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Wardowski Type Characterization of the Interpolative Berinde Weak Fixed Point Theorem

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Abstract

In [1], Wardowski introduced the F-contractions, and used it to prove the Banach contraction principle. In this paper we introduce a concept of F-interpolative Berinde weak contraction, and use it to prove the interpolative Berinde weak mapping theorem of [2].

1 Introduction and Preliminaries

Definition 1.1. Let $F : \mathbb{R}_+ \mapsto \mathbb{R}$ be a mapping satisfying

(a) F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

(b) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ iff $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(c) There exists $\lambda \in (0, 1)$ such that $\lim_{\lambda\to\ 0^+} \alpha^\lambda F(\alpha) = 0$.

A mapping $T : X \mapsto X$, will be called an F-interpolative Berinde weak contraction, if there exists $\tau > 0$ such that for all $x, y, \in X$, $x, y \notin Fix(T)$

$$d(x, y) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}}).$$

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Example 1.2. Let $F : \mathbb{R}_+ \mapsto \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. Observe that F satisfies (a), (b), and (c) of the previous definition for any $\lambda \in (0,1)$. Each mapping $T : X \mapsto X$ satisfying the implication in the previous definition is an F-interpolative Berinde weak contraction such that

$$d(Tx, Ty) \leq e^{-\tau} d(x,y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}}$$

for all $x, y \in X$, $x, y \notin Fix(T)$, $Tx \neq Ty$. Note that for all $x, y \in X$, $x, y \notin Fix(T)$ such that $Tx = Ty$, the inequality

$$d(Tx, Ty) \leq e^{-\tau} d(x,y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}}$$

still holds, that is, T is an interpolative Berinde weak contraction [2].

2 Main Result

Theorem 2.1. *Let (X, d) be a complete metric space, and let $T : X \mapsto X$ be a continuous F-interpolative Berinde weak contraction. Then T has a fixed point $x^* \in X$, and for every $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Proof. Let $x_0 \in X$ be arbitrary and fixed. Define a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$. Denote $\gamma_n = d(x_n, x_{n+1})$, for $n = 0, 1, 2, \cdots$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$, then x_{n_0} is a fixed point of T and the proof is finished. So we assume $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\gamma_n > 0$ for all $n \in \mathbb{N}$. Since T is an F-interpolative Berinde weak contraction, we deduce the following

$$\begin{aligned}
\tau + F(\gamma_n) &= \tau + F(d(Tx_{n-1}, Tx_n)) \\
&\leq F(d(x_{n-1}, x_n)^{\frac{1}{2}} d(x_{n-1}, Tx_{n-1})^{\frac{1}{2}}) \\
&= F(d(x_{n-1}, x_n)^{\frac{1}{2}} d(x_{n-1}, x_n)^{\frac{1}{2}}) \\
&= F(d(x_{n-1}, x_n)) \\
&= F(\gamma_{n-1}).
\end{aligned}$$

which implies

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq \cdots \leq F(\gamma_0) - n\tau.$$

The above implies $\lim_{n\to\infty} F(\gamma_n) = -\infty$. It now follows from (b) of Definition 1.1, that $\lim_{n\to\infty} \gamma_n = 0$. From (c) of Definition 1.1, there exists $\lambda \in (0,1)$ such that $\lim_{n\to\infty} \gamma_n^\lambda F(\gamma_n) = 0$. Since

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq \cdots \leq F(\gamma_0) - n\tau$$

we deduce the following

$$\begin{aligned}\gamma_n^\lambda F(\gamma_n) - \gamma_n^\lambda F(\gamma_0) &\leq \gamma_n^\lambda (F(\gamma_0) - n\tau) - \gamma_n^\lambda F(\gamma_0) \\ &= -\gamma_n^\lambda n\tau \\ &\leq 0.\end{aligned}$$

Since $\lim_{n\to\infty} \gamma_n = 0$ and $\lim_{n\to\infty} \gamma_n^\lambda F(\gamma_n) = 0$. If we take limits in the above inequality we deduce $\lim_{n\to\infty} n\gamma_n^\lambda = 0$. This suggests there exists $n_1 \in \mathbb{N}$ such that $n\gamma_n^\lambda \leq 1$ for all $n \geq n_1$. Consequently, we have

$$\gamma_n \leq \frac{1}{n^{\frac{1}{\lambda}}}$$

for all $n \geq 1$. Now we show that $\{x_n\}$ is Cauchy. Consider, $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. From the definition of the metric and the above inequality, we get

$$\begin{aligned}d(x_m, x_n) &\leq \gamma_{m-1} + \gamma_{m-2} + \cdots + \gamma_m \\ &\leq \sum_{i=n}^{\infty} \gamma_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}.\end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}$ is convergent, it follows from the above inequality that $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Finally, since T is continuous we deduce the following

$$d(Tx^*, x^*) = \lim_{n\to\infty} d(Tx_n, x_n) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0$$

which implies x^* is a fixed point of T, and the proof is finished. $\square$

References

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