP) \subseteq P \vee (E \wedge F);$$

(3) $\forall E, F, P \in Equiv(X)$ *then*

$$F \wedge (EP) \subseteq (E \vee (F \wedge P)) \wedge (P \vee (E \wedge F)).$$

Proof. To prove the implication $(1) \Rightarrow (2)$ suppose $E, F, P \in Equiv(X)$ are arbitrary, and $(c, a) \in F \wedge (EP)$. Then there exists an element b such that $aPbEc$. Apply the triangular scheme for these elements with P replaced by $P' = P \vee (E \wedge F)$. We get that

$$\begin{tikzcd} & a \arrow[dl, dash, "P'"'] \arrow[d, dash, "F"', "\}\; P'=P\vee(E\wedge F)"] \\ b \arrow[r, dash, "E"'] & c \end{tikzcd}$$

Now if (2) is true, then reversing the roles of E and P we get that

$$F \wedge (PE) \subseteq E \vee (P \vee F).$$

By taking the converse of this inclusion and combining with (2) we obtain condition (3). Therefore the implication $(2) \Rightarrow (3)$ is proved. $(3) \Rightarrow (2)$ Obvious.

$(2) \Rightarrow (1)$ Let R, S and T be equivalence relations on an object X such that $R \wedge S \leq T$ we show that $R \wedge ST \leq T$.

We have

$$R \wedge ST \leq T \vee (S \wedge R) \leq T \vee T = T.$$

□

Proposition 2.3. *Let $\mathcal{E}$ be a regular Mal'tsev category. Then the following conditions are equivalent:*

(1) *Equiv(X) satisfies $SD_{\wedge}(n,2)$ for X in $\mathcal{E}$;*

(2) *the scheme depicted in Figure 1 holds for $S, R_0, \cdots, R_{n-1}$ in Equiv(X) and $x_0, \cdots, x_k, y, z$, are related as in Figure 1 where $k = \frac{n(n-1)}{2} - 1$ and T stands for $R_0 \wedge R_1 \wedge \cdots \wedge R_{n-1}$.*

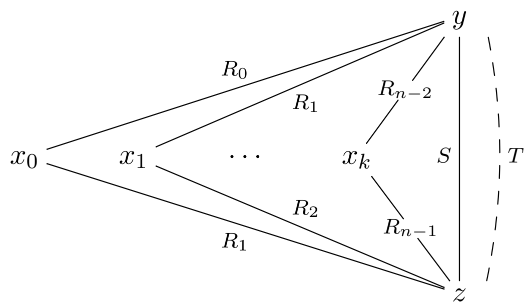


Figure 1

Proof. Suppose $SD_{\wedge}(n,2)$ holds. Using the premise of $SD_{\wedge}(n,2)$ we obtain

$$S \wedge R_0 = (S \wedge R_0) \wedge \cdots \wedge (S \wedge R_0) = S \wedge (R_0 \wedge \cdots \wedge R_{n-1}) \subseteq T,$$

whence Equiv(X) satisfies the Horn sentence

$$S \wedge R_0 = S \wedge R_1 = \cdots = S \wedge R_{n-1} \Rightarrow S \wedge \bigwedge_{0 \leq i < j < n} (R_i \vee R_j) \leq T.$$

This implies the scheme, for the situation on the left hand side in Figure 1 then gives $(y,z) \in S \wedge \bigwedge_{0 \leq i < j < n} (R_i \circ R_j) \leq T$. To show the converse, suppose that the scheme given by Figure 1 holds, $S, R_0, \cdots, R_{n-1}$ in Equiv(X) with $S \wedge R_0 = S \wedge R_1 = \cdots = S \wedge R_{n-1}$, and suppose that $(y,z) \in S \wedge \bigwedge_{0 \leq i < j < n} (R_i \circ R_j)$, there exist $x_0, x_1, \cdots, x_k$ of X such that for each $j (1 \leq j \leq k)$ there exist u, v such that $(z, x_j) \in R_u$ and $(x_j, y) \in R_v$ (according to the left hand side of Figure 1). Then the scheme applies and we conclude $(y,z) \in T$. Since $T \subseteq R_0, (y,z) \in R_0$. Hence $(y,z) \in S \wedge R_0$. This proves the " $\leq$ "

part of $SD_\wedge(n,2)$. The reverse part is simpler and does not need the scheme: $S \supseteq S \wedge R_0$ and $R_i \vee R_j \supseteq R_j \supseteq S \wedge R_i = S \wedge R_0$ clearly give $S \wedge \bigwedge_{0 \le i < j < n} (R_i \vee R_j) \supseteq S \wedge R_0$ proving the theorem. $\square$

Note that, for diagram such as Figure 1 where R_i, S or T are not symmetric, the relations are always to be considered from left to right and from top to bottom. To avoid ambiguity with the interpretation of such diagrams, from now on we will write $a \xrightarrow{E} b$ to mean that $(a,b) \in E$ whenever E is a non-symmetric relation.

Theorem 2.4. *Let $\mathcal{E}$ be a regular category. Then the following conditions are equivalent:*

(i) *$\mathcal{E}$ is an equivalence $SD_\wedge(n,2)$ Mal'tsev category;*

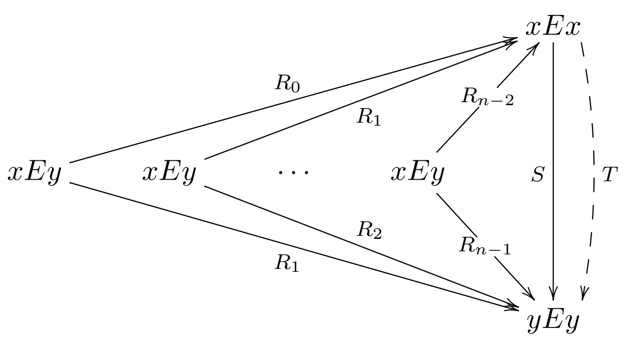
(ii) *the scheme depicted in Figure 1 holds when $S, R_0, \cdots, R_{n-1}$ and T are reflexive relations.*

Proof. $(i) \Rightarrow (ii)$ Since $\mathcal{E}$ is a Mal'tsev category, reflexive relations are necessarily equivalence relations. Since $\mathcal{E}$ is also equivalence $SD_\wedge(n,2)$, by Proposition 2.3, the the scheme depicted in Figure 1 holds for any reflexive relations in $\mathcal{E}$.

$(ii) \Rightarrow (i)$ To prove that $\mathcal{E}$ is a Mal'tsev category, we show that any reflexive relation $\langle e_1, e_2 \rangle : E \rightarrowtail X \times X$ in $\mathcal{E}$ is also symmetric (Theorem 2.1 (iii)). Suppose that $(x,y) \in E$, and consider the reflexive relations T and S on E defined as follows:

$$(aEb, cEd) \in S \Leftrightarrow (a,d) \in E, \ \text{ and } (aEb, cEd) \in T \Leftrightarrow (c,b) \in E.$$

Let $R_i = T$, for all $1 \le i \le n-1$.



(xEx and yEy by the reflexivity of the relation E). We conclude that $(xEx, yEy) \in T$ and, consequently, that $(y, x) \in E$, so that $\mathcal{E}$ is a Mal'tsev category. Since the Figure 1 holds in $\mathcal{E}$, by Proposition 2.3 the category $\mathcal{E}$ is equivalence $SD_\wedge(n, 2)$. □

3 n-permutable category

Definition 3.1. [23] A regular category $\mathcal{E}$ is an a n permutable category when the composition of (effective) equivalence relations on a given object is n-permutable: for two (effective) equivalence relations R and S on the same object, we have $(R, S)_n = (S, R)_n$. Where the composition of n alternating factors R and S is denoted by $(R, S)_n = RSRS\cdots$

Theorem 3.2. ([23], Theorem 3.5 of [5]) *Let $n \geq 2$ and let $\mathcal{E}$ be a regular category. Then the following statements are equivalent:*

(i) *$\mathcal{E}$ is n-permutable category;*

(ii) *$(P, P^\circ)_{n+1} \leq (P, P^\circ)_{n-1}$ for any relation P;*

(iii) *$(R, S)_n$ is an equivalence relation and is therefore $R \vee S$;*

(iv) *$(1_X \wedge T)T^\circ(1_X \wedge T) \leq T^{n-1}$, for any relation T on an objet X;*

(v) *for any reflexive endorelation $E \rightarrowtail X \times X$ in $\mathcal{E}$, the relation $(E, E^\circ)_{n-1}$ is an equivalence relation;*

(vi) *for any such reflexive endorelation E, the relation $(E, E^\circ)_{n-1}$ is transitive;*

(vii) *for any such reflexive endorelation E we have $(E, E^\circ)_{n-1} = (E^\circ, E)_{n-1}$;*

$(viii)$ *$E^\circ \leq E^{n-1}$ for any reflexive relation E.*

Theorem 3.3. *Let $\mathcal{E}$ be n-permutable category. Then*

(i) *$\mathcal{E}$ is an equivalence $\wedge$-semidistributive if and only if it satisfies the weak n-scheme.*

(ii) *$\mathcal{E}$ is an equivalence distributive if and only if it satisfies the n-scheme.*

Proof. For (i), suppose R, S, T in Equiv(X) with $R \wedge S = R \wedge T$ and $x, y, z_1, \cdots, z_{n-1} \in X$ such that the assumptions of the weak scheme are satisfied. Then

$$\langle x, y \rangle \in R \wedge (S, T)_n = R \wedge (S \vee T) = R \wedge T$$

due to $\wedge$-semidistributivity, i.e. it satisfies the weak n-scheme for each $n \geq 2$. Conversely, let Equiv(X) satisfy the weak n-scheme, let R, S, T in Equiv(X) with $R \wedge S = R \wedge T$. Let $\langle x, y \rangle \in R \wedge (S \vee T)$. Due to equivalence n-permutability, we have $\langle x, y \rangle \in R \wedge (S, T)_n$ with n factors. Thus it is almost evident that the assumptions of the weak n-scheme are satisfied. Applying this scheme, we conclude $\langle x, y \rangle \in R \wedge T$. We have shown $R \wedge (S \vee T) \subseteq R \wedge T$.
(ii) Obvious. □

Let n be an odd number $(n \geq 3)$. A relation $P \rightarrowtail X \times X$ on X is called positive when it is of the form $P = (E^\circ, E)_{n-1}$, for some relation $E \rightarrowtail X \times X$. In set-theoretic terms, P is positive when there exists a relation E and $x_1, x_2, \cdots, x_{n-1}$ such that $(x, x') \in P$ if $(x, x_1) \in E$, $(x_2, x_1) \in E \cdots (x', x_{n-2}) \in E$.

Proposition 3.4. *Let n be an odd number $(n \geq 3)$. A regular category $\mathcal{E}$ is an n-permutable category if and only if any reflexive and positive relation in $\mathcal{E}$ is an equivalence relation.*

Proof. Suppose that $\mathcal{E}$ is an n-permutable category and consider a reflexive and positive relation P with $1 \leq P = (E^\circ, E)_{n-1}$. Then P is symmetric. Since

$$P^\circ = (E, E^\circ)_{n-1} = (E^\circ, E)_{n+1} = (E^\circ, E)_{n-1}.$$

And As for the transitivity of P, we have

$$(E^\circ, E)_{n-1}(E^\circ, E)_{n-1} = (E^\circ, E)_{2n-2} = (E^\circ, E)_{n-1}.$$

By Theorem 3.2 $(i) \Rightarrow (ii)$. Conversely, let U be a reflexive relation on X. Then $P = (U^\circ, U)_{n-1}$ is a reflexive and positive relation, thus an equivalence relation by assumption. It follows that $\mathcal{E}$ is an n-permutable category by Theorem 3.2 (v). □

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