Earthline Journal of Mathematical Sciences
ISSN (Online): 2581-8147
Volume 6, Number 1, 2021, Pages 33-63
https://doi.org/10.34198/ejms.6121.3363

A New Generalization of the Inverse Distributions: Properties and Applications

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Abstract

In this paper the generalized inverse distribution is defined. Some properties and applications of the generalized inverse distribution are studied in some detail. Characterization theorems generalizing the new family in terms of the hazard function are obtained. Recommendation for further study concludes the paper.

1 Introduction

Suppose random variable X has CDF $F(x)$, the CDF of the inverse distribution associated with the random variable X is defined as

$$1 - F_X\left(\frac{1}{x}\right).$$

The PDF is given by

$$\frac{1}{x^2}\, f_X\left(\frac{1}{x}\right).$$

Two methods for creating probability distributions appeared in [1] and [2]. The work in [2] extends that of [1]. The CDF of the distribution generalizing that of [1] is given by

Received: September 30, 2020; Accepted: November 25, 2020

2010 Mathematics Subject Classification: 62Exx.

Keywords and phrases: inverse distribution, characterization theorem, hazard function.

$$G(x) = R_T[Q_Y(F_R(x))],$$

where R_T is the CDF of the random variable T, Q_Y is the quantile function of the random variable Y, and F_R is the CDF of the random variable R. The PDF is given by

$$g(x) = \frac{f_R(x)}{p[Q_Y(F_R(x))]}\ r_T[Q_Y(F_R(x))],$$

where p is such that $\frac{d}{dy}Q_(y) = \frac{1}{(p \circ Q)(y)}$, f_R is PDF of the random variable R, and r_T is PDF of the random variable T. This paper defines the inverse of the distributions with the above CDF and PDF and studies some of their properties with application. It should be noted that some works generalizing the inverse distributions have appeared in [3]-[16].

2 Organization of Manuscript

This manuscript is organized as follows. In Section 3, we define the new family introducing its CDF, PDF, and support. In Section 4, we discuss three sub-models according to the support of the new family, there the CDF, PDF, survival function (SF), and hazard function (HF) are highlighted in a plot. In Section 5 we obtain some statistical properties of the new family including quantile function, random number generation, rth non-central moments, and the Renyi entropy. In Section 6, the method of maximum likelihood in estimating model parameters are discussed. In Section 7, a simulation study is conducted to assess the performance of the maximum likelihood method in estimating model parameters. In Section 8, application to three real life data sets are considered, and submodels of the new family are shown to be competitive in fitting real life data. In Section 9, some characterization theorems generalizing the new family are presented in terms of the hazard function of a random variable. Some further research direction concludes the paper in Section 10.

3 The New Family Defined

3.1 The CDF of the New Family

Suppose random variable X has CDF

$$G(x) = \int_a^{Q_Y(F_R(x))} r_T(t)dt.$$

Consider the random variable $Z = \frac{1}{X}$. The CDF of Z is

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Z \leq z) \\ &= \mathbb{P}\left(\frac{1}{X} \leq z\right) \\ &= \mathbb{P}\left(X \geq \frac{1}{z}\right) \\ &= 1 - \mathbb{P}\left(X < \frac{1}{z}\right) \\ &= 1 - G\left(\frac{1}{z}\right). \end{aligned}$$

So

$$F_Z(z) = 1 - \int_a^{Q_Y\left(F_R\left(\frac{1}{z}\right)\right)} r_T(t)dt = 1 - R_T\left[Q_Y\left(F_R\left(\frac{1}{z}\right)\right)\right].$$

We call the random variable Z with above CDF a generalized inverse type distribution, or the inverse $T - R\{Y\}$ distribution.

3.2 The PDF of the New Family

The PDF of the generalized inverse type distribution is given by

$$f_Z(z) = r_T\left[Q_Y\left(F_R\left(\frac{1}{z}\right)\right)\right] \times q_Y\left(F_R\left(\frac{1}{z}\right)\right) \times \frac{1}{z^2} f_R\left(\frac{1}{z}\right),$$

where

$$R'_T = r_T,$$

$$Q'_Y = q_Y,$$

$$F'_R = f_R.$$

3.3 The Support of the New Family

Our discussion is motivated by [1] and [2]. Here we consider three cases

Case #1: T has support $[0, \infty)$

As recorded in Table 1 [2], in this case we may take Y as Exponential, Weibull, Rayleigh, Dagum, Lomax, Log-Logistic, and Exponential-Exponential.

Case #2: T has support $(-\infty, \infty)$

As recorded in Table 1 [2], in this case we may take Y as Cauchy, Extreme Value (Gumbel), Laplace, Logistic, and Generalized Logistic II.

Case #3: T has support $[0, 1]$ or bounded support in general

According to [1], we may take $Q_Y(y) = y$.

In all three cases $F_R(\frac{1}{z})$ and $f_R(\frac{1}{z})$ has same support as the support of the distribution associated with the random variable R.

4 Some Sub-Models of the New Family

In this section we introduce three sub-models according to the support of the new family. The PDF, CDF, survival function (SF), and hazard function (HF) are visualized.

Case #1: T has support $[0,\infty)$

The proposed sub-model is called the Inverse Standard Exponential-Dagum {Standard Log-Logistic} distribution. The PDF is given by

$$f(x;a,b,c)=\frac{abe^{\frac{1}{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}}\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)\left(\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}-1\right)^{2}}.$$

The CDF is given by

$$F(x;a,b,c)=e^{\frac{1}{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}}.$$

The SF is given by

$$SF(x;a,b,c)=1-F(x;a,b,c)=1-e^{\frac{1}{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}}$$

and the HF is given by

$$HF(x;a,b,c)=\frac{f(x;a,b,c)}{1-F(x;a,b,c)}=\frac{\frac{abe^{\frac{1}{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}}\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)\left(\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}-1\right)^{2}}}{1-e^{\frac{1}{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}}},$$

where $x,a,b,c>0$. We write

$$J\sim ISEDSLL(a,b,c)$$

if J is an Inverse Standard Exponential-Dagum{Standard Log-Logistic} random variable.

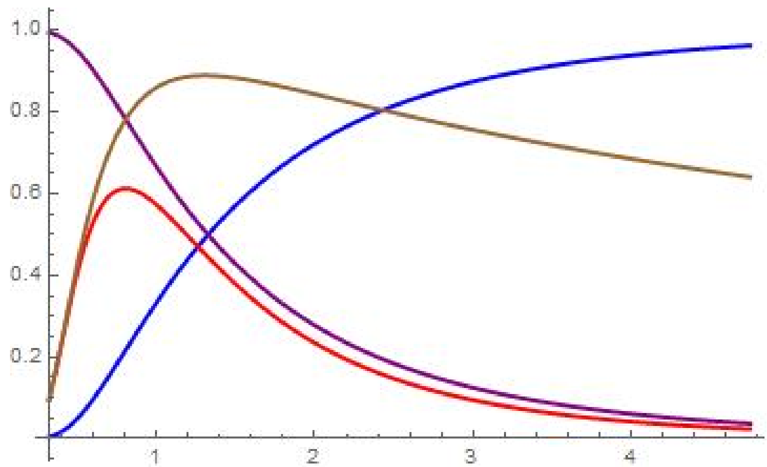

Figure 1: The CDF(blue), PDF(red), SF(purple), and HF(brown) of ISEDSLL(5.1028,1.22623,0.195919).

Case #2: T has support $(-\infty, \infty)$

The proposed sub-model is called the Inverse Standard Extreme Value-Dagum {Standard Logistic} distribution. The PDF is given by

$$k(x; a, b, c) = \frac{abe^{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}} \left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)}.$$

The CDF is given by

$$K(x; a, b, c) = 1 - e^{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}.$$

The SF function is given by

$$SF(x; a, b, c) = 1 - K(x; a, b, c) = 1 - (1 - e^{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}).$$

The HF is given by

$$HF(x;a,b,c) = \frac{k(x;a,b,c)}{K(x;a,b,c)} = \frac{\dfrac{abe^{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)}}{1-(1-e^{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}})},$$

where $x, a, b, c > 0$. We write

$$JJ \sim ISEVDSL(a, b, c)$$

if JJ is an Inverse Standard Extreme Value-Dagum{Standard Logistic} random variable.

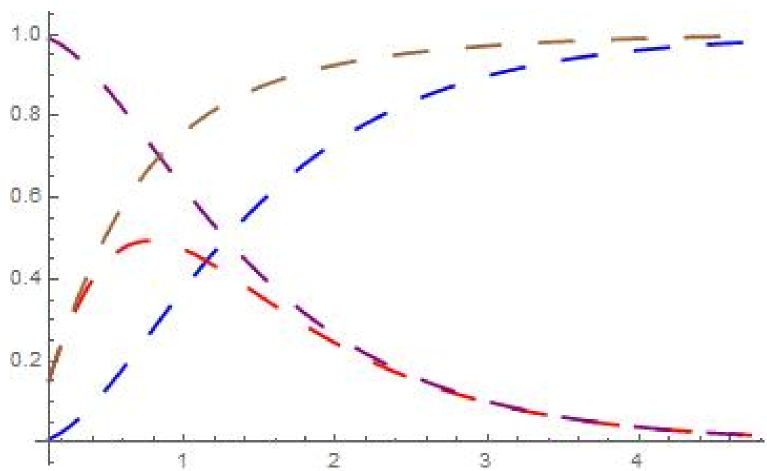

Figure 2: The CDF(blue), PDF(red), SF(purple), and HF(brown) of ISEVDSL(0.526334,1.88075,1.04701).

Case #3: T has support $[0, 1]$ or bounded support in general

The proposed sub-model is called the Inverse Beta(2,2)-Dagum{Standard Uniform} distribution. The PDF is given by

$$w(x;a,b,c)=\frac{6ab\left(\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}-1\right)\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{-3a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)}.$$

The CDF is given by

$$W(x;a,b,c)=1-I_{\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{-a}}(2,2).$$

The SF is given by

$$SF(x;a,b,c)=1-W(x;a,b,c)=1-(1-I_{\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{-a}}(2,2)).$$

The HF is given by

$$HF(x;a,b,c)=\frac{w(x;a,b,c)}{1-W(x;a,b,c)}=\frac{\frac{6ab\left(\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}-1\right)\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{-3a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)}}{1-(1-I_{\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{-a}}(2,2))},$$

where

$$I_z(a,b)=\frac{\int_0^z t^{a-1}(1-t)^{b-1}dt}{\int_0^1 t^{a-1}(1-t)^{b-1}dt}$$

and $x,a,b,c>0$. We write

$$JK\sim IBDSU(a,b,c)$$

if JK is an Inverse Beta(2,2)-Dagum{Standard Uniform} random variable.

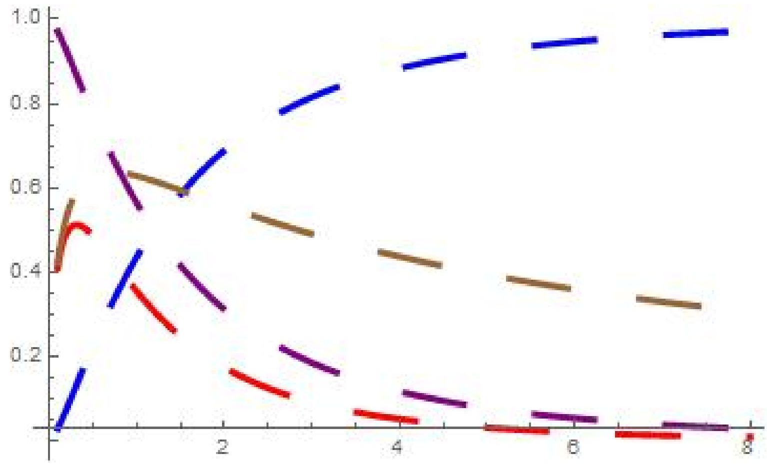

Figure 3: The CDF(blue), PDF(red), SF(purple), and HF(brown) of IBDSU(2.85163, 0.796766, 0.165976).

5 Some Mathematical Properties

5.1 Quantile Function

Theorem 5.1. *The quantile function of the generalized inverse type distribution is given by*

$$Q(p) = \frac{1}{Q_R(F_Y(Q_T(1-p)))},$$

where $0 < p < 1$.

Proof. With $0 < p < 1$, we solve the following equation for $Q(p)$

$$p = 1 - R_T\left[Q_Y\left(F_R\left(\frac{1}{Q(p)}\right)\right)\right],$$

where $Q_R = F_R^{-1}$, $F_Y^{-1} = Q_Y$, and $Q_T = R_T^{-1}$. □

5.2 Random Number Generation

If U is uniform on $(0,1)$, then random numbers from the generalized inverse type distribution can be obtained using

$$X = \frac{1}{Q_R(F_Y(Q_T(1-U)))}.$$

5.3 rth Non-Central Moments

Assuming the random variable R follows the standard log-logistic distribution, then the random variable

$$X = \frac{F_Y(Q_T(1-U)) - 1}{F_Y(Q_T(1-U))}$$

follows the generalized inverse type standard log-logistic distribution. Thus we have the following

Theorem 5.2. *The ordinary moments of the inverse type standard log-logistic distribution for $r \in \mathbb{N}$, can be expressed as*

$$\mu'_r = \sum_{k=0}^{\infty} (-1)^{r+k} \binom{r}{k} (F_Y(Q_T(1-U)))^{k-r}.$$

Proof. By the binomial series we can write

$$\left(\frac{F_Y(Q_T(1-U)) - 1}{F_Y(Q_T(1-U))}\right)^r$$

as

$$\sum_{k=0}^{\infty} (-1)^{r+k} \binom{r}{k} (F_Y(Q_T(1-U)))^{k-r}.$$

Hence the result □

5.4 Renyi Entropy

By definition the Renyi entropy is defined as

$$I_R(\delta) = \frac{1}{1-\delta} \log \left[\int_{-\infty}^{\infty} f^{\delta}(z) dz \right],$$

where $\delta > 0$ and $\delta \neq 1$.

Since the PDF of the inverse type standard log-logistic distribution is given by

$$f_Z(z) = r_T\left[Q_Y\left(\frac{1}{1+z}\right)\right] \times q_Y\left(\frac{1}{1+z}\right) \times \frac{1}{(1+z)^2}.$$

The definition immediately above implies the following

Theorem 5.3. *The Renyi entropy of the generalized inverse type log-logistic distribution can be expressed as*

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_{-\infty}^{\infty}\left\{r_T\left[Q_Y\left(\frac{1}{1+z}\right)\right] \times q_Y\left(\frac{1}{1+z}\right) \times \frac{1}{(1+z)^2}\right\}^{\delta} dz\right],$$

where $\delta > 0$ *and* $\delta \neq 1$.

6 Parameter Estimation

The method of maximum likelihood is used in this paper to estimate model parameters. Here we discuss this method for the generalized inverse type distribution. Suppose $x_1, x_2, \cdots, x_n$ is a random sample of size n from the generalized inverse type distribution. It can be shown that the total log-likelihood function is given by

$$\begin{aligned}\ln L = &\sum_{i=1}^{n}\left\{ln\left(r_T\left[Q_Y\left(F_R\left(\frac{1}{z};\xi\right)\right)\right]\right)\right\} + \sum_{i=1}^{n}\left\{ln\left(q_Y\left(F_R\left(\frac{1}{z};\xi\right)\right)\right)\right\}\\ &+\sum_{i=1}^{n}\left\{ln\left(\frac{1}{z^2}\right)\right\} + \sum_{i=1}^{n}\left\{ln\left(f_R\left(\frac{1}{z};\xi\right)\right)\right\},\end{aligned}$$

where ξ is a vector of parameters associated with the distribution of the random variable R. Partial differentiation of the total log-likelihood function with respect to model parameters gives the following as the score function

$$\frac{\partial \ln L}{\partial \xi} = \sum_{i=1}^{n} \frac{r'_T\left(Q_Y\left(F_R\left(\frac{1}{z};\xi\right)\right)\right) \times q_Y\left(F_R\left(\frac{1}{z};\xi\right)\right) \times f_R\left(\frac{1}{z};\xi\right)}{r_T\left[Q_Y\left(F_R\left(\frac{1}{z};\xi\right)\right)\right]} +$$

$$\sum_{i=1}^{n} \frac{q'_Y\left(F_R\left(\frac{1}{z};\xi\right)\right) \times f_R\left(\frac{1}{z};\xi\right)}{q_Y\left(F_R\left(\frac{1}{z};\xi\right)\right)} + \sum_{i=1}^{n} \frac{f'_R\left(\frac{1}{z};\xi\right)}{f_R\left(\frac{1}{z};\xi\right)}.$$

Equating the score function to zero and numerically solving the equation using techniques such as the quasi Newton-Raphson method, gives the maximum likelihood estimates for the model parameters. Let $\triangle = (\xi)$, for the purposes of constructing confidence intervals for the parameters in the hyperbolic tan-X family of distributions, the observed information matrix, call it $J(\triangle)$, can be used due to the complex nature of the expected information matrix. The observed information matrix is given by

$$J(\triangle) = -\left[\frac{\partial^2 lnL}{\partial\xi\partial\xi}\right].$$

When the usual regularity conditions are satisfied and that the parameters are within the interior of the parameter space, but not on the boundary, the distribution of $\sqrt{n}(\widehat{\triangle} - \triangle)$ converges to the multivariate normal distribution $N_p(0, I^{-1}(\triangle))$, where $I(\triangle)$ is the expected information matrix, and it is assumed that $\xi = (\xi_1, \cdots, \xi_p)$. The asymptotic behavior remains valid when $I(\triangle)$ is replaced by the observed information matrix evaluated at $J(\widehat{\triangle})$. The asymptotic multivariate normal distribution $N_p(0, J^{-1}(\widehat{\triangle}))$ is a very useful tool for constructing an approximate $100(1-\psi)\%$ two-sided confidence intervals for the model parameters, where ψ is the significance level.

7 Simulation Study

In this section we show that the method of maximum likelihood is adequate in estimating the parameters in the generalized inverse type distribution. For this, a Monte Carlo simulation study is carried out to assess the performance of the estimation method in the ISEVDSL sub-model. Samples of sizes 200, 400, 500, and 700, are drawn from the ISEVDSL distribution, and the samples have been drawn for the following set of parameters

(a) Set I: $(a, b, c) = (0.3, 0.5, 0.7)$,

(b) Set II: $(a, b, c) = (0.7, 0.5, 0.3)$,

(c) Set III: $(a, b, c) = (0.5, 0.5, 0.5)$.

The maximum likelihood estimators for the parameters a, b and c are obtained. The procedure has been repeated 400 times, and the mean and standard deviation for the estimates are computed, and the results are summarized in Tables 1-3 below for each of sets I, II and III, respectively, considered above

Table 1: Result of Simulation Study for Set I.

	Parameter a	
Sample Size	Average Estimate	Standard Deviation
200	0.3892565	0.3328055
400	0.349296	0.2133193
500	0.3424607	0.1929938
700	0.3191086	0.1541398
	Parameter b	
Sample Size	Average Estimate	Standard Deviation
200	0.5111777	0.06877959
400	0.5048487	0.04457237
500	0.5040079	0.04033673
700	0.5001244	0.03365005
	Parameter c	
Sample Size	Average Estimate	Standard Deviation
200	0.7595461	0.2755513
400	0.7286692	0.1859357
500	0.7165568	0.1435993
700	0.7201309	0.1159644

From the table above, we find that the simulated estimates are close to the true values of the parameters and hence the estimation method is adequate. We have also observed that the estimated standard deviation consistently decrease with increasing sample size as can been seen by plotting the standard deviation against the sample size.

Table 2: Result of Simulation Study for Set II.

	Parameter a	
Sample Size	Average Estimate	Standard Deviation
200	0.8608792	0.6430768
400	0.7671702	0.3629744
500	0.7725065	0.3436264
700	0.7729478	0.2817265
	Parameter b	
Sample Size	Average Estimate	Standard Deviation
200	0.5159465	0.08208287
400	0.5067538	0.05257483
500	0.5070602	0.04684715
700	0.5069424	0.03793475
	Parameter c	
Sample Size	Average Estimate	Standard Deviation
200	0.306919	0.07088208
400	0.3032652	0.04883596
500	0.3009655	0.04330803
700	0.2984207	0.03330566

From the table above, we find that the simulated estimates are close to the true values of the parameters and hence the estimation method is adequate. We have also observed that the estimated standard deviation consistently decrease with increasing sample size as can been seen by plotting the standard deviation against the sample size.

Table 3: Result of Simulation Study for Set III.

	Parameter a	
Sample Size	Average Estimate	Standard Deviation
200	0.6291761	0.5183241
400	0.5504443	0.2885157
500	0.575298	0.2909363
700	0.5474437	0.2274251
	Parameter b	
Sample Size	Average Estimate	Standard Deviation
200	0.5107722	0.07200797
400	0.5043487	0.04647903
500	0.5069798	0.04290096
700	0.5038033	0.03537766
	Parameter c	
Sample Size	Average Estimate	Standard Deviation
200	0.5287993	0.1546726
400	0.5128494	0.09710285
500	0.5029422	0.08321209
700	0.5037321	0.07232243

From the table above, we find that the simulated estimates are close to the true values of the parameters and hence the estimation method is adequate. We have also observed that the estimated standard deviation consistently decrease with increasing sample size as can been seen by plotting the standard deviation against the sample size.

Overall the simulation study conducted, indicated that using the method of maximum likelihood in estimating model parameters is adequate.

8 Applications

We study some new generalizations of the inverse Dagum Distribution.

8.1 Data Set #1

The proposed sub-model is called the Inverse Standard Exponential-Dagum {Standard Log-Logistic} distribution. The first application is a real data set given by [17]. It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul as recorded in [18].

8.1.1 The PDF of the Proposed Sub-Model

$$f(x;a,b,c)=\frac{abe^{\frac{1}{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}}\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)\left(\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}-1\right)^{2}},$$

where $x,a,b,c>0$.

8.1.2 The CDF of the Proposed Sub-Model

$$F(x;a,b,c)=e^{\frac{1}{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}},$$

where $x,a,b,c>0$.

8.1.3 The Competitor

The competing model is called the Inverse Standard Extreme Value-Dagum {Standard Logistic} distribution. The PDF is given by

$$k(x;a,b,c)=\frac{abe^{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)},$$

where $x,a,b,c>0$, and the CDF is given by

$$K(x;a,b,c)=1-e^{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}},$$

where $x,a,b,c>0$.

Using the R software, we report below in Table 4, the estimates for the parameters in each of the two distributions alongside their standard errors.

Table 4: Estimates for the parameter of fitted distribution.

Distribution	**Parameters**	**Estimates**	**Standard error**
ISEDSLL	$\hat{a}$	41.80316368	49.74838160
	$\hat{b}$	1.14765052	0.12907628
	$\hat{c}$	0.02635994	0.02868664
ISEVDSL	$\hat{a}$	0.4560337	0.3066097
	$\hat{b}$	2.4955535	0.8924501
	$\hat{c}$	0.9265444	0.3951511

The fitted CDF and PDF of ISEDSLL to the March precipitation data using the above table is shown below

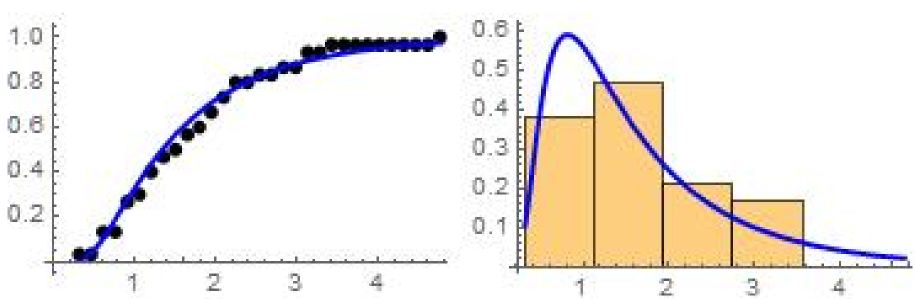

Figure 4: The CDF and PDF of ISEDSLL to the March precipitation data.

and the fitted CDF and PDF of ISEVDSL are shown below

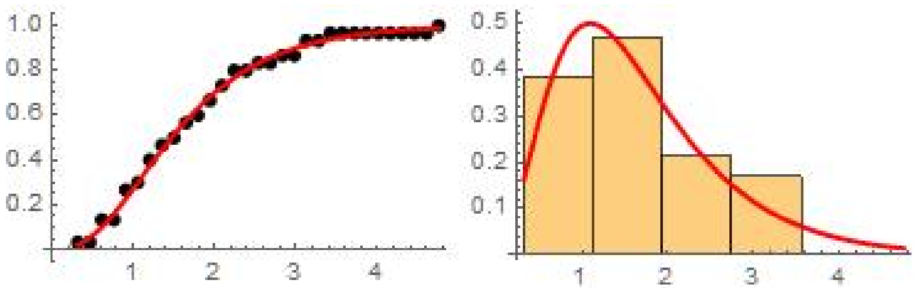

Figure 5: The CDF and PDF of ISEVDSL to the March precipitation data.

The measures of goodness of fit we consider include Bayesian information criterion (BIC), negative Log-Likelihood, Cramer von-Misses (W), Anderson Darling (A), KS (Kolmogorov Smirnov), AIC (Akaikes Information Criterion), CAIC (Consistent Akaikes Information Criterion), and HQIC (Hannan-Quinn information criterion), and they are reported in Table 5. Whilst it appears from the fits above, that all the distributions are competitive in fitting the March precipitation data, Table 5 reveals that the ISEVDSL distribution is most compatible with this data set, and hence can be considered the best in this instance.

Table 5: Goodness of fit measures.

	ISEDSLL	ISEVDSL
W	0.04518177	0.01436276
A	0.2824271	0.1062863
KS statistic	0.12341	0.060637
KS p-value	0.7508	0.9999
AIC	83.50382	82.31917
CAIC	84.4269	83.24224
BIC	87.70742	86.52276
HQIC	84.84859	83.66393
-Log(likelihood)	38.75191	38.15958

8.2 Data Set #2

The proposed sub-model is called the Inverse Standard Extreme Value-Dagum {Standard Logistic} distribution. The second application is given by [19]. The data refers to the time between failures for repairable items as recorded in [18].

8.2.1 The PDF of the Proposed Sub-Model

$$k(x;a,b,c)=\frac{abe^{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)},$$

where $x,a,b,c>0$.

8.2.2 The CDF of the Proposed Sub-Model

$$K(x;a,b,c)=1-e^{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}},$$

where $x,a,b,c>0$.

8.2.3 The Competitor

The competing model is called the Inverse Beta(2,2)-Dagum{Standard Uniform} distribution. The PDF is given by

$$w(x;a,b,c)=\frac{6ab\left(\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}-1\right)\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{-3a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)},$$

where $x,a,b,c>0$, and the CDF is given by

$$W(x;a,b,c)=1-I_{\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{-a}}(2,2),$$

where

$$I_z(a,b)=\frac{\int_0^z t^{a-1}(1-t)^{b-1}dt}{\int_0^1 t^{a-1}(1-t)^{b-1}dt}$$

and $x,a,b,c>0$.

Using the R software, we report below in Table 6, the estimates for the parameters in each of the two distributions alongside their standard errors.

Table 6: Estimates for the parameter of fitted distribution.

Distribution	**Parameters**	**Estimates**	**Standard error**
ISEVDSL	$\hat{a}$	0.526420	0.3552995
	$\hat{b}$	1.880821	0.6403359
	$\hat{c}$	1.046672	0.5831751
IBDSU	$\hat{a}$	14.15461481	29.53868234
	$\hat{b}$	1.03405481	0.15222980
	$\hat{c}$	0.04336123	0.09835706

The fitted CDF and PDF of ISEVDSL to the repairbale items data using the above table are shown below

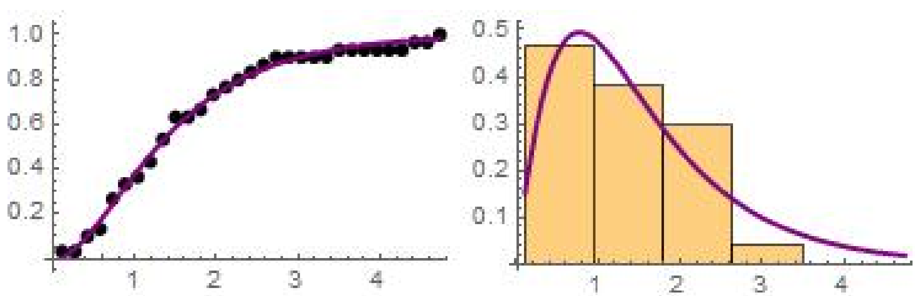

Figure 6: The CDF and PDF of ISEVDSL to the repairable items data.

and the fitted CDF and PDF of IBDSU are shown below

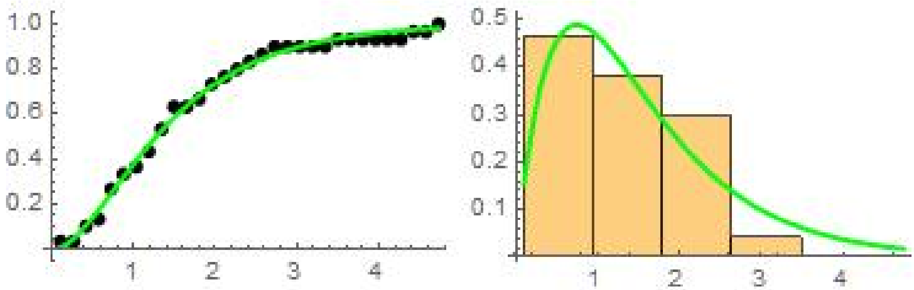

Figure 7: The CDF and PDF of IBDSU to the repairable items data.

The measures of goodness of fit we consider include Bayesian information criterion (BIC), negative Log-Likelihood, Cramer von-Misses (W), Anderson Darling (A), KS (Kolmogorov Smirnov), AIC (Akaikes Information Criterion), CAIC (Consistent Akaikes Information Criterion), and HQIC (Hannan-Quinn information criterion), and they are reported in Table 7. Whilst it appears from the fits above, that all the distributions are competitive in fitting the repairable items data, Table 7 reveals that the ISEVDSL distribution is most compatible with this data set, and hence can be considered the best in this instance.

Table 7: Goodness of fit measures.

	ISEVDSL	IBDSU
W	0.01732601	0.01757459
A	0.1290585	0.1323956
KS statistic	0.06792	0.065782
KS p-value	0.9991	0.9995
AIC	85.19467	85.25216
CAIC	86.11775	86.17524
BIC	89.39826	89.45575
HQIC	86.53943	86.59692
-Log(likelihood)	39.59733	39.62608

8.3 Data Set #3

The proposed sub-model is called the Inverse Beta(2,2)-Dagum{Standard Uniform} distribution. The third application is the vinyl chloride data obtained from clean upgrading, monitoring wells in mg/L; this data set was used by [20] and is recorded in [18].

8.3.1 The PDF of the Proposed Sub-Model

$$w(x;a,b,c) = \frac{6ab\left(\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}-1\right)\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{-3a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)},$$

where $x, a, b, c > 0$.

8.3.2 The CDF of the Proposed Sub-Model

$$W(x;a,b,c) = 1 - I_{\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{-a}}(2,2),$$

where

$$I_z(a,b) = \frac{\int_0^z t^{a-1}(1-t)^{b-1}dt}{\int_0^1 t^{a-1}(1-t)^{b-1}dt},$$

where $x, a, b, c > 0$.

8.3.3 The Competitor

The competing model is called the Inverse Standard Exponential-Dagum {Standard Log-Logistic} distribution. The PDF is given by

$$f(x;a,b,c) = \frac{abe^{\frac{1}{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}}\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}{x\left(\left(\frac{1}{cx}\right)^{b}+1\right)\left(\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}-1\right)^{2}},$$

where $x, a, b, c > 0$, and the CDF is given by

$$F(x;a,b,c) = e^{\frac{1}{1-\left(\left(\frac{1}{cx}\right)^{-b}+1\right)^{a}}},$$

where $x, a, b, c > 0$.

Using the R software, we report below in Table 8, the estimates for the parameters in each of the two distributions alongside their standard errors.

Table 8: Estimates for the parameter of fitted distribution.

Distribution	**Parameters**	**Estimates**	**Standard error**
IBDSU	$\hat{a}$	9.5796402	12.77780197
	$\hat{b}$	0.7239137	0.09656385
	$\hat{c}$	0.0227487	0.04812812
ISEDSLL	$\hat{a}$	22.427212314	13.178166097
	$\hat{b}$	0.662065711	0.067428760
	$\hat{c}$	0.007888539	0.006353141

The fitted CDF and PDF of IBDSU to the vinyl chloride data using the above table are shown below

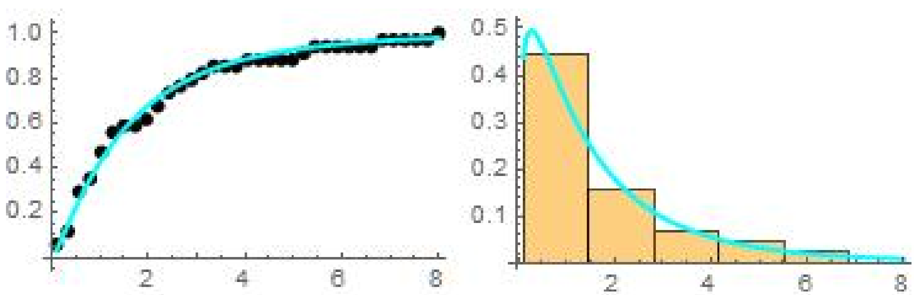

Figure 8: The CDF and PDF of IBDSU to the vinyl chloride data.

and the fitted CDF and PDF of ISEDSLL are shown below

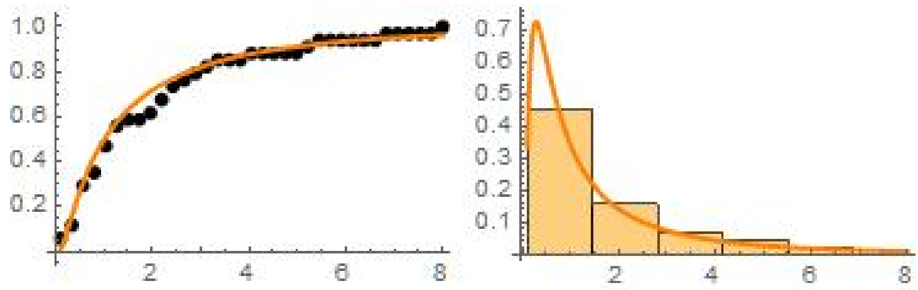

Figure 9: The CDF and PDF of ISEDSLL to the vinyl chloride data.

The measures of goodness of fit we consider include Bayesian information criterion (BIC), negative Log-Likelihood, Cramer von-Misses (W), Anderson Darling (A), KS (Kolmogorov Smirnov), AIC (Akaikes Information Criterion), CAIC (Consistent Akaikes Information Criterion), and HQIC (Hannan-Quinn information criterion), and they are reported in Table 9. Whilst it appears from the fits above, that all the distributions are competitive in fitting the vinyl chloride data, Table 9 reveals that the IBDSU distribution is most compatible with this data set, and hence can be considered the best in this instance.

Table 9: Goodness of fit measures.

	IBDSU	ISEDSLL
W	0.03134333	0.03845091
A	0.2063411	0.3086084
KS statistic	0.082004	0.10125
KS p-value	0.9762	0.8768
AIC	116.0194	116.7265
CAIC	116.8194	117.5265
BIC	120.5985	121.3056
HQIC	117.581	118.2881
-Log(likelihood)	55.00969	55.36324

9 Characterization Theorems

In this section, we present two generalizations of the generalized inverse type distributions using the hazard rate function

9.1 Hazard Function Characterization I

It is well known that the hazard function, h_F, of a twice differentiable function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

In this section we present a Kumaraswamy-generalized inverse type distribution. The result here is inspired by [21]. First let us introduce the following

Definition 9.1. We say a random variable X follows a Kumaraswamy-G type distribution if its CDF is given by

$$F(x;\xi) = 1 - (1 - G(x;\xi))^2,$$

where G is some baseline distribution, $x \in Supp(G)$, and ξ is a vector of parameters in the baseline distribution whose support depends on G.

Remark 9.2. Note that if we take $\lambda = 1$ and $\varphi = 2$ in equation (1) of [22], then we get the CDF in the above definition.

The PDF of the Kumaraswamy-G type distribution is given by

$$f(x;\xi) = 2g(x;\xi)(1 - G(x;\xi)),$$

where g is the PDF of the baseline distribution. Clearly the hazard rate function of the Kumaraswamy-G type distribution is given by

$$h(x;\xi) = \frac{2g(x;\xi)}{(1 - G(x;\xi))}.$$

Theorem 9.3. *Let $X : \Omega \mapsto \mathbb{R}$ be a continuous random variable. The PDF of X is*

$$2g(x;\xi)(1-G(x;\xi))$$

for some baseline distribution with PDF g and CDF G if and only if its hazard rate function $h(x)$ satisfies the following differential equation

$$h'(x) - \frac{g'(x)}{g(x)}h(x) = \frac{2g(x)^2}{(1-G(x))^2}$$

with boundary condition $h(0) = 2g(0)$.

Proof. If X has PDF as stated in the theorem, then the differential equation as stated holds. Now if the stated differential equation holds, then

$$\frac{d}{dx}\left\{g(x)^{-1}h(x)\right\} = 2\frac{d}{dx}\left\{(1-G(x))^{-1}\right\}$$

which implies

$$h(x) = \frac{2g(x)}{1-G(x)}$$

which is the hazard rate function of the Kumaraswamy-G type distribution. □

Clearly, a characterization of the Kumaraswamy-generalized inverse type distribution. is obtained from the above theorem by letting the baseline PDF be given as in Section 3.2, and letting the baseline CDF be given as in Section 3.1.

9.2 Hazard Function Characterization II

It is well known that the hazard function, h_F, of a twice differentiable function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

In this section we present a Weibull-generalized inverse type distribution. The result here is inspired by [23]. First let us introduce the following

Definition 9.4. We say a random variable X follows a Weibull-G distribution if its CDF is given by

$$F(x;\xi) = 1 - e^{-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}},$$

where G is some baseline distribution, $x \in Supp(G)$, and ξ is a vector of parameters in the baseline distribution whose support depends on G, and $\alpha > 0$, and $\overline{G} = 1 - G$.

The PDF of the Weibull-G distribution is given by

$$f(x;\xi) = \alpha g(x;\xi)\frac{G(x;\xi)^{\alpha-1}}{\overline{G(x;\xi)}^{\alpha+1}} e^{-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}},$$

where g is the PDF of the baseline distribution. Clearly the hazard rate function of the Weibull-G distribution is given by

$$h_F(x;\xi) = \alpha g(x;\xi)\frac{G(x;\xi)^{\alpha-1}}{\overline{G(x;\xi)}^{\alpha+1}}.$$

Theorem 9.5. *Let $X : \Omega \mapsto \mathbb{R}$ be a continuous random variable. The PDF of X is*

$$\alpha g(x;\xi)\frac{G(x;\xi)^{\alpha-1}}{\overline{G(x;\xi)}^{\alpha+1}} e^{-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\alpha}}$$

for some baseline distribution with PDF g, CDF G, $\alpha > 0$, and $\overline{G} = 1 - G$, if and only if its hazard rate function $h_F(x)$ satisfies the following differential equation

$$h'_F(x) - g'(x)g(x)^{-1}h_F(x) = \alpha g(x)\frac{d}{dx}\frac{G(x)^{\alpha-1}}{\overline{G(x)}^{\alpha+1}}$$

with $x \in \mathbb{R}$, with initial condition $h_F(0) = 0$ for $\alpha > 1$.

Proof. If X has PDF as stated in the theorem, then the differential equation as stated in the theorem holds. Now if the stated differential equation holds, then

$$\frac{d}{dx}\left\{g(x)^{-1}h_F(x)\right\} = \alpha\frac{d}{dx}\frac{G(x)^{\alpha-1}}{\overline{G(x)}^{\alpha+1}}$$

or

$$h_F(x;\xi) = \alpha g(x;\xi)\frac{G(x;\xi)^{\alpha-1}}{\overline{G(x;\xi)}^{\alpha+1}}$$

which is the hazard function of Weibull-G □

Clearly, a characterization of the Weibull-generalized inverse type distribution. is obtained from the above theorem by letting the baseline PDF and CDF be given as in Section 3.2, and Section 3.1, respectively.

10 Further Recommendation

In the sense of [24] and [25], the "CDF" of the quantile generated family of distributions is given by

$$Q_T[V[F(x)]],$$

where Q_T is a quantile function, V is an appropriate weight depending on the support of T, and $F(x)$ is some baseline distribution. The truncated distribution in the sense of [26] has CDF

$$\frac{Q_T[V[F(x)]] - Q_T[V(0)]}{Q_T[V(1)] - Q_T[V(0)]}.$$

Suppose the random variable X has CDF given as above, and consider the random variable $Z = \frac{1}{X}$, the CDF of Z is given by

$$F_Z(z) = 1 - \frac{Q_T[V[F(1/z)]] - Q_T[V(0)]}{Q_T[V(1)] - Q_T[V(0)]}.$$

We call the random variable Z with the above CDF the inverse truncated quantile generated random variable. Obviously the PDF can be obtained by differentiating the CDF. A future interesting problem is to obtain some properties and applications of the inverse truncated quantile generated family of probability distributions.

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