

***t*-norms over Fuzzy Multigroups**

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Abstract

In this paper, we propose the notion of fuzzy multigroups under t -norms. Some properties of them are explored and some related results are obtained. Also inverse, product, intersection and sum of them will be defined and investigated properties of them. Finally under group homomorphisms, image and pre image of them will be introduced and investigated.

1. Introduction

Classical set theory is a basic concept used to represent various situations in mathematical notations where repeated occurrences of elements are not allowed. This theory was formulated by a German Mathematician George Ferdinand Ludwig Cantor (1845-1918). Cantor defined a set as a collection into a whole, of definite, well-distinguished objects (called elements) of our intuition or of our thought. For a set, the order of succession of its elements is ignored and the elements shall not be allowed to appear more than once. Mathematics requires that all mathematical notions including sets must be exact. The issue of vagueness or imperfection knowledge has been a problem for a long time for philosophers, mathematicians, logicians and computer scientists, particularly in the area of artificial intelligence. Multiset in particular is necessary because in various circumstances repetition of elements become mandatory to

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the system, for example considering a graph with loops in chemical bonding, molecules of a substance, repeated roots of polynomial equations in mathematics, repeated readings in volumetric analysis experiment, repeated observations in statistical samples and so on. Taking these facts into consideration, the term multiset as (Knuth, 1981) noted was first suggested by De Bruijn in 1970 in one of their private communications. The development of multiset theory is in fact, one small part of the remarkable proliferation of non-classical or non-standard set theory. A multiset (mset), which is a generalization of classical or standard (Cantorian) set, is a set where an element can occur more than once. Fuzzy set is a mathematical model of vague qualitative or quantitative data, frequently generated by means of the natural language. The model is based on the generalization of the classical concepts of set and its characteristic function. The concept of fuzzy sets proposed by L. A. Zadeh [34] is a mathematical tool for representing vague concepts. The idea of fuzzy multisets was conceived by Yager [33] as the generalization of fuzzy sets in multisets framework. For some details on fuzzy multisets see [3, 6, 31]. Recently, by Shinoj et al. [28], the concept of fuzzy multigroups was introduced as an application of fuzzy multisets to group theory, and some properties of fuzzy multigroups were presented. In fact, fuzzy multigroup is a generalization of fuzzy groups. The theory of fuzzy sets has grown stupendously over the years giving birth to fuzzy groups proposed by Rosenfeld [27]. In mathematics, a t -norm (also T -norm or, unabbreviated, triangular norm) is a kind of binary operation used in the framework of probabilistic metric spaces and in multi-valued logic, specifically in fuzzy logic. A t -norm generalizes intersection in a lattice and conjunction in logic. The name triangular norm refers to the fact that in the framework of probabilistic metric spaces t -norms are used to generalize triangle inequality of ordinary metric spaces. The author by using norms, investigated some properties of fuzzy algebraic structures [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. The study of fuzzy multigroup structure under t -norms is very natural. The organization of this paper is as follows: Section 2 provides some preliminaries of multisets, fuzzy multisets, sum of fuzzy multigroups. In Section 3, we introduce fuzzy multigroups under t -norms and investigate some properties and results about them. In Section 4, we define inverse, product, intersection and sum of two fuzzy multigroups under t -norms and we relationship between them and obtain some results. In Section 5, we define group homomorphisms on fuzzy multigroups under t -norms and we prove that image and pre image of fuzzy multigroups under t -norms is also fuzzy multigroups under t -norms.

2. Preliminaries

Definition 2.1 ([29]). Let $X = \{x_1, x_2, \dots, x_n, \dots\}$ be a set. A multiset A over X is a cardinal-valued function, that is, $C_A : X \rightarrow \mathbb{N}$ such that $x \in \text{Dom}(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$, denotes the number of times an object x occur in A . Whenever $C_A(x) = 0$, implies $x \notin \text{Dom}(A)$. The set X is called the *ground* or *generic set* of the class of all multisets (for short, msets) containing objects from X .

A multiset $A = [a, a, b, b, c, c, c]$ can be represented as $A = [a, b, c]_{2,2,3}$ or $A = [a^2, b^2, c^3]$ or $\left\{\frac{a}{2}, \frac{b}{2}, \frac{c}{3}\right\}$. Different forms of representing multiset exist other than this. See [10, 20, 30] for details. We denote the set of all multisets by $MS(X)$.

Definition 2.2 ([30]). Let A and B be two multisets over X . Then A is called a *submultiset* of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a *proper submultiset* of B and denoted as $A \subset B$. Note that a multiset is called the parent in relation to its submultiset. Also two multisets A and B over X are comparable to each other if $A \subseteq B$ or $B \subseteq A$.

Definition 2.3 ([6]). Let X be a set. A fuzzy multiset A of X is characterized by a count membership function

$$CM_A : X \rightarrow [0, 1]$$

of which the value is a multiset of the unit interval $I = [0, 1]$. That is,

$$CM_A(x) = \{\mu^1, \mu^2, \dots, \mu^n, \dots\} \forall x \in X,$$

where $\mu^1, \mu^2, \dots, \mu^n, \dots \in [0, 1]$ such that

$$(\mu^1 \geq \mu^2 \geq \dots \geq \mu^n \geq \dots).$$

Whenever the fuzzy multiset is finite, we write

$$CM_A(x) = \{\mu^1, \mu^2, \dots, \mu^n\},$$

where $\mu^1, \mu^2, \dots, \mu^n \in [0, 1]$ such that

$$(\mu^1 \geq \mu^2 \geq \dots \geq \mu^n),$$

or simply

$$CM_A(x) = \{\mu^i\},$$

for $\mu^i \in [0, 1]$ and $i = 1, 2, \dots, n$.

Now, a fuzzy multiset A is given as

$$A = \left\{ \frac{CM_A(x)}{x} : x \in X \right\} \text{ or } A = \{(CM_A(x), x) : x \in X\}.$$

The set of all fuzzy multisets is depicted by $FMS(X)$.

Example 2.4. Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{1, 0.5, 0.4\}$ and $CM_A(b) = \{0.9, 0.6\}$ and $CM_A(c) = \{0\}$ we get that A is a fuzzy multiset of X written as

$$A = \left\{ \frac{1, 0.5, 0.4}{a}, \frac{0.9, 0.6}{b} \right\}.$$

Definition 2.5 ([6]). Let $A, B \in FMS(X)$. Then A is called a *fuzzy submultiset* of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a *proper fuzzy submultiset* of B and denoted as $A \subset B$.

Definition 2.6 ([2]). Let $A, B \in FMS(X)$. Then the sum of A and B denoted as $A + B$, is defined by the addition operation in $X \times [0, 1]$ for crisp multiset. That is,

$$CM_{A+B}(x) = CM_A(x) + CM_B(x)$$

for all $x \in X$. The meaning of the addition operation here is not as in the case of crisp multiset.

Example 2.7. Assume that $X = \{a, b, c\}$ is a set and $A, B \in FMS(X)$ such that

$$A = \left\{ \frac{1, 0.5, 0.4}{a}, \frac{0.9, 0.6, 0.3}{b}, \frac{0.9, 0.7, 0.2}{c} \right\}$$

and

$$B = \left\{ \frac{0.9, 0.8, 0.3}{a}, \frac{1, 0.8}{b}, \frac{0.3, 0.1}{c} \right\}.$$

Then

$$A + B = \left\{ \frac{1, 0.9, 0.8, 0.5, 0.4, 0.3}{a}, \frac{1, 0.9, 0.8, 0.6, 0.3}{b}, \frac{0.9, 0.7, 0.3, 0.2, 0.1}{c} \right\}.$$

Definition 2.8 ([4]). A group is a non-empty set G on which there is a binary operation $(a, b) \rightarrow ab$ such that

- (1) if a and b belong to G , then ab is also in G (closure),
- (2) $a(bc) = (ab)c$ for all $a, b, c \in G$ (associativity),
- (3) there is an element $e \in G$ such that $ae = ea = a$ for all $a \in G$ (identity),
- (4) if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a, b \in G$.

Remark 2.9. There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \rightarrow a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \rightarrow ab$ for which the identity is denoted by e :

Proposition 2.10 ([4]). *Let G be a group. Let H be a non-empty subset of G . The following are equivalent:*

- (1) H is a subgroup of G .
- (2) $x, y \in H$ implies $xy^{-1} \in H$ for all x, y .

Definition 2.11 ([1]). A t -norm T is a function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ having the following four properties:

- (T1) $T(x, 1) = x$ (neutral element),
- (T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
- (T3) $T(x, y) = T(y, x)$ (commutativity),
- (T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),

for all $x, y, z \in [0, 1]$.

We say that T be *idempotent* if $T(x, x) = x$ for all $x \in [0, 1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.12. (1) Standard intersection t -norm $T_m(x, y) = \min\{x, y\}$.

(2) Bounded sum t -norm $T_b(x, y) = \max\{0, x + y - 1\}$.

(3) algebraic product t -norm $T_p(x, y) = xy$.

(4) Drastic T -norm

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1, \\ x & \text{if } y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t -norm

$$T_{nM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product t -norm

$$T_{H_0}(x, y) = \begin{cases} 0 & \text{if } x = y = 0, \\ \frac{xy}{x + y - xy} & \text{otherwise.} \end{cases}$$

The drastic t -norm is the pointwise smallest t -norm and the minimum is the pointwise largest t -norm: $T_D(x, y) \leq T(x, y) \leq T_{\min}(x, y)$ for all $x, y \in [0, 1]$.

Lemma 2.13 ([1]). *Let T be a t -norm. Then*

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

3. Fuzzy Multigroups under t -norms

Definition 3.1. Let $A \in FMS(G)$. Then A is said to be a *fuzzy multigroup* of G under t -norm T if it satisfies the following two conditions:

(1) $CM_A(xy) \geq T(CM_A(x), CM_A(y))$,

$$(2) CM_A(x^{-1}) \geq CM_A(x),$$

for all $x, y \in G$.

The set of all fuzzy multisets of G under t -norm T is depicted by $TFMS(G)$.

Example 3.2. Let $G = \{e, a, b, c\}$ be a Klein 4-group such that

$$ab = c, ac = b, bc = a, a^2 = b^2 = c^2 = e.$$

Again, let

$$A = \left\{ \frac{1, 0.9}{e}, \frac{0.7, 0.5}{a}, \frac{0.8, 0.6}{b}, \frac{0.7, 0.5}{c} \right\}.$$

Then $A \in FMS(G)$. Let T be a standard intersection t -norm as $T(x, y) = T_m(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$. Now

$$CM_A(ea) = CM_A(a) = 0.7, 0.5 \geq T(CM_A(e), CM_A(a)) = 0.7, 0.5,$$

$$CM_A(eb) = CM_A(b) = 0.8, 0.6 \geq T(CM_A(e), CM_A(b)) = 0.8, 0.6,$$

$$CM_A(ec) = CM_A(c) = 0.7, 0.5 \geq T(CM_A(e), CM_A(c)) = 0.7, 0.5,$$

$$CM_A(ab) = CM_A(c) = 0.7, 0.5 \geq T(CM_A(a), CM_A(b)) = 0.7, 0.5,$$

$$CM_A(ac) = CM_A(b) = 0.8, 0.6 \geq T(CM_A(a), CM_A(c)) = 0.8, 0.6,$$

$$CM_A(bc) = CM_A(a) = 0.7, 0.5 \geq T(CM_A(b), CM_A(c)) = 0.7, 0.5,$$

$$CM_A(aa) = CM_A(e) = 1, 0.9 \geq T(CM_A(a), CM_A(a)) = 1, 0.9,$$

$$CM_A(bb) = CM_A(e) = 1, 0.9 \geq T(CM_A(b), CM_A(b)) = 1, 0.9,$$

$$CM_A(cc) = CM_A(e) = 1, 0.9 \geq T(CM_A(c), CM_A(c)) = 1, 0.9,$$

$$CM_A(ee) = CM_A(e) = 1, 0.9 \geq T(CM_A(e), CM_A(e)) = 1, 0.9,$$

$$CM_A(a^{-1}) = CM_A(a) = 0.7, 0.5 \text{ and } CM_A(b^{-1}) = CM_A(b) = 0.8, 0.6$$

$$CM_A(c^{-1}) = CM_A(c) = 0.7, 0.5 \text{ and } CM_A(e^{-1}) = CM_A(e) = 1, 0.9.$$

Thus $A \in TFMS(G)$.

Lemma 3.3. Let $A \in FMS(G)$ and G be a finite group and T be idempotent. If A satisfies condition (1) of Definition 3.1, then $A \in TFMS(G)$.

Proof. Let $x \in G$, $x \neq e$. As G is finite, so x has finite order, say $n > 1$. Then $x^n = e$ and $x^{-1} = x^{n-1}$. Now by using condition (1) repeatedly, we have that

$$\begin{aligned} CM_A(x^{-1}) &= CM_A(x^{n-1}) = CM_A(x^{n-2}x) \\ &\geq T(CM_A(x^{n-2}), CM_A(x)) \\ &\geq T(\underbrace{CM_A(x), CM_A(x), \dots, CM_A(x)}_n) \\ &= CM_A(x). \end{aligned}$$

Thus $A \in TFMS(G)$. □

Theorem 3.4. Let $A \in TFMS(G)$. If T be idempotent, then for all $x \in G$, and $n \geq 1$,

- (1) $CM_A(e) \geq CM_A(x)$;
- (2) $CM_A(x^n) \geq CM_A(x)$;
- (3) $CM_A(x) = CM_A(x^{-1})$.

Proof. Let $x \in G$ and $n \geq 1$.

(1)

$$\begin{aligned} CM_A(e) &= CM_A(xx^{-1}) \geq T(CM_A(x), CM_A(x^{-1})) \\ &\geq T(CM_A(x), CM_A(x)) = CM_A(x). \end{aligned}$$

(2)

$$CM_A(x^n) = CM_A(\underbrace{xx \dots x}_n) \geq T(\underbrace{CM_A(x), CM_A(x), \dots, CM_A(x)}_n) = CM_A(x).$$

(3) $CM_A(x) = CM_A((x^{-1}))^{-1} \geq CM_A(x^{-1}) \geq CM_A(x)$. Then $CM_A(x) = CM_A(x^{-1})$. □

Corollary 3.5. *Let T be an idempotent t -norm. Then $A \in TFMS(G)$ if and only if*

$$CM_A(xy^{-1}) \geq T(CM_A(x), CM_A(y))$$

for all $x, y \in G$.

Proof. Let $x, y \in G$. If $A \in TFMS(G)$, then

$$CM_A(xy^{-1}) \geq T(CM_A(x), CM_A(y^{-1})) \geq T(CM_A(x), CM_A(y)).$$

Conversely, let $CM_A(xy^{-1}) \geq T(CM_A(x), CM_A(y))$ for all $x, y \in G$. Then

$$\begin{aligned} CM_A(x^{-1}) &\geq CM_A(ex^{-1}) \geq T(CM_A(e), CM_A(x)) \\ &\geq T(CM_A(x), CM_A(x)) = CM_A(x). \end{aligned} \text{ (Theorem 3.4 (part 1))}$$

Also

$$CM_A(xy) = CM_A(x(y^{-1})^{-1}) \geq T(CM_A(x), CM_A(y^{-1})) \geq T(CM_A(x), CM_A(y)).$$

Then $A \in TFMS(G)$. □

Proposition 3.6. *Let $A \in TFMS(G)$ and $x \in G$. If T be idempotent-norm, then $CM_A(xy) = CM_A(y) \forall y \in G$ if and only if $CM_A(x) = CM_A(e)$.*

Proof. Let $CM_A(xy) = CM_A(y) \forall y \in G$. Then by letting $y = e$, we get that $CM_A(x) = CM_A(e)$.

Conversely, suppose that $CM_A(x) = CM_A(e)$. By Theorem 3.4 we have that $CM_A(x) \geq CM_A(xy)$ and $CM_A(x) \geq CM_A(y)$. Now

$$\begin{aligned} CM_A(xy) &\geq T(CM_A(x), CM_A(y)) \geq T(CM_A(y), CM_A(y)) \\ &= CM_A(y) = CM_A(x^{-1}xy) \geq T(CM_A(x), CM_A(xy)) \\ &\geq T(CM_A(xy), CM_A(xy)) = CM_A(xy). \end{aligned}$$

Therefore $CM_A(xy) = CM_A(y) \forall y \in G$. □

Proposition 3.7. *Let $A \in TFMS(G)$ and T be idempotent t -norm and $CM_A(xy^{-1}) = CM_A(e)$ for all $x, y \in G$. Then $CM_A(x) = CM_A(y)$.*

Proof. Let $CM_A(xy^{-1}) = CM_A(e)$ for all $x, y \in G$. Then

$$\begin{aligned} CM_A(x) &= CM_A(xy^{-1}y) \geq T(CM_A(xy^{-1}), CM_A(y)) \\ &= T(CM_A(e), CM_A(y)) \geq T(CM_A(y), CM_A(y)) = CM_A(y) \\ &= CM_A(y^{-1}) = CM_A(x^{-1}xy^{-1}) \geq T(CM_A(x^{-1}), CM_A(xy^{-1})) \\ &= T(CM_A(x^{-1}), CM_A(e)) = CM_A(x^{-1}) = CM_A(x) \end{aligned}$$

and then $CM_A(x) = CM_A(y)$.

Proposition 3.8. Let $A \in TFMS(G)$ and $CM_A(x) \neq CM_A(y)$ for all $x, y \in G$. Then $CM_A(xy) \geq T(CM_A(x), CM_A(y))$.

Proof. Let $CM_A(x) > CM_A(y)$ for all $x, y \in G$ and we get that $CM_A(x) > CM_A(xy)$ and then

$$CM_A(y) = T(CM_A(x), CM_A(y))$$

and

$$CM_A(xy) = T(CM_A(x), CM_A(xy)).$$

Now

$$\begin{aligned} CM_A(xy) &\geq T(CM_A(x), CM_A(y)) = CM_A(y) \\ &= CM_A(x^{-1}xy) \geq T(CM_A(x^{-1}), CM_A(xy)) \\ &= T(CM_A(x), CM_A(xy)) = CM_A(xy) \end{aligned}$$

and then

$$CM_A(xy) = CM_A(y) = T(CM_A(x), CM_A(y)). \quad \square$$

Proposition 3.9. Let $A \in TFMS(G)$. Then

- (1) $A^* = \{x \in G : CM_A(x) = CM_A(e)\}$ is a subgroup of G .
- (2) $A_* = \{x \in G : CM_A(x) > 0\}$ is a subgroup of G .

(3) If *T* be idempotent *t*-norm, then

$$A^{[\alpha]} = \{x \in G : CM_A(x) \geq \alpha\}$$

is a subgroup of *G* for all $\alpha \in [0, 1]$.

Proof. Let $x, y \in G$.

(1) If $x, y \in A^*$, then $CM_A(x) = CM_A(y) = CM_A(e)$. Now

$$\begin{aligned} CM_A(xy^{-1}) &\geq T(CM_A(x), CM_A(y^{-1})) \geq T(CM_A(x), CM_A(y)) \\ &= T(CM_A(e), CM_A(e)) = CM_A(e) = CM_A(xy^{-1}yx^{-1}) \\ &\geq T(CM_A(xy^{-1}), CM_A(yx^{-1})) = T(CM_A(xy^{-1}), CM_A(xy^{-1})^{-1}) \\ &\geq T(CM_A(xy^{-1}), CM_A(xy^{-1})) = CM_A(xy^{-1}). \end{aligned}$$

Thus $CM_A(xy^{-1}) = CM_A(e)$ and then $xy^{-1} \in A^*$. Now as Proposition 2.10 we get that A^* is a subgroup of *G*.

(2) Let $x, y \in A_*$, then $CM_A(x) > 0$ and $CM_A(y) > 0$. Then

$$\begin{aligned} CM_A(xy^{-1}) &\geq T(CM_A(x), CM_A(y^{-1})) \\ &\geq T(CM_A(x), CM_A(y)) > T(0, 0) = 0 \end{aligned}$$

and so $CM_A(xy^{-1}) > 0$ and then $xy^{-1} \in A_*$. Thus Proposition 2.10 give us that A_* is a subgroup of *G*.

(3) Let $x, y \in A^{[\alpha]}$, then $CM_A(x) \geq \alpha$ and $CM_A(y) \geq \alpha$. Now

$$\begin{aligned} CM_A(xy^{-1}) &\geq T(CM_A(x), CM_A(y^{-1})) \\ &\geq T(CM_A(x), CM_A(y)) \geq T(\alpha, \alpha) = \alpha \end{aligned}$$

and so $xy^{-1} \in A^{[\alpha]}$ and from Proposition 2.10 we get that $A^{[\alpha]}$ is a subgroup of *G*. □

4. Inverse, Product, Intersection and Sum of Fuzzy Multigroups under t -norms

Definition 4.1. Let $A \in TFMS(G)$. Then A^{-1} is called *inverse* of A and defined as $CM_{A^{-1}}(x) = CM_A(x^{-1})$ for all $x \in G$.

Corollary 4.2. $A \in TFMS(G)$ if and only if $A^{-1} \in TFMS(G)$.

Proof. Let $x, y \in G$. If $A \in TFMS(G)$, then

(1)

$$\begin{aligned} CM_{A^{-1}}(xy) &= CM_A(xy)^{-1} = CM_A(y^{-1}x^{-1}) \\ &\geq T(CM_A(y^{-1}), CM_A(x^{-1})) \\ &= T(CM_{A^{-1}}(y), CM_{A^{-1}}(x)) \\ &= T(CM_{A^{-1}}(x), CM_{A^{-1}}(y)). \end{aligned}$$

(2)

$$CM_{A^{-1}}(x^{-1}) = CM_A(x^{-1})^{-1} \geq CM_A(x^{-1}) = CM_{A^{-1}}(x).$$

Thus $A^{-1} \in TFMS(G)$.

Conversely, let $A^{-1} \in TFMS(G)$. Then

(1)

$$\begin{aligned} CM_A(xy) &= CM_A((xy)^{-1})^{-1} = CM_{A^{-1}}((xy)^{-1}) \\ &= CM_{A^{-1}}(y^{-1}x^{-1}) \geq T(CM_{A^{-1}}(y^{-1}), CM_{A^{-1}}(x^{-1})) \\ &= T(CM_A(y), CM_A(x)) = T(CM_A(x), CM_A(y)). \end{aligned}$$

(2)

$$CM_A(x^{-1}) = CM_{A^{-1}}(x) = CM_{A^{-1}}(x^{-1})^{-1} \geq CM_{A^{-1}}(x^{-1}) = CM_A(x).$$

Therefore $A \in TFMS(G)$.

Definition 4.3. Let $A, B \in TFMS(G)$. Then the product of A and B denoted as $A \circ B$ is governed by

$$CM_{A \circ B}(x) = \begin{cases} \sup_{x=yz} T(CM_A(y), CM_B(z)) & \text{if } x = yz, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$CM_{A \circ B}(x) = \sup_{y \in G} T(CM_A(y), CM_B(y^{-1}x)) = \sup_{y \in G} T(CM_A(xy^{-1}), CM_B(y)).$$

Definition 4.4. Let $A, B \in TFMS(G)$. Then the intersection of A and B denoted as $A \cap B$ is governed by

$$CM_{A \cap B}(x) = T(CM_A(x), CM_B(x))$$

for all $x \in G$.

Proposition 4.5. Let $A, B \in TFMS(G)$. Then $A \cap B \in TFMS(G)$.

Proof. Let $x, y \in G$.

(1)

$$\begin{aligned} CM_{A \cap B}(xy) &= T(CM_A(xy), CM_B(xy)) \\ &\geq T(T(CM_A(x), CM_A(y)), T(CM_B(x), CM_B(y))) \\ &= T(T(CM_A(x), CM_B(x)), T(CM_A(y), CM_B(y))) \text{ (Lemma 2.13)} \\ &= T(CM_{A \cap B}(x), CM_{A \cap B}(y)). \end{aligned}$$

(2)

$$\begin{aligned} CM_{A \cap B}(x^{-1}) &= T(CM_A(x^{-1}), CM_B(x^{-1})) \\ &\geq T(CM_A(x), CM_B(x)) = CM_{A \cap B}(x). \end{aligned}$$

Therefore $A \cap B \in TFMS(G)$.

Corollary 4.6. Let $I_n = \{1, 2, \dots, n\}$. If $\{A_i \mid i \in I_n\} \subseteq TFMS(G)$. Then $A = \bigcap_{i \in I_n} A_i \in TFMS(G)$.

Proposition 4.7. Let $A, B \in TFMS(G)$. Then the following assertions hold:

$$(1) (A^{-1})^{-1} = A.$$

$$(2) \text{ If } A \subseteq B, \text{ then } A^{-1} \subseteq B^{-1}.$$

$$(3) (A \circ B)^{-1} = B^{-1} \circ A^{-1}.$$

$$(4) (A \cap B)^{-1} = A^{-1} \cap B^{-1}.$$

Proof. Let $x, y, z \in G$. Then

$$(1) CM_{(A^{-1})^{-1}}(x) = CM_{A^{-1}}(x^{-1}) = CM_A((x^{-1})^{-1}) = CM_A(x) \text{ and so } (A^{-1})^{-1} = A.$$

$$(2) \text{ As } A \subseteq B \text{ so } CM_A(x^{-1}) \leq CM_B(x^{-1}) \text{ and then}$$

$$CM_{A^{-1}}(x) = CM_A(x^{-1}) \leq CM_B(x^{-1}) = CM_{B^{-1}}(x).$$

Thus $A^{-1} \subseteq B^{-1}$.

(3)

$$\begin{aligned} CM_{(A \circ B)^{-1}}(x) &= CM_{(A \circ B)}(x^{-1}) \\ &= \sup_{x^{-1}=y^{-1}z^{-1}} T(CM_A(y^{-1}), CM_B(z^{-1})) \\ &= \sup_{x^{-1}=(zy)^{-1}} T(CM_A(y^{-1}), CM_B(z^{-1})) \\ &= \sup_{x^{-1}=(zy)^{-1}} T(CM_B(z^{-1}), CM_A(y^{-1})) \\ &= \sup_{x=zy} T(CM_{B^{-1}}(z), CM_{A^{-1}}(y)) \\ &= CM_{(B^{-1} \circ A^{-1})}(x). \end{aligned}$$

Then $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$.

(4)

$$\begin{aligned}
 CM_{(A \cap B)^{-1}}(x) &= CM_{(A \cap B)}(x^{-1}) \\
 &= T(CM_A(x^{-1}), CM_B(x^{-1})) \\
 &= T(CM_{A^{-1}}(x), CM_{B^{-1}}(x)) \\
 &= CM_{(A^{-1} \cap B^{-1})}(x).
 \end{aligned}$$

Thus $(A \cap B)^{-1} = A^{-1} \cap B^{-1}$.

Proposition 4.8. *A ∈ TFMS(G) if and only if A satisfies the following conditions:*

(1) $A \circ A \subseteq A$,

(2) $A^{-1} = A$.

Proof. Let $x, y, z \in G$ such that $x = yz$. If $A \in TFMS(G)$, then

(1)

$$CM_A(x) = CM_A(yz) \geq T(CM_A(y), CM_A(z)) = (A \circ A)(x)$$

so $A \circ A \subseteq A$.

(2) $CM_{A^{-1}}(x) = CM_A(x^{-1}) = CM_A(x)$ and then $A^{-1} = A$.

Conversely, let $x \in G$. As $A \circ A \subseteq A$ so

$$\begin{aligned}
 CM_A(yz) &= CM_A(x) \geq (CM_{A \circ A})(x) \\
 &= \sup_{x=yz} T(CM_A(y), CM_A(z)) \\
 &\geq T(CM_A(y), CM_A(z)).
 \end{aligned}$$

Also since $A^{-1} = A$ so $CM_A(x^{-1}) = CM_{A^{-1}}(x) = CM_A(x)$.

Therefore $A \in TFMS(G)$.

Proposition 4.9. *Let A, B ∈ TFMS(G). Then (A ∘ B) ∈ TFMS(G) if and only if*

$A \circ B = B \circ A$.

Proof. Let $A, B \in TFMS(G)$, then from Proposition 4.8 we get that $A \circ A \subseteq A$ and $B \circ B \subseteq B$ and $A^{-1} = A$ and $B^{-1} = B$.

If $(A \circ B) \in TFMS(G)$, then from Proposition 4.7 and Proposition 4.8 we get that

$$B \circ A = B^{-1} \circ A^{-1} = (A \circ B)^{-1} = A \circ B.$$

Conversely, let $A \circ B = B \circ A$. As

(1)

$$\begin{aligned} (A \circ B) \circ (A \circ B) &= A \circ (B \circ A) \circ B \\ &= A \circ (A \circ B) \circ B \\ &= (A \circ A) \circ (B \circ B) \\ &\subseteq A \circ B \end{aligned}$$

(2)

$$(A \circ B)^{-1} = B^{-1} \circ A^{-1} = B \circ A = A \circ B$$

so Proposition 4.8 gives us that $(A \circ B) \in TFMS(G)$.

Proposition 4.10. Let $A, B \in TFMS(G)$ and T be idempotent t -norm. Then $A \subseteq A \circ B$ if and only if $CM_A(e) \leq CM_B(e)$.

Proof. Let $x, y, z \in G$ and $CM_A(e) \leq CM_B(e)$. Then

$$\begin{aligned} CM_{(A \circ B)}(x) &= \sup_{x=yz} T(CM_A(y), CM_B(z)) \\ &\geq \sup_{x=xe} T(CM_A(x), CM_B(e)) \\ &\geq T(CM_A(x), CM_B(e)) \\ &\geq T(CM_A(x), CM_A(e)) \\ &\geq T(CM_A(x), CM_A(x)) \\ &= CM_A(x) \end{aligned}$$

and so $CM_{(A \circ B)}(x) \geq CM_A(x)$ and then $A \subseteq A \circ B$.

Conversely, let $A \subseteq A \circ B$. If $CM_A(e) > CM_B(e)$, then we will have

$$\begin{aligned} CM_{(A \circ B)}(e) &= \sup_{e=xx^{-1}} T(CM_A(x), CM_B(x^{-1})) \\ &\leq T(CM_A(e), CM_B(e)) \\ &\leq T(CM_A(e), CM_A(e)) \\ &= CM_A(e) \end{aligned}$$

and we get that $A \circ B \subset A$ and this is a contradiction. Therefore $CM_A(e) \leq CM_B(e)$. \square

Proposition 4.11. *Let $A, B \in TFMS(G)$ and $CM_A(e) = CM_B(e)$ and T be idempotent *t*-norm. Then $A \subseteq A \circ B$ and $B \subseteq A \circ B$.*

Proof. Let $x, y, z \in G$ and $CM_A(e) = CM_B(e)$. Then

$$\begin{aligned} CM_{(A \circ B)}(x) &= \sup_{x=yz} T(CM_A(y), CM_B(z)) \\ &\geq T(CM_A(x), CM_B(e)) \\ &= T(CM_A(x), CM_A(e)) \\ &\geq T(CM_A(x), CM_A(x)) = CM_A(x) \end{aligned}$$

and then $CM_{(A \circ B)}(x) \geq CM_A(x)$ that is $A \subseteq A \circ B$.

Also

$$\begin{aligned} CM_{(A \circ B)}(x) &= \sup_{x=yz} T(CM_A(y), CM_B(z)) \\ &\geq T(CM_A(e), CM_B(x)) \\ &= T(CM_B(e), CM_B(x)) \\ &\geq T(CM_B(x), CM_B(x)) = CM_B(x) \end{aligned}$$

therefore $CM_{(A \circ B)}(x) \geq CM_B(x)$ and so $B \subseteq A \circ B$.

Proposition 4.12. *Let $A, B \in TFMS(G)$ and $CM_A(e) = CM_B(e)$ and T be idempotent *t*-norm. If $A \circ B \in TFMS(G)$, then $A \circ B$ is generated by A and B .*

Proof. Suppose that $A \circ B \in TFMS(G)$. Then we show that $A \circ B$ is the smallest containing A and B . As Proposition 4.11 we get that $A \subseteq A \circ B$ and $B \subseteq A \circ B$. Let $C \in TFMS(G)$ such that $A, B \subseteq C$ and $x, y, z \in G$. Then

$$\begin{aligned} CM_{(A \circ B)}(x) &= \sup_{x=yz} T(CM_A(y), CM_B(z)) \\ &\leq \sup_{x=yz} T(CM_C(y), CM_C(z)) \\ &= CM_{(C \circ C)}(x) = C(x) \end{aligned}$$

and then $A \circ B \subseteq C$. Thus $A \circ B$ is generated by A and B . □

Proposition 4.13. Let $A, B \in TFMS(G)$. Then $A + B \in TFMS(G)$.

Proof. Let $x, y \in G$. Then

(1)

$$\begin{aligned} CM_{(A+B)}(xy) &= CM_A(xy) + CM_B(xy) \\ &\geq T(CM_A(x), CM_A(y)) + T(CM_B(x), CM_B(y)) \\ &= T(CM_A(x) + CM_B(x), CM_A(y) + CM_B(y)) \\ &= T(CM_{A+B}(x), CM_{A+B}(y)). \end{aligned}$$

(2)

$$\begin{aligned} CM_{(A+B)}(x^{-1}) &= CM_A(x^{-1}) + CM_B(x^{-1}) \\ &\geq CM_A(x) + CM_B(x) \\ &= CM_{(A+B)}(x). \end{aligned}$$

Thus $A + B \in TFMS(G)$.

Remark 4.14. Let $\{A_i\}_{i \in I} \in TFMS(G)$. Then $\sum_{i \in I} A_i \in TFMS(G)$.

5. Group Homomorphisms and Fuzzy Multigroups under t -norms

Definition 5.1. Let G and H be groups and $f : G \rightarrow H$ be a homomorphism. Let

$A \in FMS(G)$ and $B \in FMS(H)$. Define $f(A) \in FMS(H)$ and $f^{-1}(B) \in FMS(G)$ as

$$f(CM_A)(h) = (CM_{f(A)})(h) = \begin{cases} \sup\{CM_A(g) \mid g \in G, f(g) = h\} & \text{if } f^{-1}(h) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^{-1}(CM_B(g)) = CM_{f^{-1}(B)}(g) = CM_B(f(g))$$

for all $g \in G$.

Proposition 5.2. *Let G and H be groups and $f : G \rightarrow H$ be an epimorphism. If $A \in TFMS(G)$, then $f(A) \in TFMS(H)$.*

Proof. Let $u, v \in H$ and $x, y \in G$ such that $u = f(x)$ and $v = f(y)$, then

$$\begin{aligned} f(CM_A)(uv) &= \sup\{CM_A(xy) \mid u = f(x), v = f(y)\} \\ &\geq \sup\{T(CM_A(x), CM_A(y)) \mid u = f(x), v = f(y)\} \\ &= T(\sup\{CM_A(x) \mid u = f(x)\}, \sup\{CM_A(y) \mid v = f(y)\}) \\ &= T(f(CM_A)(u), f(CM_A)(v)). \end{aligned}$$

Also

$$\begin{aligned} f(CM_A)(u^{-1}) &= \sup\{CM_A(x^{-1}) \mid u^{-1} = f(x^{-1})\} \\ &= \sup\{CM_A(x^{-1}) \mid u^{-1} = f^{-1}(x)\} \\ &\geq \sup\{CM_A(x) \mid u = f(x)\} = f(CM_A)(u). \end{aligned}$$

Thus $f(A) \in TFMS(H)$. □

Proposition 5.3. *Let G and H be groups and $f : G \rightarrow H$ be a homomorphism. If $B \in TFMS(H)$, then $f^{-1}(B) \in TFMS(G)$.*

Proof. Let $x, y \in G$. Then

$$f^{-1}(CM_B)(xy) = CM_B(f(xy)) = CM_B(f(x)f(y))$$

$$\begin{aligned} &\geq T(CM_B(f(x)), CM_B(f(y))) \\ &= T(f^{-1}(CM_B)(x), f^{-1}(CM_B)(y)). \end{aligned}$$

Also

$$\begin{aligned} f^{-1}(CM_B)(x^{-1}) &= CM_B(f(x^{-1})) = CM_B(f^{-1}(x)) \\ &\geq CM_B(f(x)) = f^{-1}(CM_B)(x). \end{aligned}$$

Therefore $f^{-1}(B) \in TFMS(G)$.

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References

- [1] M. T. Abu Osman, On some products of fuzzy subgroups, *Fuzzy Sets and Systems* 24 (1987), 79-86. [https://doi.org/10.1016/0165-0114\(87\)90115-1](https://doi.org/10.1016/0165-0114(87)90115-1)
- [2] P. A. Ejegwa, On fuzzy multigroups and fuzzy submultigroups, *Journal of Fuzzy Mathematics* 26(3) (2018), 641-654.
- [3] P. A. Ejegwa, Correspondence between fuzzy multisets and sequences, *Global Journal of Science Frontier Research: Mathematics and Decision Science* 14 (2014), 61-66.
- [4] T. Hungerford, *Algebra*, Graduate Texts in Mathematics, Springer, 2003.
- [5] D. S. Malik and J. N. Mordeson, *Fuzzy Commutative Algebra*, World Science Publishing Co. Pte. Ltd., 1995.
- [6] S. Miyamoto, The basic operation of the fuzzy multi-set, *Japan Fuzzy Journal* 8 (1996), 639-645. https://doi.org/10.3156/jfuzzy.8.4_47
- [7] R. Rasuli, Fuzzy ideals of subtraction semigroups with respect to a t -norm and a t -conorm, *The Journal of Fuzzy Mathematics Los Angeles* 24(4) (2016), 881-892.
- [8] R. Rasuli, Fuzzy modules over a t -norm, *Int. J. Open Problems Compt. Math.* 9(3) (2016), 12-18. <https://doi.org/10.12816/0033740>
- [9] R. Rasuli, Fuzzy subrings over a t -norm, *The Journal of Fuzzy Mathematics Los Angeles* 24(4) (2016), 995-1000.

- [10] R. Rasuli, Norms over intuitionistic fuzzy subrings and ideals of a ring, *Notes on Intuitionistic Fuzzy Sets* 22(5) (2016), 72-83.
- [11] R. Rasuli, Norms over fuzzy Lie algebra, *Journal of New Theory* 15 (2017), 32-38.
- [12] R. Rasuli, Fuzzy subgroups on direct product of groups over a t -norm, *Journal of Fuzzy Set Valued Analysis* 3 (2017), 96-101. <https://doi.org/10.5899/2017/jfsva-00339>
- [13] R. Rasuli, Characterizations of intuitionistic fuzzy subsemirings of semirings and their homomorphisms by norms, *Journal of New Theory* 18 (2017), 39-52.
- [14] R. Rasuli, *Intuitionistic Fuzzy Subrings and Ideals of a Ring Under Norms*, LAP LAMBERT Academic Publishing, 2017.
- [15] R. Rasuli, Characterization of Q -fuzzy subrings (anti Q -fuzzy subrings) with respect to a t -norm (t -conorms), *J. Inf. Optim. Sci.* 39 (2018), 827-837. <https://doi.org/10.1080/02522667.2016.1228316>
- [16] R. Rasuli, T -fuzzy submodules of $R \times M$, *Journal of New Theory* 22 (2018), 92-102.
- [17] R. Rasuli, Fuzzy subgroups over a t -norm, *Journal of Information and Optimization Sciences* 39 (2018), 1757-1765. <https://doi.org/10.1080/02522667.2018.1427028>
- [18] R. Rasuli, Fuzzy sub-vector spaces and sub-bivector spaces under t -norms, *General Letters in Mathematics* 5 (2018), 47-57. <https://doi.org/10.31559/glm2018.5.1.6>
- [19] R. Rasuli, Anti fuzzy submodules over a t -conorm and some of their properties, *The Journal of Fuzzy Mathematics Los Angeles* 27 (2019), 229-236.
- [20] R. Rasuli, Artinian and Noetherian Fuzzy Rings, *Int. J. Open Problems Compt. Math.* 12 (2019), 1-7.
- [21] R. Rasuli and H. Narghi, t -norms over Q -fuzzy subgroups of a group, *Jordan Journal of Mathematics and Statistics (JJMS)* 12 (2019), 1-13.
- [22] R. Rasuli, Fuzzy equivalence relation, fuzzy congruence relation and fuzzy normal subgroups on group G over t -norms, *Asian Journal of Fuzzy and Applied Mathematics* 7 (2019), 14-28. <https://doi.org/10.24203/ajfam.v7i2.5736>
- [23] R. Rasuli, Norms over anti fuzzy G -submodules, *MathLAB Journal* 2 (2019), 56-64.
- [24] R. Rasuli, Norms over bifuzzy bi-ideals with operators in semigroups, *Notes on Intuitionistic Fuzzy Sets* 25 (2019), 1-11. <https://doi.org/10.7546/nifs.2019.25.1.1-11>
- [25] R. Rasuli, Norms over basic operations on intuitionistic fuzzy sets, *The Journal of Fuzzy Mathematics Los Angeles* 27(3) (2019), 561-582.

-
- [26] R. Rasuli, *T*-fuzzy bi-ideals in semirings, *Earthline J. Math. Sci.* 2(1) (2019), 241-263. <https://doi.org/10.34198/ejms.2119.241263>
- [27] A. Rosenfeld, Fuzzy groups, *Journal of Mathematical Analysis and Applications* 35 (1971), 512-517. [https://doi.org/10.1016/0022-247X\(71\)90199-5](https://doi.org/10.1016/0022-247X(71)90199-5)
- [28] T. K. Shinoj, A. Baby and J. J. Sunil, On some algebraic structures of fuzzy multisets, *Annals of Fuzzy Mathematics and Informatics* 9 (2015), 77-90.
- [29] D. Singh, A. M. Ibrahim, T. Yohanna and J. N. Singh, An overview of the applications of multisets, *Novi Sad J. Math.* 37 (2007), 73-92.
- [30] A. Syropoulos, *Mathematics of Multisets*, Springer-Verlag, Berlin, Heidelberg, 2001, pp. 347-358. https://doi.org/10.1007/3-540-45523-X_17
- [31] A. Syropoulos, On generalized fuzzy multisets and their use in computation, *Iranian Journal of Fuzzy Systems* 9 (2012), 113-125.
- [32] N. J. Wildberger, *A New Look at Multisets*, School of Mathematics, UNSW Sydney, 2052, Australia, 2003.
- [33] R. R. Yager, On the theory of bags, *International Journal of General Systems* 13 (1986), 23-37. <https://doi.org/10.1080/03081078608934952>
- [34] L. A. Zadeh, Fuzzy sets, *Inform. Control.* 8 (1965), 338-353. [https://doi.org/10.1016/S0019-9958\(65\)90241-X](https://doi.org/10.1016/S0019-9958(65)90241-X)