The Weibull-exponential {Rayleigh} Distribution: Theory and Applications

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Abstract

This study introduces a new distribution in the family of generalized exponential distributions generated using the transformed-transformer method. Some properties of the distribution are presented. The new distribution has three parameters and they are estimated numerically using the BGFS iterative method implemented in R software. Two real sets of data are adopted to demonstrate the flexibility and potential applications of the new distribution.

1. Introduction

Many sets of data are being generated in various fields of human endeavour on daily basis. Some of these data sets are quite unique in nature that there are yet no known existing statistical distributions that adequately describe such data. This has given rise to new probability distributions. In doing this, many methods have been adopted by various researchers. These methods, based on the period of development, have been majorly grouped into two: methods before 1980 which include differential equation, translation and quantile methods and those adopted since 1980 which are referred to as contemporary methods (these include those obtained by adding parameters to a known distribution and/or compounding of distributions (Lee et al. [17]). Although the methods used in generating new distributions prior to 1980 are still being explored by researchers in

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generating new distributions, most of the new distributions are generated using the contemporary methods.

Among the recent methods, this study will adopt the transformed-transformer method developed by Alzaatreh et al. [5]. Here, they used a random variable X, with pdf, $f(x)$ and cdf, $F(x)$ to transform another variable with support $(-\infty, \infty)$ having pdf and cdf, $r(t)$ and $R(t)$ respectively to obtain the distribution function of a new class of distributions as

$$K(x) = \int_{a}^{Z(F(x))} r(t)\,dt = R\{Z(F(x))\}, \tag{1}$$

where $Z(F(x)) \in [a,b]$ is monotonically increasing, $Z'(F(x)) < \infty$, $\lim\limits_{x \to -\infty}(Z(F(x)))$ $= a$ and $\lim\limits_{x \to \infty}(Z(F(x))) = b$, with the corresponding density function given as

$$k(x) = Z'(F(x))r\{Z(F(x))\}. \tag{2}$$

Some of the studies using the transformed-transformer method include those by Alzaatreh et al. [6], Al-Aqtash et al. [1], Merovci and Elbatal [18], Oguntunde et al. [21], Fatima and Ahmad [13], Osatohanmwen et al. [22].

On replacing the function, $Z(F(x))$ with the quantile function of another random variable, Y in terms of $F(x)$, Aljarrah et al. [3] obtained a new class of distributions called T-X{Y} family of distributions as

$$K(x) = \int_{a}^{Q_Y(F(x))} r(t)\,dt = R\{Q_Y(F(x))\}. \tag{3}$$

If $F(x) = \lambda$ and $p(y) > 0$ all y in a neighbourhood of $Q_Y(\lambda)$ where $\lambda \in (0,1)$, then $Q'_y(\lambda) < \infty$ and is equal to $[p(Q_Y(\lambda))]^{-1}$ (Shorack and Wellner [24]).

Hence, the associated pdf to $k(x)$ is

$$k(x) = \frac{f(x)}{p\{Q_Y(F(x))\}} r\{Q_Y(F(x))\}. \tag{4}$$

Due to the confusion that may arise with the use of $F(x)$ and $K(x)$ as the cdfs of the random variable X, Alzaatreh et al. [7] redefined the T-X{Y} as T-R{Y}. The T-R{Y} approach is defined as follows.

Let T, R and Y be random variables with respective cumulative distribution functions, $F_T(x), F_R(x)$ and $F_Y(x)$ and the corresponding quantile functions as $Q_T(p), Q_R(p)$ and $Q_Y(p)$, where the quantile function is defined as $Q_V(p) = F_V^{-1}(p) = \inf\{v : F_V(v) \geq p\}$, $0 < p < 1$. Again, assuming the probability density function of the three random variables exist and are represented respectively as $F_T(x), F_R(x)$ and $F_Y(x)$ with $T, Y \in [a, b]$, and $R \in [c, d]$ for $-\infty \leq a < b \leq \infty$ and $-\infty \leq c < d \leq \infty$, then the cdf of the new class of distributions is given by

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)\,dt = F_T\{Q_Y(F_R(x))\} \tag{5}$$

with the corresponding pdf as

$$f_X(x) = f_R(x) \cdot Q_Y'(F_R(x)) \cdot f_T(Q_Y(F_R(x))) = f_R(x)\frac{f_T(Q_Y(F_R(x)))}{f_Y(Q_Y(F_R(x)))}. \tag{6}$$

The concept of T-R{Y} framework was also adopted in subsequent studies by Alzahal et al. [10], Almheidat et al. [4], Tahir et al. [25], Alzaatreh et al. [8], Alzaatreh et al. [9], Aldeni et al. [2], Jamal et al. [16], Hamed et al. [15], Ekum et al. [11].

Other sections in this article are organized as follows. In Section 2, the Weibull-Exponential {Rayleigh} is introduced. Some features of the distribution and the estimation of its parameters are studied and presented in Sections 3 and 4 respectively. The use of data to demonstrate the flexibility and application of the new distribution is achieved in Section 5. Finally, the article is summarized and concluded in Section 6.

2. The Weibull-exponential {Rayleigh}

Suppose T follows the Weibull (λ, α) distribution with cdf, pdf and quantile function respectively given as

$$F_T(x) = 1 - e^{-(\lambda x)^\sigma} \tag{7}$$

$$f_T(x) = \sigma\lambda(\lambda x)^{\sigma-1} e^{-(\lambda x)^\sigma} \tag{8}$$

$$Q_T(p) = \frac{1}{\lambda}[-\log(1-p)]^{1/\sigma} \tag{9}$$

$$\lambda, \sigma > 0, \, x > 0.$$

Similarly, let R follow the exponential distribution with cdf, pdf and quantile function respectively given as

$$F_R(x) = 1 - e^{-\beta x} \tag{10}$$

$$f_R(x) = \beta e^{-\beta x} \tag{11}$$

$$Q_R(p) = -\frac{1}{\beta}\log(1-p) \tag{12}$$

$$\beta > 0, \, x > 0.$$

Finally, let Y follow the standard Rayleigh distribution with cdf, pdf and quantile function respectively given as

$$F_Y(x) = 1 - e^{-x^2/2} \tag{13}$$

$$f_Y(x) = xe^{-x^2/2} \tag{14}$$

$$Q_Y(p) = (-2\log(1-p))^{1/2} \tag{15}$$

where $x > 0$, $p \in (0,1)$.

From (15),

$$Q_Y(F_R(x)) = (-2\log(1-F_R(x)))^{1/2}.$$

From (10),

$$Q_Y(F_R(x)) = \left(-2\log\left(1-\left(1-e^{-\beta x}\right)\right)\right)^{1/2}$$

$$= (2\beta x)^{1/2}.$$

Therefore $F_T(Q_Y(F_R(x))) = F_T\left((2\beta x)^{1/2}\right).$

From (7)

$$F_T\left(Q_Y\left(F_R\left(x\right)\right)\right) = F_T\left((2\beta x)^{1/2}\right) = 1 - e^{-\left(\lambda(2\beta x)^{1/2}\right)^{\sigma}}.$$

Hence, the cdf of the Weibull-exponential {Rayleigh} is given by

$$F_X\left(x\right) = 1 - e^{-\left(\lambda(2\beta x)^{1/2}\right)^{\sigma}}. \tag{16}$$

From (8)

$$f_T\left(Q_Y\left(F_R\left(x\right)\right)\right) = f_T\left((2\beta x)^{1/2}\right) = \sigma\lambda\left(\lambda(2\beta x)^{1/2}\right)^{\sigma-1} e^{-\left(\lambda(2\beta x)^{1/2}\right)^{\sigma}}. \tag{17}$$

From (14)

$$f_Y\left(Q_Y\left(F_R\left(x\right)\right)\right) = f_Y\left((2\beta x)^{1/2}\right) = (2\beta x)^{1/2} e^{-\beta x}. \tag{18}$$

Substituting (11), (17) and (18) into (6) gives the corresponding pdf of the Weibull-Exponential {Rayleigh} distribution as

$$f_X\left(x\right) = f_{WERD} = \beta\sigma\lambda^{\sigma}\left(2\beta x\right)^{\sigma/2-1} e^{-\left(\lambda(2\beta x)^{1/2}\right)^{\sigma}}. \tag{19}$$

Remark 1.

(i) Weibull $\left(2\beta, \frac{1}{4}\right)$ is a special case of WER distribution when $\sigma = \dfrac{1}{2}$ and $\lambda = 1$

(ii) Rayleigh $\left(\frac{\sqrt{2}}{4\beta}\right)$ is a special case of WER distribution when $\sigma = 4$ and $\lambda = 1$

(iii) Exponential (2β) results from WER distribution when $\sigma = 2$ and $\lambda = 1$

(iv) When $\sigma = 1$, the WER distribution reduces to Exponential-Exponential {Rayleigh} with parameters λ and 2β

Equations (16) and (19) respectively give the cdf and pdf of the 3-parameter Weibull-exponential {Rayleigh} distribution with all the three parameters (β, σ, λ) as shape parameters.

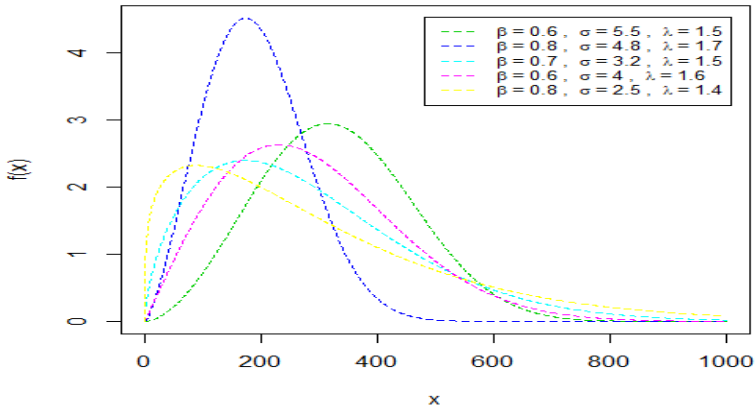Some shapes of the distribution density function are given in Figure 1.

Figure 1. The PDFs of WERD for some values of the parameters.

The superimposed density plots in Figure 2 for different values of the parameters of the WERD indicate that the WERD can be right-skewed or approximately normal.
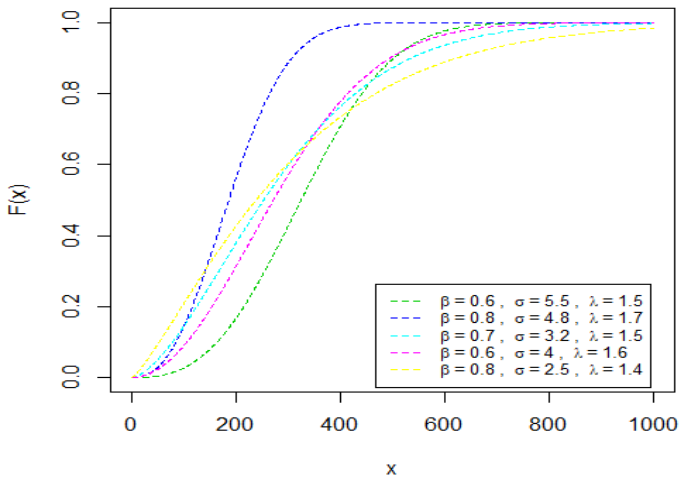

Figure 2. The CDFs of WERD for some values of the parameters.

3. Properties of the Weibull-Exponential {Rayleigh} Distribution

Here, we discuss some of the properties of the Weibull-exponential {Rayleigh} distribution

a. Hazard Function: From (16) and (19), the hazard function of the Weibull-exponential {Rayleigh} distribution is obtained as

$$h_{WERD}(x) = \beta\sigma\lambda^{\sigma}(2\beta x)^{\sigma/2-1}. \tag{20}$$

b. The Relationship between the Weibull-Exponential {Rayleigh} Distributions and the Weibull Distribution

Theorem 1. *If a random variable T follows the Weibull distribution, then the random variable,* $X = \dfrac{T^2}{2\beta}$ *follows Weibull-Exponential {Rayleigh} distribution.*

Proof.

(i) Given that T is a Weibull random variable with pdf

$$f_T(t) = \sigma\lambda(\lambda t)^{\sigma-1}e^{-(\lambda t)^{\sigma}}.$$

Also,

$$x = \frac{t^2}{2\beta}\beta(2\beta x)^{-1/2}$$

$$t = (2\beta x)^{1/2}$$

$$\frac{dt}{dx} = \beta(2\beta x)^{-1/2}$$

$$f_T(x) = \sigma\lambda\left(\lambda(2\beta x)^{1/2}\right)^{\sigma-1}e^{-\left(\lambda(2\beta x)^{1/2}\right)^{\sigma}}$$

$$f_X(x) = f_T(x)\left|\frac{dt}{dx}\right|$$

$$f_T(x) = \beta\sigma\lambda^{\sigma}(2\beta x)^{\frac{\sigma}{2}-1}e^{-\left(\lambda(2\beta x)^{1/2}\right)^{\sigma}}$$

which is the result given in (19). Thus X assumes the Weibull-Exponential {Rayleigh} distribution with parameters β, σ and λ.

c. Quantile Function: The quantile function of the T-R{Y} distribution defined in (5) is

$$Q_{T-R\{Y\}}(p) = Q_R(F_Y(Q_T(p))), \tag{21}$$

where $Q_R(.)$ and $Q_T(.)$ are the quantile functions of R and T with distribution functions $F_R(x)$ and $F_T(x)$ respectively.

Theorem 2. *The Weibull-exponential {Rayleigh} distribution has its quantile function obtained as*

$$Q_{WERD}(p) = \frac{\left[\frac{1}{\lambda}(-\log(1-p))\frac{1}{\sigma}\right]^2}{2\beta}. \tag{22}$$

Proof. From (16)

$$F_X(x) = p = 1 - e^{-\left(\lambda(2\beta x)^{1\!/\!2}\right)^{\sigma}}$$

$$e^{-\left(\lambda(2\beta x)^{1\!/\!2}\right)^{\sigma}} = 1 - p.$$

Taking log of both sides

$$-\left(\lambda(2\beta x)^{1\!/\!2}\right)^{\sigma} = \log(1-p)$$

$$x = \frac{1}{2\beta}\left[\frac{1}{\lambda}(-\log(1-p))^{1\!/\!2}\right]^2.$$

Thus,

$$Q_{WERD}(p) = \frac{\left[\frac{1}{\lambda}(-\log(1-p))^{1\!/\!2}\right]^2}{2\beta}.$$

To obtain the median of the Weibull-exponential {Rayleigh} distribution, $p = 0.5$ is substituted in (22).

$$Q_{WERD}(p = 0.5) = \frac{\left[\frac{1}{\lambda}(-\log(0.5))^{\frac{1}{\sigma}}\right]^2}{2\beta}.$$

d. Moments: The rth crude moment of a random variable X, say m_r' is given by

$$m_r' = \int_0^\infty x^r f_X(x)\,dx.$$

Theorem 3. *The rth crude moment of Weibull-exponential {Raylegh} is given by*

$$m_r' = \frac{\Gamma\left(\frac{2r}{\sigma} + 1\right)}{\left(2\lambda^2\beta\right)^r}. \tag{23}$$

Proof. From Theorem 1,

$$X = \frac{T^2}{2\beta}$$

$$\therefore\ m_r' = E\left[X^r\right] = E\left[\left(\frac{T^2}{2\beta}\right)^r\right]$$

$$= \left(\frac{1}{2\beta}\right)^r E\left[T^{2r}\right]$$

$$= \left(\frac{1}{2\beta}\right)^r \int_0^\infty t^{2r}\,(\sigma\lambda)^{\sigma-1}\,e^{-(\lambda t)^\sigma}\,dt$$

$$= \sigma\lambda^\sigma \left(\frac{1}{2\beta}\right)^r \int_0^\infty t^{2r+\sigma-1} e^{-(\lambda t)^\sigma}\,dt$$

$$= \frac{\Gamma\left(\frac{2r}{\sigma} + 1\right)}{\left(2\lambda^2\beta\right)^r},$$

where

$$\int_0^\infty t^{2r+\sigma-1} e^{-(\lambda t)^\sigma}\, dt = \frac{\Gamma\left(\dfrac{2r}{\sigma}+1\right)}{\sigma \lambda^{2r+\sigma}} \qquad \text{(Grandshteyn and Ryzhik [14]).}$$

Therefore, the first four crude moments of Weibull-Exponential {Rayleigh} distribution are

$$m_1' = \frac{\Gamma\left(\dfrac{2}{\sigma}+1\right)}{2\beta\lambda^2}$$

$$m_2' = \frac{\Gamma\left(\dfrac{4}{\sigma}+1\right)}{4\beta^2\lambda^4}$$

$$m_3' = \frac{\Gamma\left(\dfrac{6}{\sigma}+1\right)}{8\beta^3\lambda^6}$$

$$m_4' = \frac{\Gamma\left(\dfrac{8}{\sigma}+1\right)}{16\beta^4\lambda^8}.$$

a. The **standard deviation (SD)** is given by

$$SD = \left(m_2' - (m_1')^2\right)^{1/2}.$$

Therefore, the standard deviation of Weibull-exponential {Rayleigh} is

$$SD = \frac{1}{\sqrt{\sigma}\beta\lambda^2}\left(\Gamma\left(\frac{4}{\sigma}\right) - \frac{1}{\sigma}\left(\Gamma\left(\frac{2}{\sigma}\right)\right)^2\right)^{1/2}. \tag{24}$$

b. Coefficient of Skewness (CS) and Coefficient of Kurtosis (CK):

Based on moments, the coefficient of skewness is given by

$$CS = \frac{m_3' - 3m_1'm_2' - 2(m_1')^2}{\left(m_2' - (m_1')^2\right)^{3/2}}.$$

Hence, the coefficient of skewness for the Weibull-exponential {Rayleigh} is

$$CS = \frac{\sqrt{\sigma}\left(3\Gamma\left(\frac{6}{\sigma}\right) - \frac{12}{\sigma}\Gamma\left(\frac{2}{\sigma}\right)\Gamma\left(\frac{4}{\sigma}\right) - \frac{4}{\sigma}\left(\Gamma\left(\frac{2}{\sigma}\right)\right)^2\right)}{4\left(\Gamma\left(\frac{4}{\sigma}\right) - \left(\Gamma\left(\frac{2}{\sigma}\right)\right)^2\right)^{3/2}}. \tag{25}$$

Similarly, the coefficient of kurtosis is given by

$$CS = \frac{m_4' - 4m_1'm_3' + 6(m_1')^2 m_2' - 3(m_1')^4}{\left(m_2' - (m_1')^2\right)^2}.$$

Thus, the coefficient of kurtosis for Weibull-exponential {Rayleigh} is

$$CS = \frac{\sigma\left(\Gamma\left(\frac{8}{\sigma}\right) - \frac{6}{\sigma}\Gamma\left(\frac{2}{\sigma}\right)\Gamma\left(\frac{6}{\sigma}\right) + \frac{12}{\sigma^2}\left(\Gamma\left(\frac{2}{\sigma}\right)\right)^2\Gamma\left(\frac{4}{\sigma}\right) - \frac{6}{\sigma^3}\left(\Gamma\left(\frac{2}{\sigma}\right)\right)^4\right)}{2\left(\Gamma\left(\frac{4}{\sigma}\right) - \left(\Gamma\left(\frac{2}{\sigma}\right)\right)^2\right)^2}. \tag{26}$$

c. Shannon Entropy: This measures the level of variation of uncertainty of a random variable, Shannon [23]. The Shannon entropy of the T-R{Y} class of distributions is given by

$$\eta_{T-R\{Y\}} = \eta_T + E[\log(f_Y(T))] - E[\log(f_R(X))], \tag{27}$$

where η_T is the Shannon entropy of T.

Theorem 4. *The Shannon entropy of the Weibull-exponential {Rayleigh} distribution is*

$$\eta_{WERD} = 1 + \beta\mu + \gamma\left(1 - \frac{2}{\sigma}\right) - \log\left(\sigma\beta\lambda^2\right) - \frac{1}{\sigma\lambda^2}\Gamma\left(\frac{2}{\sigma}\right). \tag{28}$$

Proof.

In (20), since $T \sim$ Weibull, the Shannon entropy of T is given by

$$\eta_T = 1 - \log(\sigma\lambda) + \gamma\left(1 - \frac{1}{\sigma}\right), \tag{29}$$

where γ is Euler's constant.

Recall

$$f_Y(x) = xe^{-\frac{x^2}{2}}$$

$$\Rightarrow f_Y(T) = Te^{-\frac{T^2}{2}}$$

$$\log(f_Y(T)) = \log(T) - \frac{T^2}{2}$$

$$\mathrm{E}[\log(f_Y(T))] = \mathrm{E}[\log(T)] - \frac{1}{2}\mathrm{E}[T^2]$$

$$\mathrm{E}[T^2] = \int_0^\infty t^2\left(\lambda\sigma(\lambda t)^{\sigma-1}e^{-(\lambda\sigma)^\sigma}\right)dt$$

$$\mathrm{E}\left[\log(T) - \frac{1}{2}T^2\right] = \lambda\sigma\int_0^\infty\left(\log(t) - \frac{1}{2}t^2\right)(\lambda t)^{\sigma-1}e^{-(\lambda t)^\sigma}dt$$

$$= \sigma\lambda^\sigma\int_0^\infty t^{\sigma-1}e^{-\lambda^\sigma t^\sigma}\log(t)\,dt - \frac{\sigma\lambda^\sigma}{2}\int_0^\infty t^{\sigma+1}e^{-\lambda^\sigma t^\sigma}dt.$$

Recall:

$$\int_0^\infty x^m e^{-sx^b}\log(x)\,dx = \frac{\Gamma(a)}{b^2 s^a}(\psi(a) - \log(s)), \quad \text{(Moll, 2007)}$$

where $a = \dfrac{m+1}{b}$.

Therefore,

$$\mathrm{E}\left[\log(T) - \frac{1}{2}T^2\right] = \sigma\lambda^\sigma\left[\frac{\Gamma(1)}{\sigma^2\lambda^\sigma}(\psi(1) - \sigma\log(\lambda))\right] - \frac{\sigma\lambda^\sigma}{2}\left[\frac{\Gamma\left(\frac{2}{\sigma}+1\right)}{\sigma\lambda^{\sigma+2}}\right]$$

$$\therefore \mathrm{E}[\log(f_Y(T))] = \frac{1}{\sigma}(-\gamma - \sigma\log(\lambda)) - \frac{1}{\sigma\lambda^2}\Gamma\left(\frac{2}{\sigma}\right). \tag{30}$$

Also, recall

$$f_R(x) = \beta e^{-\beta x}$$

$$\therefore f_R(X) = \beta e^{-\beta X}$$

$$\log(f_R(X)) = \log(\beta) - \beta X$$

$$E[\log(f_R(x))] = E[\log(\beta) - \beta X]$$

$$= \log(\beta) - \beta E[X]$$

$$E\Big[\log\big(f_R(x)\big)\Big] = \log(\beta) - \beta\mu. \tag{31}$$

On substituting (29), (30) and (31) into (27) we have

$$\eta_{WERD} = 1 + \beta\mu + \gamma\left(1 - \frac{2}{\sigma}\right) - \log\left(\sigma\beta\lambda^2\right) - \frac{1}{\sigma\lambda^2}\Gamma\left(\frac{2}{\sigma}\right).$$

Theorem 5.

d. Mode: *The mode of Weibull-exponential {Rayleigh} is given by*

$$x = \frac{1}{2\beta\lambda^2}\left(\frac{\sigma - 2}{\sigma}\right)^{2/\sigma}.$$

Proof.

The first derivative of (19) is

$$f_X'(x) = \left(\frac{\sigma}{2} - 1\right)x^{\sigma/2 - 2}e^{-\left(\lambda[2\beta x]^{1/2}\right)^{\sigma}} - \frac{\sigma}{2}\left(2\beta\lambda^2\right)^{\sigma/2}e^{-\left(\lambda[2\beta x]^{1/2}\right)^{\sigma}}$$

factorizing and equating to zero

$$x^{\sigma/2 - 1}e^{-\left(\lambda[2\beta x]^{1/2}\right)^{\sigma}}\left[\left(\frac{\sigma}{2} - 1\right) - \frac{\sigma}{2}\left(2\beta\lambda^2\right)^{\sigma/2}x^{\sigma/2}\right]$$

$$x = \frac{1}{2\beta\lambda^2}\left(\frac{\sigma - 2}{\sigma}\right)^{2/\sigma}.$$

4. Estimation of Weibull-exponential {Rayleigh} Parameters using Maximum Likelihood Method

Suppose $x_1, x_2, ..., x_n$ is a random sample of size n, the log-likelihood function of the Weibull-exponential {Rayleigh} distribution is

$$L = \sum_{i=1}^{n} \ln(f_X(x_i)) = n\ln\beta + n\ln\sigma + n\sigma\ln\lambda + n\left(\frac{\sigma}{2} - 1\right)\{\ln(2) + \ln\beta\}$$

$$+ \left(\frac{\sigma}{2} - 1\right)\sum_{i=1}^{n}\ln x_i - \lambda^{\sigma}(2\beta)^{\frac{1}{2}}\sum_{i=1}^{n} x_i^{\frac{\sigma}{2}}. \tag{32}$$

By partially differentiating (32) with respect to each of the parameters, we obtain

$$\frac{\partial L}{\partial \lambda} = \frac{n\sigma}{\lambda} - \sigma(2\beta)^{\frac{\sigma}{2}}\lambda^{\sigma-1}\sum_{i=1}^{n} x^{\frac{\sigma}{2}} \tag{33}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \frac{n\left(\frac{\sigma}{2} - 1\right)}{\beta} - \sigma\lambda^{\sigma}(2\beta)^{\frac{\sigma}{2}-1}\sum_{i=1}^{n} x^{\frac{\sigma}{2}} \tag{34}$$

$$\frac{\partial L}{\partial \sigma} = \frac{n}{\sigma} + \frac{n}{2}[\ln 2 + \ln\beta] + \frac{1}{2}\sum_{i=1}^{n}\ln x_i$$

$$-\left(2\lambda^2\beta\right)^{\frac{\sigma}{2}}\left\{\ln\left(2\lambda^2\beta\right)^{\frac{1}{2}}\sum x_i^{\frac{\sigma}{2}} + \sum_{i=1}^{n} x^{\frac{\sigma}{2}}\ln x^{\frac{1}{2}}\right\}. \tag{35}$$

Since equating (33)-(35) to zero and simultaneously solving for each of the parameters cannot result to a close form of solution, the numerical method, Broyden-Fletcher-Goldfarb-Shanno (BFGS) iterative method embedded in R is adopted.

5. Applications

This section presents an application of the Weibull-Exponential {Rayleigh} distribution using two real datasets. The fit of the Weibull-Exponential {Rayleigh} distribution (WERD) to the data sets is compared with those of Weibull-Exponential (WED) (Oguntunde et al. [21], Weibull-Rayleigh distribution (WRD) (Merovci and Elbatal [18]), Exponential-Rayleigh distribution (ERD (Fatima and Ahmad [13]), Exponential distribution (ED) and Rayleigh distribution (RD) with respective PDFs

$$f_{WED}(x) = \alpha\beta\theta\left(1 - e^{-\theta x}\right)^{\beta-1} \exp\left\{\theta\beta x - \alpha\left(e^{\theta x} - 1\right)^{\beta}\right\}; \quad x > 0, \; \alpha, \beta, \theta > 0 \tag{36}$$

$$f_{WRD}(x) = \alpha\beta\lambda x e^{\frac{\lambda}{2}x^2}\left(e^{\frac{\lambda}{2}x^2} - 1\right)^{\beta-1} \exp\left(-\alpha\left(e^{\frac{\lambda}{2}x^2} - 1\right)^{\beta}\right); \quad x > 0, \; \alpha, \beta, \lambda > 0 \tag{37}$$

$$f_{ERD}(x) = \lambda\beta e^{\frac{\lambda}{2}x^2} \exp\left(-\beta\left(e^{\frac{\lambda}{2}x^2} - 1\right)\right); \quad x > 0, \; \beta, \lambda > 0 \tag{38}$$

$$f_{ED}(x) = \frac{1}{\lambda}e^{-\frac{x}{\lambda}}; \quad x > 0, \lambda > 0. \tag{39}$$

The measures adopted for the goodness-of-fit testing are the parameter estimates, log-likelihood function, Akaike Information Criteria (AIC) and Kolmogorov-Smirnov Statistic (K-s) and p-values obtained with the aid of R software. A distribution with smaller AIC and K-S but higher log-likelihood and p-values relative to other distributions gives a better fit to the dataset.

Dataset 1. The dataset represents the time between failures for a repairable item obtained from Murthy et al. [20]

1.43,0.11,0.71,0.77,2.63,1.49,3.46,2.46,0.59,0.74,1.23,0.94,

4.36,0.40,1.74,4.73,2.23,0.45,0.70,1.06,1.46,0.30,1.82,2.37,

0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

Dataset 2. This dataset contains the tensile strength, measured in GPa, of 69 carbon fibres tested under tension at gauge lengths of 20 mm and reported by Bader and Priest [12]

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997,
2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270,
2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490,
2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684,
2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012,
3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Table 1. Estimates and performance of the distributions for data set I.

Distribution	Parameter	Std Error	LL	AIC	K-s	p-Value
WERD	$\hat{\sigma} = 2.9270$	0.4058	-39.9104	85.8207	0.0749	0.7630
	$\hat{\lambda} = 0.4632$	10.0908				
	$\hat{\beta} = 1.3627$	59.3697				
WED	$\hat{\alpha} = 2.5825$	2.5369	-40.5934	87.1869	0.1024	0.8798
	$\hat{\theta} = 0.2079$	0.1212				
	$\hat{\beta} = 1.2113$	0.2038				
WRD	$\hat{\alpha} = 8.1794$	11.0738	-40.0404	86.0807	0.0811	0.9800
	$\hat{\lambda} = 0.0320$	0.0529				
	$\hat{\beta} = 0.6989$	0.1027				
ERD	$\hat{\beta} = 238.8614$	63.3332	-42.9890	89.9780	0.1899	0.2020
	$\hat{\lambda} = 63.3332$	0.0004				
ED	$\hat{\lambda} = 1.5427$	0.2817	-43.0054	88.0108	0.1845	0.2290
RD	$\hat{\lambda} = 1.3433$	0.1226	-42.9183	87.8366	0.1864	0.2189

The results reported in Table 1 reveal that the Weibull-Exponential {Rayleigh} distribution performed better than the other distributions compared with as it has the highest value of log-likelihood (LL) and p-value. Also, the AIC and K-s values of the distribution are the smallest when compared with those of other distributions in fitting the data.
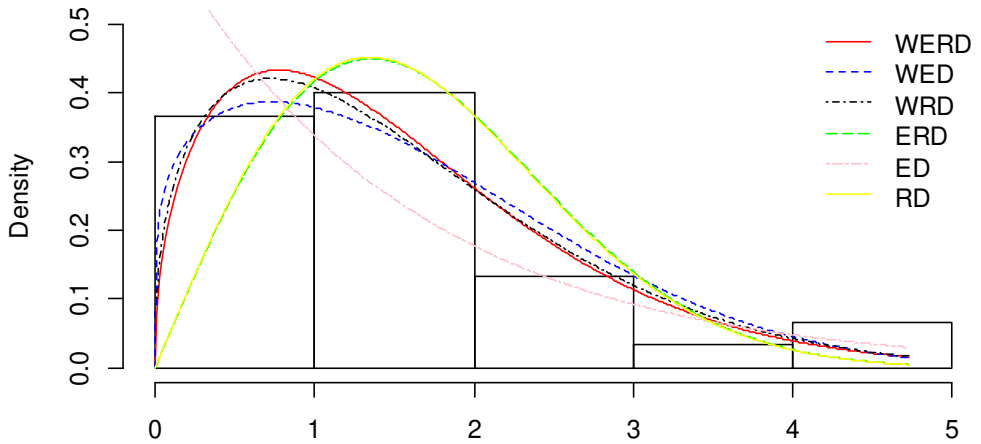
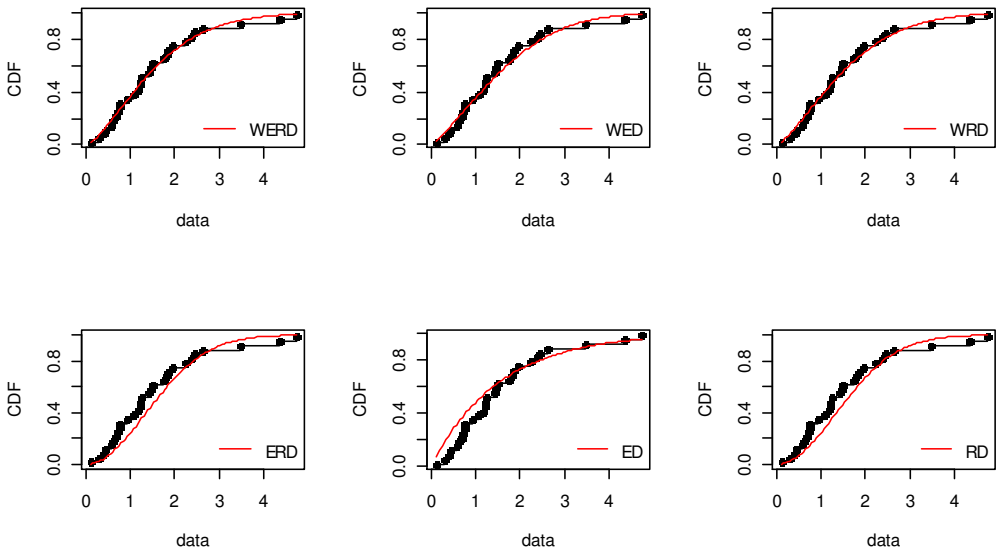**Figure 3.** The histogram and superimposed fitted PDFs for the data on failure times of repairable item.

Figure 4. The CDFs for the data on failure times of repairable item.

Table 2. Estimates and performance of the distributions for data set II.

Distribution	Parameter	Std Error	LL	AIC	K-s	p-Value
WERD	$\hat{\sigma} = 11.0106$	1.0012	-49.5961	105.1923	0.0562	0.9727
	$\hat{\lambda} = 0.6041$	1.1813				
	$\hat{\beta} = 0.5169$	2.0215				
WED	$\hat{\alpha} = 1.7577$	3.6464	-50.3682	106.6894	0.0651	0.9137
	$\hat{\theta} = 0.2351$	0.0859				
	$\hat{\beta} = 4.0883$	0.5473				
WRD	$\hat{\alpha} = 9.4721$	4.3632	-50.3682	106.7365	0.0666	0.8994
	$\hat{\lambda} = 0.0909$	0.0447				
	$\hat{\beta} = 2.3316$	0.2608				
ERD	$\hat{\beta} = 0.1107$	0.0418	-56.1930	116.3861	0.1101	0.3476
	$\hat{\lambda} = 0.6309$	0.0760				
ED	$\hat{\lambda} = 2.4514$	0.2951	-130.8676	263.7352	0.4483	0.0000
RD	$\hat{\lambda} = 1.7678$	0.1064	-87.2454	176.4931	0.3385	0.0000

Values in Table 2 indicate that, for the given set of data, the Weibull-Exponential {Rayleigh} distribution (WERD) gives the best fit based on the values of the LL, AIC, K-s and p-value.
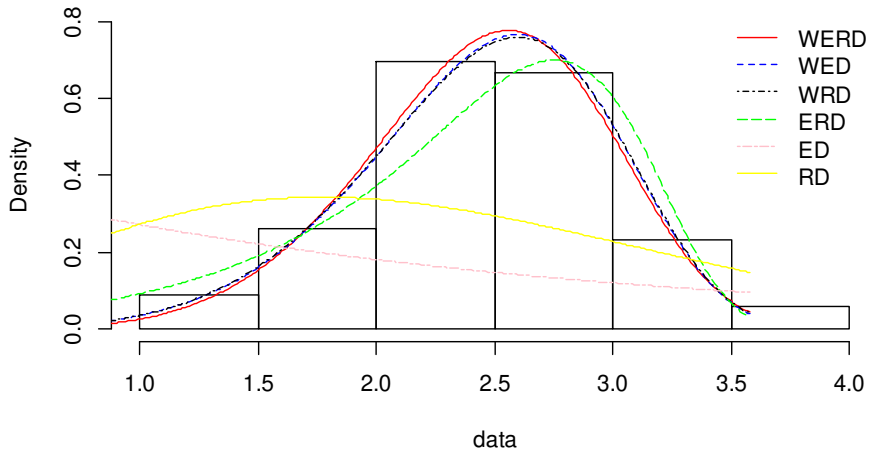
Figure 5. The histogram and superimposed fitted PDFs for the data on carbon fibres.
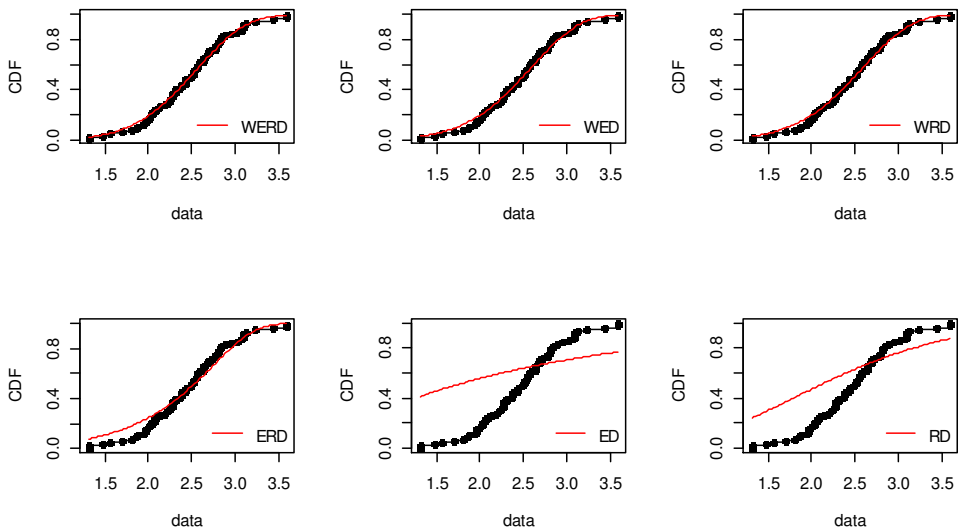

Figure 6. The CDFs for the data on carbon fibres.

6. Summary and Conclusion

In this article, a new submodel belonging to the class of generalized exponential distributions is defined and studied using the T-R{Y} approach introduced by Aljarrah et al. [3]. Some features of the submodel are derived. The estimates of the parameters of the submodel are obtained with use of maximum likelihood method. Two real data sets are adopted in demonstrating the flexibility and potential applications of the new distribution.

References

[1]   R. Al-Aqtash, C. Lee and F. Famoye, Gumbel-Weibull distribution: Properties and application, *Journal of Modern Applied Statistical Methods* 13(2) (2014), 201-225. https://doi.org/10.22237/jmasm/1414815000

[2]   M. Aldeni, C. Lee and F. Famoye, Families of distributions arising from the quantile of generalized lambda distribution, *Journal of Statistical Distributions and Applications* 4(25) (2017), 1-18. https://doi.org/10.1186/s40488-017-0081-4.

[3]   M.A. Aljarrah, C. Lee and F. Famoye, A new method of generating T-X family of distributions using quantile functions, *Journal of Statistical Distributions and Applications* 1(2) (2014), 1-17. http://www.jsdajournal.com/content/1/1/2

[4]   M. Almheidat, F. Famoye and C. Lee, Some generalized families of Weibull distribution: Properties and applications, *International Journal of Statistics and Probability* 4(3) (2015), 18-35. http://dx.doi/10.5539/ijsp.v4n3p18

[5]   A. Alzaatreh, C. Lee and F. Famoye, A new method for generating families of continuous distributions, *Metron* 71 (2013a), 63-79. https://doi.org/10.1007/s40300-013-0007-y

[6]   A. Alzaatreh, F. Famoye and C. Lee, Weibull-Pareto distribution and applications, *Comm. Statist. Theory Methods* 42(9) (2013b), 1673-1691. http://dx.doi.org/10.1080/03610926.2011.599002

[7]   A. Alzaatreh, C. Lee and F. Famoye, T-normal family of distributions: a new approach to generalize the normal distribution, *Journal of Statistical Distributions and Applications* 1(16) (2014), 1-18. http://www.jsdajournal.com/content/1/1/16

[8]   A. Alzaatreh, C. Lee and F. Famoye, Family of generalized gamma distributions: Properties and Applications, *Hacet. J. Math. Stat.* 45(3) (2016a), 869-886. https://doi.org/10.15672/HJMS.20156610980

[9]   A. Alzaatreh, C. Lee, F. Famoye and I. Ghosh, The generalized Cauchy family of distributions with applications, *Journal of Statistical Distributions and Applications* 3 (2016b), Article No. 12. https://doi.org/10.1186/s40488-016-0050-3

[10]   A. Alzahal, F. Famoye and C. Lee, Exponentiated T-X family of distributions with some applications, *International Journal of Statistics and Probability* 2(3) (2013), 31-49. http://dx.doi.org/10.5539/ijsp.v2n3p31

[11]   M.I. Ekum, M.O. Adamu and E.E. Akwarawak, T-Dagum: A way of generalizing Dagum distribution using Lomax quantile function, *J. Probab. Stat.* (2020), Art. ID 1641207, 17 pp. https://doi.org/10.1155/2020/1641207

[12]   M.G. Bader and A.M. Priest, Statistical aspects of fiber and bundle strength in hybrid

composites, T. Hayashi, K. Kawata and S. Umekawa, eds., Progress in Science and Engineering Composites, pp. 1129-1136, Tokyo: ICCM-IV, 1982.

[13] K. Fatima and S.P. Ahmad, Statistical properties of Exponential Rayleigh distribution and its application to medical sciences and engineering, International Conference on Recent Innovations in Science, Agriculture, Engineering and Management University College of Computer Applications Guru Kashi University, Bathinda, Punjab (India) (2017), 491-506.

[14] I.S. Grandshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Product*, 8th ed., Elsevier, Inc., 2015.

[15] D. Hamed, F. Famoye, and C. Lee, On families of generalized Pareto distributions: properties and applications, *Journal of Data Science* (2018), 377-396. https://doi.org/10.6339/JDS.201804_16(2).0008

[16] F. Jamal, M.A. Aljarrah, M.H. Tahir and M.A. Nasir, A new extended generalized Burr-III family of distributions, *Tbilis Math. J.* 11 (2018), 59-78.

[17] C. Lee, F. Famoye and A. Alzaatreh, Methods for generating families of univariate continuous distributions in the recent decades, *WIREs Computational Statistics* 5 (2013), 219-238. https://doi.org/10.1002/wics.1255

[18] F. Merovci and I. Elbatal, Weibull Rayleigh distribution: Theory and Applications, *Appl. Math. Inf. Sci.* 9(4) (2015), 2127-2137. http://dx.doi.org/10.12785/amis/090452

[19] V.H. Moll, The integrals in Gradshteyn and Ryzhik. Part 3: Combinations of logarithms and exponentials, *Sci. Ser. A Math. Sci. (N.S.)* 15(1) (2007), 31-36. http://arxiv.org/abs/0705.0175v1

[20] D.N.P. Murthy, M. Xie and R. Jiang, *Weibull Models*, John Wiley, New Jersey, 2004.

[21] P. Oguntunde, O.S. Balogun, H.I. Okagbue and S.A. Bishop, The Weibull-Exponential distribution: Its properties and applications, *Journal of Applied Sciences* 15(11) (2015), 1305-1311. https://doi.org/10.3923/jas.2015.1305.1311

[22] P. Osatohanmwen, F.O. Oyegue and S.M. Ogbonmwan, A new member from the T-X family of distributions: The Gumbel-Burr XII distribution and its Properties, *Sankhya A* 81 (2019), 298-322. https://doi.org/10.1007/s13171-017-0110-x

[23] C.E. Shanon, A mathematical theory of communication, *Mobile Computing and Communications Review* 5 (1948). https://doi.org/10.1145/584091.584093

[24] G.R. Shorack and J.A. Wellner, *Empirical Processes with Applications to Statistics*, John Wiley and Sons, New York, 1986.

[25] M.H. Tahir, G.M. Cordeiro, A. Alzaatreh, M. Monsoor and M. Zubair, The logistic-X family of distributions and its applications, *Comm. Statist. Theory Methods* 45 (2016), 7326-7349. http://dx.doi.org/10.1080/03610926.2014.980516