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Coefficient Estimates of Bi-Starlike and Bi-Convex Functions with Respect to Symmetrical Points Associated with the Second Kind Chebyshev Polynomials

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Abstract

In this paper, by making use the second kind Chebyshev polynomials, we introduce and study a certain class of bi-starlike and bi-convex functions with respect to symmetrical points defined in the open unit disk. We find upper bounds for the second and third coefficients of functions belong to this class.

1. Introduction

The importance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. Several researchers dealing with orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their numerous uses in different applications one can refer [5, 7, 9]. The Chebyshev polynomials of the first and second kinds are well known and they are defined by

$$T_n(t) = \cos n\theta \quad \text{and} \quad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta} \quad (-1 < t < 1),$$

where n indicates the polynomial degree and $t = \cos n\theta$.

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Let $\mathcal{A}$ stand for the family of functions *f* which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Also, let *S* be the subclass of $\mathcal{A}$ consisting of the form (1.1) which are univalent in *U*. It is well known (see [6]) that every function $f \in S$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z, (z \in U)$ and $f(f^{-1}(w)) = w, (|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in *U* if both *f* and f^{-1} are univalent in *U*. Let Σ stand for the class of bi-univalent functions in *U* given by (1.1). For a brief history and interesting examples of functions that are in (or are not in) the class Σ, together with various other properties of the bi-univalent functions class Σ, one can refer the work of Srivastava et al. [13] and the references stated therein. Recently, many authors introduced various subclasses of the bi-univalent functions class Σ and investigated non sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) (see [1, 2, 3, 4, 8, 12]).

Sakaguchi [11] introduced the class S_s^* of functions starlike with respect to symmetric points, which consists of functions $f \in S$ satisfying the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \qquad z \in U.$$

Also, Wang et al. [15] introduced the class K_s of functions convex with respect to symmetric points, which consists of functions $f \in S$ satisfying the condition

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{(f(z) - f(-z))'}\right\} > 0, \qquad z \in U.$$

With a view to recalling the principal of subordination between analytic functions, let the functions *f* and *g* be analytic in *U*. We say that the function *f* is said to be subordinate to g, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. It is well known that (see [10]), if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We consider the function

$$H(z, t) = \frac{1}{1 - 2tz + z^2}, \quad t \in \left(\frac{1}{2}, 1\right], \; z \in U.$$

We note that if $t = \cos\beta$, where $\beta \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, then

$$H(z, t) = \frac{1}{1 - 2\cos\beta z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\beta}{\sin\beta} z^n, \quad z \in U.$$

Therefore

$$H(z, t) = 1 + 2\cos\beta z + (3\cos^2\beta - \sin^2\beta) z^2 + \cdots, \qquad z \in U.$$

In view of [16], we can write

$$H(z, t) = 1 + U_1(t) z + U_2(t) z^2 + \cdots \quad (z \in U, \, t \in (-1, 1)),$$

where

$$U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1 - t^2}} \quad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

are the Chebyshev polynomials of the second kind. Also, it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$$

and

$$U_1(t) = 2t, \;\; U_2(t) = 4t^2 - 1, \;\; U_3(t) = 8t^3 - 4t, \; \ldots . \tag{1.3}$$

The generating function of the first kind of Chebyshev polynomial $T_n(t)$, $t \in [-1, 1]$ is given by

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1 - tz}{1 - 2tz + z^2}, \qquad z \in U.$$

The Chebyshev polynomials of first kind $T_n(t)$ and of the second kind $U_n(t)$ are connected by

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),\ T_n(t) = U_n(t) - tU_{n-1}(t),\ 2T_n(t) = U_n(t) - U_{n-2}(t).$$

2. Main Results

Definition 2.1. For $\gamma \geq 0$ and $t \in \left(\frac{1}{2}, 1\right]$, a function $f \in \Sigma$ is said to be in the class $\mathcal{D}_{\Sigma}^{s}(\gamma, t)$ if it satisfies the subordinations:

$$\left(\frac{2zf'(z)}{f(z) - f(-z)}\right)^{\gamma}\left(\frac{2(zf'(z))'}{(f(z) - f(-z))'}\right)^{1-\gamma} \prec H(z, t) = \frac{1}{1 - 2tz + z^2}$$

and

$$\left(\frac{2wg'(w)}{g(w) - g(-w)}\right)^{\gamma}\left(\frac{2(wg'(w))'}{(g(w) - g(-w))'}\right)^{1-\gamma} \prec H(w, t) = \frac{1}{1 - 2tw + w^2},$$

where the function $g = f^{-1}$ is given by (1.2).

Theorem 2.1. *For* $\gamma \geq 0$ *and* $t \in \left(\frac{1}{2}, 1\right]$, *let f be in the class* $\mathcal{D}_{\Sigma}^{s}(\gamma, t)$. *Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|2(\gamma - 1)t^2 - (\gamma - 2)^2(2t^2 - 1)|}}$$

and

$$|a_3| \leq \frac{t^2}{(\gamma - 2)^2} + \frac{t}{3 - 2\gamma}.$$

Proof. Let $f \in \mathcal{D}_{\Sigma}^{s}(\gamma, t)$. Then there exists two analytic functions $u, v : U \to U$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \quad (z \in U) \tag{2.1}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \ (w \in U), \tag{2.2}$$

with $u(0) = v(0) = 0, |u(z)| < 1, |v(w)| < 1, w \in U$ such that

$$\left(\frac{2zf'(z)}{f(z) - f(-z)}\right)^{\gamma}\left(\frac{2(zf'(z))'}{(f(z) - f(-z))'}\right)^{1-\gamma} = 1 + U_1(t)u(z) + U_2(t)u^2(z) + \cdots \tag{2.3}$$

and

$$\left(\frac{2wg'(w)}{g(w) - g(-w)}\right)^{\gamma}\left(\frac{2(wg'(w))'}{(g(w) - g(-w))'}\right)^{1-\gamma} = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots. \tag{2.4}$$

Combining (2.1), (2.2), (2.3) and (2.4), we obtain

$$\left(\frac{2zf'(z)}{f(z) - f(-z)}\right)^{\gamma}\left(\frac{2(zf'(z))'}{(f(z) - f(-z))'}\right)^{1-\gamma}$$

$$= 1 + U_1(t)u_1 z + [U_1(t)u_2 + U_2(t)u_1^2]z^2 + \cdots \tag{2.5}$$

and

$$\left(\frac{2wg'(w)}{g(w) - g(-w)}\right)^{\gamma}\left(\frac{2(wg'(w))'}{(g(w) - g(-w))'}\right)^{1-\gamma}$$

$$= 1 + U_1(t)v_1 w + [U_1(t)v_2 + U_2(t)v_1^2]w^2 + \cdots \tag{2.6}$$

It is well-known that if $|u(z)| < 1$ and $|v(w)| < 1, z, w \in U$, then

$$|u_i| \le 1 \text{ and } |v_i| \le 1 \text{ for all } i \in \mathbb{N}. \tag{2.7}$$

Comparing the corresponding coefficients in (2.5) and (2.6), after simplifying, we have

$$-2(\gamma - 2)a_2 = U_1(t)u_1, \tag{2.8}$$

$$2[(\gamma - 2)^2 + (3\gamma - 4)]a_2^2 + 2(3 - 2\gamma)a_3 = U_1(t)u_2 + U_2(t)u_1^2, \tag{2.9}$$

$$2(\gamma - 2)a_2 = U_1(t)v_1 \tag{2.10}$$

and

$$2[(\gamma-2)^2+(5-3\gamma)+(2\gamma-3)]a_2^2+2(2\gamma-3)a_3=U_1(t)v_2+U_2(t)v_1^2. \quad (2.11)$$

It follows from (2.8) and (2.10) that

$$u_1=-v_1 \quad (2.12)$$

and

$$8(\gamma-2)^2a_2^2=U_1^2(t)(u_1^2+v_1^2). \quad (2.13)$$

If we add (2.9) to (2.11), we find that

$$2(2(\gamma-2)^2+2(\gamma-1))a_2^2=U_1(t)(u_2+v_2)+U_2(t)(u_1^2+v_1^2). \quad (2.14)$$

Substituting the value of $u_1^2+v_1^2$ from (2.13) in the right hand side of (2.14), we get

$$4\left[(\gamma-2)^2\left(1-\frac{2U_2(t)}{U_1^2(t)}\right)+\gamma-1\right]a_2^2=U_1(t)(u_2+v_2). \quad (2.15)$$

Further computations using (1.3), (2.7) and (2.15), we obtain

$$|a_2|\le\frac{t\sqrt{2t}}{\sqrt{|2(\gamma-1)t^2-(\gamma-2)^2(2t^2-1)|}}.$$

Next, if we subtract (2.11) from (2.9), we deduce that

$$4(3-2\gamma)(a_3-a_2^2)=U_1(t)(u_2-v_2)+U_2(t)(u_1^2-v_1^2). \quad (2.16)$$

In view of (2.12) and (2.13), we get from (2.16)

$$a_3=\frac{U_1^2(t)}{8(\gamma-2)^2}(u_1^2+v_1^2)+\frac{U_1(t)}{4(3-2\gamma)}(u_2-v_2).$$

Thus applying (1.3), we obtain

$$|a_3|\le\frac{t^2}{(\gamma-2)^2}+\frac{t}{3-2\gamma}.$$

For $\gamma=1,$ the class $\mathcal{D}_{\Sigma}^{s}(\gamma, t)$ reduced to the class $\mathcal{D}_{\Sigma}^{s}(1, t)$ of bi-starlike functions with respect to symmetrical points. For functions belongs to this class, we conclude the following result.

Corollary 2.1. *For* $t \in \left(\frac{1}{2}, 1\right]$, *let f be in the class* $\mathcal{D}_{\Sigma}^{s}(1, t)$. *Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|2t^2 - 1|}}$$

and

$$|a_3| \leq t(t+1).$$

For $\gamma = 0$, the class $\mathcal{D}_{\Sigma}^{s}(\gamma, t)$ reduced to the class $\mathcal{F}_{\Sigma}^{sc}(t)$ which was considered recently by Wanas and Majeed [14].

Corollary 2.2 [14]**.** *For* $t \in \left(\frac{1}{2}, 1\right]$, *let f be in the class* $\mathcal{F}_{\Sigma}^{sc}(t)$. *Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|2 - 5t^2|}}$$

and

$$|a_3| \leq \frac{t(3t+4)}{12}.$$

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