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Subclass of p-valent Function with Negative Coefficients Applying Generalized Al-Oboudi Differential Operator

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Abstract

In this paper we introduce a new subclass $\mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$ of p-valent functions with negative coefficient defined by Hadamard product associated with a generalized differential operator. Radii of close-to-convexity, starlikeness and convexity of the class $\mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$ are obtained. Also, distortion theorem, growth theorem and coefficient inequalities are established.

1 Introduction and Definitions

Let $\mathcal{G}$ be class of functions $f(z)$ of the form

$$f(z) = z + \sum_{w=2}^{\infty} l_w z^w \tag{1.1}$$

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which are holomorphic in the open unit disk $\triangle = \{z \in \mathbb{C} : |z| < 1\}$.

For $f(z)$ belongs to $\mathcal{G}$, Opoola [6] (see also [11, 12]) has introduced the following differential operator:

$$D^{0,\gamma}_{\phi,\zeta} f(z) = f(z)$$

$$D^{1,\gamma}_{\phi,\zeta} f(z) = (1 + (\phi - \gamma - 1)\zeta) f(z) - z(\gamma - \phi)\zeta + z\zeta f'(z) = D^{\gamma}_{\phi,\zeta} f(z), \quad (1.2)$$

$$D^{2,\gamma}_{\phi,\zeta} f(z) = D^{\gamma}_{\phi,\zeta}(D^{1,\gamma}_{\phi,\zeta} f(z)),$$

$$D^{h,\gamma}_{\phi,\zeta} f(z) = D^{\gamma}_{\phi,\zeta}(D^{h+1,\gamma}_{\phi,\zeta} f(z)), \quad (1.3)$$

if $f(z)$ is given by (1.1), then by (1.2) and (1.3), we see that

$$D^{h,\gamma}_{\phi,\zeta} f(z) = z + \sum_{w=2}^{\infty} (1 + (w + \phi - \gamma - 1)\zeta)^h l_w z^w, \quad (1.4)$$

where $0 \leq \phi \leq \gamma, \zeta \geq 0$ and $h \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$.

Let $\mathcal{T}$ denote the subclass of $\mathcal{G}$ consisting of the form

$$f(z) = z - \sum_{w=2}^{\infty} l_w z^w, \quad (1.5)$$

where $l_w \geq 0$ and $w \in \mathbb{N}$. This class has introduced and studied by Silverman [9].

The Hadamard product of two power series

$$f(z) = z - \sum_{w=2}^{\infty} l_w z^w, \qquad g(z) = z - \sum_{w=2}^{\infty} j_w z^w$$

and it is defined in $\mathcal{T}$ as follows:

$$(f * g) = f(z) * g(z) = f(z) = z - \sum_{w=2}^{\infty} l_w j_w z^w.$$

Let $\mathcal{G}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{w=1}^{\infty} l_{p+w} z^{p+w} \quad (1.6)$$

that are holomorphic and p-valent in $|z| < 1$.

Also let $\mathcal{T}_p$ denote the subclass of $\mathcal{G}_p$ consisting of functions that can be expressed as

$$f(z) = z^p - \sum_{w=1}^{\infty} l_{p+w} z^{p+w}. \tag{1.7}$$

The Hadamard product of two power series

$$f(z) = z^p - \sum_{w=1}^{\infty} l_{p+w} z^{p+w}, \qquad g(z) = z^p - \sum_{w=1}^{\infty} j_{p+w} z^{p+w}$$

and it is defined in $\mathcal{T}_p$ as follows:

$$(f * g) = f(z) * g(z) = f(z) = z - \sum_{w=1}^{\infty} l_{p+w} j_{p+w} z^{p+w}.$$

From the above differential operator, the convolution of two power series $f(z)$ and $g(z)$ is given by

$$D^{h,\gamma}_{\phi,\zeta,p}(f * g)(z) = p^h z^p - \sum_{w=1}^{\infty} (1 + (p + w + \phi - \gamma - 1)\zeta)^h l_{p+w} j_{p+w} z^{p+w}, \tag{1.8}$$

where $p \in \mathbb{N} = \{1, 2, 3, \cdots\}$. Motivated by [2], [10], [7], we define a new subclass $\mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$ of the class $\mathcal{T}_p$.

Definition 1.1. For $0 \le \psi < 1$, $\varrho \ge 0$ and $0 \le \phi \le \gamma$, $0 \le \beta \le \frac{1}{2}$, $\zeta \ge 0$, we let $\mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$ be subclass of the class $\mathcal{T}_p$ consisting of functions of the form (1.7) and satisfying the analytic criterion

$$\Re\left\{ \frac{z D^{h,\gamma}_{\phi,\zeta,p}(f * g)'(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f * g)(z)} + \beta \frac{z^2 D^{h,\gamma}_{\phi,\zeta,p}(f * g)''(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f * g)(z)} - \psi \right\} \ge \varrho \left| \frac{z D^{h,\gamma}_{\phi,\zeta,p}(f * g)'(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f * g)(z)} + \beta \frac{z^2 D^{h,\gamma}_{\phi,\zeta,p}(f * g)''(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f * g)(z)} - 1 \right|. \tag{1.9}$$

The main purpose of this paper is to investigate some geometric properties of the class $\mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$ such as the coefficient bounds, growth and radii of starlikeness, distortion properties, convexity and close to convexity for the class $\mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$. [5], [9], [1], [4], [8], [3], study the univalent functions for different classes.

2 Coefficient Inequalities

In the following theorem we obtain necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$. We have the following lemma useful for this work.

Lemma 2.1. [8] *Let $\psi \geq 0$ and ν be any complex number. Then $\Re(\nu) \geq \psi$ if and only if*

$$|\nu - (1 + \psi)| < |\nu + (1 - \psi)|.$$

Lemma 2.2. [8] *Let $\varrho \geq 0$, $0 \leq \psi$ and $\theta \in \mathbb{R}$. Then*

$$\Re(\nu) > \varrho|\nu - 1| + \psi$$

if and only if

$$\Re\left(\nu(1 + \varrho e^{i\theta}) - \varrho e^{i\theta}\right) > \psi,$$

where ν is a complex number.

Theorem 2.3. *Let $f(z) \in \mathcal{T}_p$ be given by (1.7). Then $f(z) \in \mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$ if and only if*

$$\sum_{w=1}^{\infty} \{(p+k)[1+(p+w-1)\beta](1+\psi) - (\varrho+\psi)\}[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w} j_{p+w} \leq p^h[p - \psi]. \quad (2.1)$$

The result is sharp for the function

$$f(z) = z - \frac{p^h(p-\psi) + p^h(p-1)\varrho + \beta p^{h+1}(p-1)z^h(1-\varrho)}{\{(p+k)[1+(p+w-1)\beta](1+\psi) - (\varrho+\psi)\} [1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}}.$$

Proof. If $f(z) \in \mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$ and $|z| = 1$, then by Definition 1.1

$$\Re\left\{\frac{zD^{h,\gamma}_{\phi,\zeta,p}(f*g)'(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)} + \beta\frac{z^2D^{h,\gamma}_{\phi,\zeta,p}(f*g)''(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)} - \psi\right\} \geq \varrho\left|\frac{zD^{h,\gamma}_{\phi,\zeta,p}(f*g)'(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)} + \beta\frac{z^2D^{h,\gamma}_{\phi,\zeta,p}(f*g)''(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)} - 1\right|.$$

Using Lemma 2.2, it is sufficient to show that

$$\Re\left\{\left(\frac{zD^{h,\gamma}_{\phi,\zeta,p}(f*g)'(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)} + \beta\frac{z^2D^{h,\gamma}_{\phi,\zeta,p}(f*g)''(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)}\right)(1+\varrho e^{i\theta}) - \varrho e^{i\theta}\right\} \geq \psi,$$

$$\Re\left[\frac{[zD^{h,\gamma}_{\phi,\zeta,p}(f*g)'(z)](1+\varrho e^{i\theta}) + [\beta z^2D^{h,\gamma}_{\phi,\zeta,p}(f*g)''(z)](1+\varrho e^{i\theta}) - \varrho e^{i\theta}D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)}\right] > \psi. \tag{2.2}$$

For convenience, let

$$\mathcal{A}(z) = [zD^{h,\gamma}_{\phi,\zeta,p}(f*g)'(z)](1+\varrho e^{i\theta}) + [\beta z^2D^{h,\gamma}_{\phi,\zeta,p}(f*g)''(z)](1+\varrho e^{i\theta}) - \varrho e^{i\theta}D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)$$

and

$$\mathcal{B}(z) = D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z).$$

That is equation (2) is equivalent to

$$\Re\left(\frac{\mathcal{A}(z)}{\mathcal{B}(z)}\right) \geq \psi$$

applying Lemma 2.1

$$\left|\frac{\mathcal{A}(z)}{\mathcal{B}(z)} - (1+\psi)\right| \leq \left|\frac{\mathcal{A}(z)}{\mathcal{B}(z)} + (1-\psi)\right| \implies \left|\frac{\mathcal{A}(z) - (1+\psi)\mathcal{B}(z)}{\mathcal{B}(z)}\right| < \left|\frac{\mathcal{A}(z) + (1-\psi)\mathcal{B}(z)}{\mathcal{B}(z)}\right|$$

$$|\mathcal{A}(z)+(1-\psi)\mathcal{B}(z)|-|\mathcal{A}(z)-(1+\psi)\mathcal{B}(z)|.$$

Now,

$$|\mathcal{A}(z)+(1-\psi)\mathcal{B}(z)| = \Bigg|\Bigg[p^{h+1}z^p-\sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h (p+w)l_{p+w}j_{p+w}z^{p+w}\Bigg](1+\varrho e^{i\theta})+\Bigg[p^{h+1}(p-1)z^p-\sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h (p+w)(p+w-1)l_{p+w}j_{p+w}z^{p+w}\Bigg]\beta(1+\varrho e^{i\theta})-\Bigg[p^n z^p-\sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w}j_{p+w}z^{p+w}\Bigg]\varrho e^{i\theta}+(1-\psi)\Bigg[p^n z^p-\sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w}j_{p+w}z^{p+w}\Bigg]\Bigg|$$

$$=\Bigg|p^{h+1}z^p+p^{h+1}z^p\varrho e^{i\theta}-\sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h(p+w)(1+\varrho e^{i\theta})l_{p+w}j_{p+w}z^{p+w} +p^{h+1}(p-1)z^p\beta(1+\varrho e^{i\theta})-\sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h(p+w)(p+w-1)(1+\varrho e^{i\theta}) \beta l_{p+w}j_{p+w}z^{p+w}-p^h z^p\varrho e^{i\theta}+\sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h\varrho e^{i\theta}l_{p+w}j_{p+w}z^{p+w}+p^h z^p-\psi p^h z^p -(1-\psi)\sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w}j_{p+w}z^{p+w}\Bigg|$$

$$=\Bigg|p^h z^p[p-\psi+1]+p^h z^p\varrho e^{i\theta}[p-1]+p^{h+1}(p-1)z^p\beta(1+\varrho e^{i\theta})-\sum_{w=1}^{\infty}\Bigg[(p+w)(1+\varrho e^{i\theta}) +\beta(p+w)(p+w-1)(1+\varrho e^{i\theta})-\varrho e^{i\theta}+(1-\psi)\Bigg]\Bigg[[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w}j_{p+w}z^{p+w}\Bigg]\Bigg|.$$

Now with $|z| = 1$

$$\geq p^h[p-\psi+1]+p^h\varrho[p-1]+p^{h+1}(p-1)\beta(1+\varrho)-\sum_{w=1}^{\infty}\Bigg[(p+w)[1+\beta(p+w-1)](1+\varrho) - \varrho + (1-\psi)\Bigg]\Bigg[[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w}j_{p+w}\Bigg]. \quad (2.3)$$

Also,

$$|\mathcal{A}(z)-(1+\psi)\mathcal{B}(z)| = \Bigg|\Bigg[p^{h+1}z^p - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h (p+w)l_{p+w}j_{p+w}z^{p+w}\Bigg](1+\varrho e^{i\theta}) + \Bigg[p^{h+1}(p-1)z^p - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h (p+w)(p+w-1)l_{p+w}j_{p+w}z^{p+w}\Bigg]\beta(1+\varrho e^{i\theta}) - \Bigg[p^n z^p - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w}j_{p+w}z^{p+w}\Bigg]\varrho e^{i\theta} - (1+\psi)\Bigg[p^n z^p - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w}j_{p+w}z^{p+w}\Bigg]\Bigg|$$

$$= \Bigg|p^{h+1}z^p + p^{h+1}z^p\varrho e^{i\theta} - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h(p+w)(1+\varrho e^{i\theta})l_{p+w}j_{p+w}z^{p+w} + p^{h+1}(p-1)z^p\beta(1+\varrho e^{i\theta}) - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h(p+w)(p+w-1)(1+\varrho e^{i\theta})\beta l_{p+w}j_{p+w}z^{p+w} - p^h z^p\varrho e^{i\theta} + \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h\varrho e^{i\theta}l_{p+w}j_{p+w}z^{p+w} - p^h z^p - \psi p^h z^p - (1-\psi)\sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w}j_{p+w}z^{p+w}\Bigg|$$

$$= \Bigg|p^h z^p[p-\psi-1] + p^h z^p\varrho e^{i\theta}[p-1] + p^{h+1}(p-1)z^p\beta(1+\varrho e^{i\theta}) - \sum_{w=1}^{\infty}\Bigg[(p+w)(1+\varrho e^{i\theta}) + \beta(p+w)(p+w-1)(1+\varrho e^{i\theta}) - \varrho e^{i\theta} - (1+\psi)\Bigg]\Bigg[[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w}j_{p+w}z^{p+w}\Bigg]\Bigg|.$$

Now with $|z| = 1$

$$\leq p^h[\psi+1-p]+p^h\varrho[p-1]+p^{h+1}(p-1)\beta(1+\varrho)+\sum_{w=1}^{\infty}\Bigg[(p+w)[1+\beta(p+w-1)](1+\varrho) - \varrho - (1+\psi)\Bigg]\Bigg[[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w} j_{p+w}\Bigg]. \quad (2.4)$$

It is easy to show that

$$|\mathcal{A}(z)+(1-\psi)\mathcal{B}(z)|-|\mathcal{A}(z)-(1+\psi)\mathcal{B}(z)| = 2p^h[p-\psi]-2\sum_{w=1}^{\infty}\Bigg[(p+w)[1+\beta(p+w-1)] (1+\varrho)-(\varrho+\psi)\Bigg]\Bigg[[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w} j_{p+w}\Bigg] \geq 0$$

$$-2\sum_{w=1}^{\infty}\Bigg[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)\Bigg]\Bigg[[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w} j_{p+w}\Bigg] \geq -2p^h[p-\psi]$$

$$\sum_{w=1}^{\infty}\Bigg[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)\Bigg]\Bigg[[1+(p+w+\phi-\gamma-1)\zeta]^h l_{p+w} j_{p+w}\Bigg] \leq p^h[p-\psi].$$

Conversely, suppose the inequality (2.5) holds, we need to show that

$$\Re\left[\frac{[zD^{h,\gamma}_{\phi,\zeta,p}(f*g)'(z)](1+\varrho e^{i\theta})+[\beta z^2 D^{h,\gamma}_{\phi,\zeta,p}(f*g)''(z)](1+\varrho e^{i\theta}) - \varrho e^{i\theta}D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)}\right] > \psi,$$

$$\Re\left[\frac{[zD^{h,\gamma}_{\phi,\zeta,p}(f*g)'(z)](1+\varrho e^{i\theta})+[\beta z^2 D^{h,\gamma}_{\phi,\zeta,p}(f*g)''(z)](1+\varrho e^{i\theta}) - \varrho e^{i\theta}D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)-\psi D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)}{D^{h,\gamma}_{\phi,\zeta,p}(f*g)(z)}\right] > \psi.$$

Since $|e^{i\theta}| = 1$, hence $\Re(e^{i\theta}) \leq |e^{i\theta}| = 1$, letting $|z| \longrightarrow 1^{-1}$ yields, we let $H = [1 + (p + w + \phi - \gamma - 1)\zeta]^h$

$$\Re\left[\frac{p^h(p-\psi) + p^h[p-1]\varrho + \beta p^{h+1}(p-1)(1-\varrho) - \sum_{w=1}^{\infty} H l_{p+w} j_{p+w} \left[(p+w)[1+\beta(p+w-1)](1+\varrho) - (\varrho+\psi)\right]}{p^h - \sum_{w=1}^{\infty} H l_{p+w} j_{p+w}}\right] > 0, \tag{2.5}$$

then we have

$$\sum_{w=1}^{\infty} H l_{p+w} j_{p+w} \left[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)\right] \leq p^h(p-\psi)+p^h[p-1]\varrho + \beta p^{h+1}(p-1)z^p(1-\varrho)$$

$$f(z) = z - \frac{p^h(p-\psi) + p^h[p-1]\varrho + \beta p^{h+1}(p-1)z^p(1-\varrho)}{\sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h \left[(p+w)[1+\beta(p+w-1)](1+\varrho) - (\varrho+\psi)\right] j_{p+w}} \tag{2.6}$$

which completes the proof. □

Corollary 2.4. *Let $f(z) \in \mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$. Then*

$$l_{p+w} \leq \frac{p^h(p-\psi) + p^h[p-1]\varrho + \beta p^{h+1}(p-1)z^p(1-\varrho)}{[1+(p+w+\phi-\gamma-1)\zeta]^h \left[(p+w)[1+\beta(p+w-1)](1+\varrho) - (\varrho+\psi)\right] j_{p+w}}.$$

Taking $\beta = 0$ in Theorem 2.3, we have the following corollary.

Corollary 2.5. *Let $f(z) \in \mathcal{T}_p$. Then $f(z) \in \mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$ if and only if*

$$\sum_{w=1}^{\infty}\{(p+k)(1+\psi) - (\varrho+\psi)\}[1+(p+w+\phi-\gamma-1)\zeta]^h \leq p^h[p-\psi].$$

3 Growth Theorem and Distortion Theorem

Theorem 3.1. *If $f(z) \in \mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$ and $j_{p+w} \geq j_2$, then*

$$r^p - \frac{p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^{p+1} \leq |f(z)|$$
$$\leq r^p + \frac{p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^{p+1}$$

and

$$pr^{p-1} - \frac{(p+1)p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^p \leq |f(z)|$$
$$\leq pr^{p-1} + \frac{(p+1)p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^p.$$

The result is sharp for, $(|z| = r < 1)$

$$f(z) = z^p - \frac{p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^{p+1}.$$

Proof. Since

$$f(z) = z^p - \sum_{w=1}^{\infty} l_{p+w} z^{p+w},$$

we have

$$|f(z)| = |z^p - \sum_{w=1}^{\infty} l_{p+w} z^{p+w}| \leq |z|^p + \sum_{w=1}^{\infty} l_{p+w}|z|^{p+w} \leq r^p + r^{p+1}\sum_{w=1}^{\infty} l_{p+w} \quad (3.1)$$

$$[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}$$
$$\leq [(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)][1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}.$$

Using Theorem 2.3, we have

$$[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}\sum_{w=1}^{\infty} l_{p+w}$$
$$\leq \sum_{w=1}^{\infty} l_{p+w} j_{p+w}[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)][1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}$$
$$\leq p^h(p-\psi)$$

that is

$$\sum_{w=1}^{\infty} l_{p+w} \leq \frac{p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}}$$

using the above equation in 3.2, we have

$$|f(z)| \leq r^p + \frac{p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^{p+1}$$

and

$$|f(z)| \geq r^p - \frac{p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^{p+1}.$$

The result is sharp for

$$|f(z)| = z^p - \frac{p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^{p+1}.$$

Similarly, since

$$f'(z) = pz^{p-1} - \sum_{w=1}^{\infty}(p+w) l_{p+w} z^{p+w-1}$$

we have that

$$|f'(z)| = |pz^{p-1} - \sum_{w=1}^{\infty}(p+w) l_{p+w} z^{p+w-1}| \leq p|z|^{p-1} + \sum_{w=1}^{\infty}(p+w) l_{p+w} |z|^{p+w-1}$$
$$\leq pr^{p-1} + (p+1) r^p \sum_{w=1}^{\infty} l_{p+w} \quad (3.2)$$

$$[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}$$
$$\leq [(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)][1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}.$$

Using Theorem 2.3, we have

$$[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1} \sum_{w=1}^{\infty} l_{p+w}$$

$$\leq \sum_{w=1}^{\infty} l_{p+w} j_{p+w} [(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)][1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}$$
$$\leq p^h(p-\psi)$$

that is

$$\sum_{w=1}^{\infty} l_{p+w} \leq \frac{p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}}$$

using the above equation in 3.2,

$$|f'(z)| \leq pr^{p-1} + \frac{(p+1)p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^p$$

and

$$|f'(z)| \geq pr^{p-1} - \frac{(p+1)p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^p$$

$$|f'(z)| \geq 1 - \frac{(p+1)p^h(p-\psi)}{[(p+1)[1+\beta p](1+\varrho)-(\varrho+\psi)][1+(p+\phi-\gamma)\zeta]^h j_{p+1}} r^p.$$

This completes the proof. □

4 Radii of Univalent Starlikeness, Convexity and Close to Convexity

Theorem 4.1. *If $f(z) \in \mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$, then $f(z)$ is starlike of order $\sigma(0 \leq \sigma < 1)$ in the disc $|z| < r_1(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma)$, where*

$$r_1(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma) =$$

$$inf_{p+w} \left\{ \frac{(1-\sigma)[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)] \, [1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}}{(p+w-\sigma)[p^h(p-\psi)+p^h[p-1]\varrho+\beta p^{h+1}(p-1)z^p(1-\varrho)]} \right\}^{\frac{1}{p+w-1}}, \tag{4.1}$$

where $p \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \sigma \quad (0 \leq \sigma < 1)$$

for $|z| < r_1(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma)$, we have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &= \left|\frac{z[pz^{p-1} - \sum_{w=1}^{\infty}(p+w)l_{p+w}z^{p+w-1}]}{z[z^{p-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}]} - 1\right| \\ &= \left|\frac{pz^{p-1} - \sum_{w=1}^{\infty}(p+w)l_{p+w}z^{p+w-1}}{z^{p-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}} - 1\right| \\ &= \left|\frac{[pz^{p-1} - \sum_{w=1}^{\infty}(p+w)l_{p+w}z^{p+w-1}] - [z^{p-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}]}{z^{p-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}}\right| \\ &= \left|\frac{pz^{p-1} - \sum_{w=1}^{\infty}(p+w)l_{p+w}z^{p+w-1} - z^{p-1} + \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}}{z^{p-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}}\right| \\ &= \frac{(p-1)z^{p-1} - \sum_{w=1}^{\infty}(p+w-1)l_{p+w}z^{p+w-1}}{z^{p-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}} \leq 1-\sigma \qquad (4.2)\end{aligned}$$

if

$$\sum_{w=1}^{\infty} \frac{(p+w-\sigma)l_{p+w}|z|^{p+w-1}}{1-\sigma} \leq 1.$$

Hence by Theorem 2.3, (4.2) will be true if

$$\frac{(p+w-\sigma)l_{p+w}|z|^{p+w-1}}{1-\sigma} \leq \frac{[(p+w)[1+\beta(p+w-1)](1+\varrho) - (\varrho+\psi)][1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}}{[p^h(p-\psi) + p^h[p-1]\varrho + \beta p^{h+1}(p-1)z^p(1-\varrho)]}$$

and hence

$$|z| \leq \left\{\frac{(1-\sigma)[(p+w)[1+\beta(p+w-1)](1+\varrho) - (\varrho+\psi)][1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}}{(p+w-\sigma)[p^h(p-\psi) + p^h[p-1]\varrho + \beta p^{h+1}(p-1)z^p(1-\varrho)]}\right\}^{\frac{1}{p+w-1}} \qquad (4.3)$$

setting

$$|z| = r_1(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma)$$

we get the desired result. □

Theorem 4.2. *If $f(z) \in \mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$, then $f(z)$ is convex of order $\sigma(0 \leq \sigma < 1)$ in the disc $|z| < r_2(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma)$, where*

$$r_2(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma) =$$
$$inf_{p+w} \left\{ \frac{(1-\sigma)[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)] \quad [1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}}{(p+w)(p+w-\sigma)[p^h(p-\psi)+p^h[p-1]\varrho+\beta p^{h+1}(p-1)z^p(1-\varrho)]} \right\}^{\frac{1}{p+w-1}}, \tag{4.4}$$

where $p \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Proof. It is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} + 1 - 1 \right| = \left| \frac{p(p-1)z^{p-1} - \sum_{w=1}^{\infty}(p+w)(p+w-1)l_{p+w}z^{p+w-1}}{pz^{p-1} - \sum_{w=1}^{\infty}(p+w)l_{p+w}z^{p+w-1}} \right|$$
$$= \frac{p(p-1) - \sum_{w=1}^{\infty}(p+w)(p+w-1)l_{p+w}z^p}{p - \sum_{w=1}^{\infty}(p+w)l_{p+w}z^p} \leq 1-\sigma \tag{4.5}$$

then

$$\sum_{w=1}^{\infty} \frac{(p+w)(p+w-\sigma)l_{p+w}|z|^{p+w-1}}{1-\sigma} \leq 1-\sigma.$$

Hence, by Theorem 2.3, (4.5) will be

$$\frac{(p+w)(p+w-\sigma)l_{p+w}|z|^{p+w-1}}{1-\sigma} \leq \frac{1}{\sum_{w=1}^{\infty} l_{p+w}}$$

$$\frac{(p+w)(p+w-\sigma)l_{p+w}|z|^{p+w-1}}{1-\sigma} \leq$$
$$\frac{[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)][1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}}{[p^h(p-\psi)+p^h[p-1]\varrho+\beta p^{h+1}(p-1)z^p(1-\varrho)]}$$

and hence

$$|z| \leq \left\{ \frac{(1-\sigma)[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)] \quad [1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}}{[p^h(p-\psi)+p^h[p-1]\varrho+\beta p^{h+1}(p-1)z^p(1-\varrho)]} \right\}^{\frac{1}{p+w-1}} \tag{4.6}$$

setting

$$|z| = r_2(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma)$$

we get the desired result. □

Theorem 4.3. *If $f(z) \in \mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta)$, then $f(z)$ is close to convex of order $\sigma (0 \leq \sigma < 1)$ in the disc $|z| < r_3(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma)$, where*

$$r_3(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma) = \\ inf_{p+w} \left\{ \frac{(1-\sigma)[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)] \\ [1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}}{(p+w)[p^h(p-\psi)+p^h[p-1]\varrho+\beta p^{h+1}(p-1)z^p(1-\varrho)]} \right\}^{\frac{1}{p+w-1}}, \tag{4.7}$$

where $p \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Proof. It is sufficient to show that

$$|f'(z) - 1| \leq 1 - \sigma \quad (0 \leq \sigma < 1)$$

for $|z| \leq r_3(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma)$ we have

$$\left|f'(z) - 1\right| = \left|pz^{p-1} - \sum_{w=1}^{\infty} (p+w) l_{p+w} z^{p+w-1} - 1\right| \leq p|z|^{p-1} \\ - \sum_{w=1}^{\infty} (p+w) l_{p+w} |z|^{p+w-1} - 1 \leq 1 - \sigma. \tag{4.8}$$

If

$$\sum_{w=1}^{\infty} \frac{(p+w) l_{p+w} |z|^{p+w-1}}{1-\sigma} \leq 1,$$

then by Theorem 2.3, (4.8) will be true if

$$\frac{(p+w) l_{p+w} |z|^{p+w-1}}{1-\sigma} \leq \\ \frac{[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)][1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}}{[p^h(p-\psi)+p^h[p-1]\varrho+\beta p^{h+1}(p-1)z^p(1-\varrho)]}$$

and hence

$$|z| \leq \left\{ \frac{\begin{array}{c}[(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)] \\ [1+(p+w+\phi-\gamma-1)\zeta]^h j_{p+w}\end{array}}{(p+w)[p^h(p-\psi)+p^h[p-1]\varrho+\beta p^{h+1}(p-1)z^p(1-\varrho)]} \right\}^{\frac{1}{p+w-1}} \tag{4.9}$$

setting

$$|z| = r_3(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma)$$

we get the desired result. The proof is complete. □

Remark 4.4. If we put $p = 1$ in Theorems 2.3, 3.1 and 4.1, we obtain the corresponding result studied by Godwin and Opoola [7].

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