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T-Fuzzy Ideals in Coupled Ordered Γ-Semirings : Some Properties

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Abstract

In this paper, we introduce the notions of T-fuzzy ideal, T-fuzzy quasi ideal, T-fuzzy bi-ideal, and T-fuzzy interior ideal. Some related properties are obtained. in coupled Γ semirings. Our work is inspired by [1].

1 Some New Notions and Notations

Remark 1.1. Throughout we make the following assumptions:

(a) M is an ordered Γ-semiring as defined in [1].

(b) T is a triangular norm as defined in [1].

(c) The triangular norm T is a combined translation as defined in [1].

(d) The imaginable property is as defined in [1], except μ is a fuzzy subset of $M \times M$.

Definition 1.2. Let M be an ordered Γ-semiring. A fuzzy subset μ of $M \times M$ will be called a T-fuzzy left (right) ideal, if it satisfies the following conditions for all $(x, m), (y, v) \in M \times M$, and $\alpha \in \Gamma$

(a) $\mu(x + y, m + v) \geq T(\mu(x, m), \mu(y, v))$;

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(b) $\mu(x\alpha y, m\alpha v) \geq \mu(y, v)$ $[\mu(x, m)]$;

(c) If $x \leq y$ and $m \leq v$, then $\mu(x, m) \geq \mu(y, v)$.

Definition 1.3. Let M be an ordered Γ-semiring. If μ is both a T-fuzzy right and left ideal of $M \times M$, then we say μ is a T-fuzzy ideal of $M \times M$.

Definition 1.4. Let M be an ordered Γ-semiring, and μ be a nonempty fuzzy subset of $M \times M$. We say μ is a fuzzy left (right) ideal of $M \times M$, if it satisfies the following conditions for all $(x, m), (y, v) \in M \times M$ and $\alpha \in \Gamma$

(a) $\mu(x + y, m + v) \geq \min\{\mu(x, m), \mu(y, v)\}$;

(b) $\mu(x\alpha y, m\alpha v) \geq \mu(y, v)$ $[\mu(x, m)]$;

(c) If $x \leq y$ and $m \leq v$, then $\mu(x, m) \geq \mu(y, v)$.

Definition 1.5. Let M be a nonempty set, and A be a nonempty subset of $M \times M$, we define the characteristic function of A by

$$\mathbb{1}_A(x, m) = \begin{cases} 1 & (x, m) \in A \\ 0 & (x, m) \notin A. \end{cases}$$

Definition 1.6. Let M be an ordered Γ-semiring, and $(x, m) \in M \times M$. By $M_{(x,m)}$ we mean the set

$$M_{(x,m)} = \{(y, z), (v, z') \in M \times M | x \leq y\alpha z, m \leq v\alpha z', \alpha \in \Gamma\}.$$

In particular, for any fuzzy subsets μ and ω of $M \times M$, we define $\mu \circ \omega : M \times M \mapsto [0, 1]$ by

$$(\mu \circ \omega)(x, m) = \begin{cases} \sup_{(y,z),(v,z') \in M_{(x,m)}} T[\mu(y, v), \omega(z, z')] & \text{if } M_{(x,m)} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.7. Let M be an ordered Γ-semiring, and A be a nonempty subset of $M \times M$.

(a) We say A is a left(right) ideal of $M \times M$, if A is closed under addition, and

$$(M \times M)\Gamma A \subseteq A(A\Gamma(M \times M) \subseteq A)$$

and for any $(x, m) \in M \times M$, $(y, v) \in A$, $x \leq y$ and $m \leq v$, implies $(x, m) \in A$.

(b) We say A is an ideal of $M \times M$, if it is both a left and right ideal of $M \times M$.

Definition 1.8. Let M be an ordered Γ-semiring, and μ, ω be two fuzzy subsets of $M \times M$. We define

$$(\mu \wedge \omega)(x, m) = T[\mu(x, m), \omega(x, m)]$$

and

$$(\mu \cap \omega)(x, m) = \min\{\mu(x, m), \omega(x, m)\}$$

for all $(x, m) \in M \times M$.

Definition 1.9. Let M be an ordered Γ-semiring. A fuzzy subset μ of $M \times M$ will be called a T-fuzzy left (right) k-ideal of $M \times M$ if it satisfies the following conditions for all $(x, m), (y, v) \in M \times M$, and $\alpha \in \Gamma$

(a) $\mu(x + y, m + v) \geq T[\mu(x, m), \mu(y, v)]$;

(b) $\mu(x\alpha y, m\alpha v) \geq \mu(y, v)$ $[\mu(x, m)]$;

(c) If $x \leq y$ and $m \leq v$, then $\mu(x, m) \geq \mu(y, v)$.

Definition 1.10. Let M be an ordered Γ-semiring. If μ is a T-fuzzy left k-ideal and a T-fuzzy right k-ideal of $M \times M$, then we say μ is a T-fuzzy k-ideal of $M \times M$.

Definition 1.11. Let μ be a fuzzy subset of $X \times X$, $a \in [0, 1 - \sup\{\mu(x, m) | (x, m) \in X \times X\}$, and $b \in [0, 1]$. The mapping

$$\mu_a^T : X \times X \mapsto [0, 1]$$

will be called a fuzzy translation of μ, if

$$\mu_a^T(x, m) = \mu(x, m) + a.$$

Definition 1.12. Let μ be a fuzzy subset of $X \times X$, $a \in [0, 1 - \sup\{\mu(x,m)|(x,m) \in X \times X\}$, and $b \in [0,1]$. The mapping

$$\mu_b^T : X \times X \mapsto [0,1]$$

will be called a fuzzy multiplication of μ, if

$$\mu_b^T(x,m) = b\mu(x,m).$$

Definition 1.13. Let μ and β be fuzzy subsets of $X \times X$. The cartesian product of μ and β will be defined as

$$(\mu \times \beta)((x,m),(y,v)) = T[\mu(x,m), \beta(y,v)]$$

for all $((x,m),(y,v)) \in X^2 \times X^2$.

Definition 1.14. Let M be an ordered Γ-semiring, and μ and β be T-fuzzy k-ideals of $M \times M$. We say $\mu \times \beta$ is a T fuzzy left (right) k-ideal of $M^2 \times M^2$ if the following holds for all $((x,m),(z,z')),((y,v),(w,w')) \in M^2 \times M^2$

(a) $(\mu \times \beta)[((x,m),(z,z'))+((y,v),(w,w'))] \geq T[\mu(x+y,m+v), \beta(z+w,z'+w')]$

(b) $(\mu \times \beta)[((x,m),(z,z'))\alpha((y,v),(w,w'))] \geq T[(\mu \times \beta)((y,v),(w,w')),(\mu \times \beta)((x,m),(z,z'))]$

(c) If $((x,m),(z,z')) \leq ((y,v),(w,w'))$, then

$$(\mu \times \beta)((x,m),(z,z')) \geq (\mu \times \beta)((y,v),(w,w'))$$

(d) $(\mu \times \beta)((x,m),(z,z')) \geq T[(\mu \times \beta)((x+y,m+v),(z+y,z'+v)),(\mu \times \beta)((y,v),(y,v))]$.

Definition 1.15. Let M be an ordered Γ semiring. A fuzzy subset μ of $M \times M$ will be called a T fuzzy quasi ideal if it satisfies the following conditions

(a) $\mu(x+y,m+v) \geq T[\mu(x,m), \mu(y,v)))$;

(b) $\mu \circ \mathbb{1}_M \wedge \mathbb{1}_M \circ \mu \subseteq \mu$;

(*c*) If $x \leq y$ and $m \leq v$, then $\mu(x, m) \geq \mu(y, v)$ for all $(x, m), (y, v) \in M \times M$.

Definition 1.16. Let M be an ordered Γ-semiring. A fuzzy subset μ of $M \times M$ will be called a fuzzy bi-ideal if the following holds

(*a*) $\mu(x + y, m + v) \geq \min\{\mu(x, m), \mu(z, z')\}$

(*b*) $\mu(x\alpha y\beta z, m\alpha v\beta z') \geq \min\{\mu(x, m), \mu(z, z')\}$

(*c*) If $x \leq y$ and $m \leq v$, then $\mu(x, m) \geq \mu(y, v)$

for all $(x, m), (y, v), (z, z') \in M \times M$, $\alpha, \beta \in \Gamma$.

Definition 1.17. Let M be an ordered Γ-semiring. A fuzzy subset μ of $M \times M$ will be called a T-fuzzy bi-ideal if the following holds

(*a*) $\mu(x + y, m + v) \geq T[\mu(x, m), \mu(z, z')]$

(*b*) $\mu(x\alpha y\beta z, m\alpha v\beta z') \geq T[\mu(x, m), \mu(z, z')]$

(*c*) If $x \leq y$ and $m \leq v$, then $\mu(x, m) \geq \mu(y, v)$

for all $(x, m), (y, v), (z, z') \in M \times M$, $\alpha, \beta \in \Gamma$.

Definition 1.18. Let M be an ordered Γ semiring. A fuzzy subset μ of $M \times M$ will be called a fuzzy interior ideal if the following holds for all $(x, m), (y, v), (z, z') \in M \times M$, and $\alpha, \beta \in \Gamma$.

(*a*) $\mu(x + y, m + v) \geq \min\{\mu(x, m), \mu(y, v)\}$;

(*b*) $\mu(x\alpha y\beta z, m\alpha v\beta z') \geq \mu(y, v)$;

(*c*) If $x \leq y$ and $m \leq v$, then $\mu(x, m) \geq \mu(y, v)$.

Definition 1.19. Let M be an ordered Γ semiring. A fuzzy subset μ of $M \times M$ will be called a T-fuzzy interior ideal if the following holds for all $(x, m), (y, v), (z, z') \in M \times M$, and $\alpha, \beta \in \Gamma$.

(*a*) $\mu(x + y, m + v) \geq T[\mu(x, m), \mu(y, v)]$.

(*b*) $\mu(x\alpha y\beta z, m\alpha v\beta z') \geq \mu(y, v)$.

(*c*) If $x \leq y$ and $m \leq v$, then $\mu(x, m) \geq \mu(y, v)$.

2 Some Properties

Proposition 2.1. *Let M be a Γ-semiring. If μ is a T-fuzzy right ideal of $M \times M$, $T(\mu(x,m),\mu(x,m)) \geq \mu(x,m)$ for all $(x,m) \in M \times M$, and $(a.b) \in M \times M$, then $I_{(a,b)}$ is a right ideal of $M \times M$ where*

$$I_{(a,b)} = \{(c,d) \in M \times M | \mu(c,d) \geq \mu(a,b)\}.$$

Proof. Let M be an ordered Γ-semiring, and μ be a T-fuzzy right ideal of $M \times M$, and $T(\mu(x,m),\mu(x,m)) \geq \mu(x,m)$ for all $(x,m) \in M \times M$. Then $I_{(a,b)} \neq \emptyset$, since $(a,b) \in I_{(a,b)}$. Let $(x_1,x_2),(y_1,y_2) \in I_{(a,b)}$, and $\alpha \in \Gamma$. Then $\mu(x_1,x_2) \geq \mu(a,b)$, and $\mu(y_1,y_2) \geq \mu(a,b)$. Thus,

$$\begin{aligned}\mu(x_1+y_1,x_2+y_2) &\geq T(\mu(x_1,x_2),\mu(y_1,y_2))\\ &\geq T[\mu(a,b),\mu(a,b)]\\ &\geq \mu(a,b).\end{aligned}$$

So $(x_1+y_1,x_2+y_2) \in I_{(a,b)}$. On the other hand

$$\mu(x_1\alpha x, x_2\alpha m) \geq \mu(x_1,x_2) \geq \mu(a,b)$$

for all $(x,m) \in M \times M$. Hence $(x_1\alpha x, x_2\alpha m) \in I_{(a,b)}$. Now let $(x_1,x_2) \in I_{(a,b)}$, $y_1 \leq x_1$, and $y_2 \leq x_2$. Then. $\mu(x_1,x_2) \geq \mu(a,b)$, and $\mu(y_1,y_2) \leq \mu(x_1,x_2)$. Thus, we have

$$\mu(y_1,y_2) \geq \mu(x_1,x_2) \geq \mu(a,b).$$

So $(y_1,y_2) \in I_{(a,b)}$. It now follows that $I_{(a,b)}$ is a right ideal of $M \times M$. □

Theorem 2.2. *Let M be an ordered Γ-semiring. Every fuzzy ideal of $M \times M$ is a T-fuzzy ideal of $M \times M$.*

Proof. Let μ be a fuzzy ideal of $M \times M$, $(x,m),(y,v) \in M \times M$, and $\alpha \in \Gamma$. Observe

$$\begin{aligned}\mu(x+y,m+v) &\geq \min\{\mu(x,m),\mu(y,v)\}\\ &\geq T(\mu(x,m),\mu(y,v)).\end{aligned}$$

Also

$$\begin{aligned}\mu(x\alpha y, m\alpha v) &\geq \max\{\mu(x,m), \mu(y,v)\}\\ &\geq T(\mu(x,m), \mu(y,v)).\end{aligned}$$

It follows that μ is a T-fuzzy ideal of $M \times M$. □

Theorem 2.3. *Let M be an ordered Γ-semiring, and μ be a fuzzy subset of $M \times M$. Then μ is a T-fuzzy left ideal of $M \times M$ iff μ satisfies the following conditions*

(*a*) $\mathbb{1}_{M\times M} \circ \mu \subseteq \mu$

(*b*) $\mu(x+y, m+v) \geq T[\mu(x,m), \mu(y,v)]$

for all $(x,m), (y,v) \in M \times M$.

Proof. Let M be an ordered Γ-semiring, $(x,m) \in M \times M$, and μ be a T-fuzzy ideal of $M \times M$.

Case 1: If $(\mathbb{1}_{M\times M} \circ \mu)(x,m) = 0$, then

$$(\mathbb{1}_{M\times M} \circ \mu)(x,m) = 0 \leq \mu(x,m).$$

So $\mathbb{1}_{M\times M} \circ \mu \subseteq \mu$.

Case 2: If $(\mathbb{1}_{M\times M} \circ \mu)(x,m) \neq 0$, then there exists $(y,v),(z,z') \in M \times M$ such that $x \leq y\alpha z$, $m \leq v\alpha z'$, and $\alpha \in \Gamma$. Thus

$$\begin{aligned}(\mathbb{1}_{M\times M} \circ \mu)(x,m) &= \sup_{(y,v),(z,z')\in M_{(x,m)}} T[\mathbb{1}_{M\times M}(y,v), \mu(z,z')]\\ &= \sup_{(y,v),(z,z')\in M_{(x,m)}} \mu(z,z')\\ &\leq \mu(y\alpha z, v\alpha z')\\ &\leq \mu(x,m)\end{aligned}$$

which implies $(\mathbb{1}_{M\times M}\circ\mu)\subseteq\mu$. For the converse, assume the given conditions hold, and let $(x,m),(y,v)\in M\times M$, $\alpha\in\Gamma$. Since $(\mathbb{1}_{M\times M}\circ\mu)\subseteq\mu$,

$$\begin{aligned}\mu(x\alpha y, m\alpha v) &\geq (\mathbb{1}_{M\times M}\circ\mu)(x\alpha y, m\alpha v)\\ &= \sup_{(a,b),(c,d)\in M_{(x\alpha y, m\alpha v)}} T[\mathbb{1}_{M\times M}(a,b),\mu(c,d)]\\ &\geq T[\mathbb{1}_{M\times M}(x,m),\mu(y,v)]\\ &= \mu(y,v).\end{aligned}$$

Now let $(x,m),(y,v)\in M\times M$ and $x\leq y$, $m\leq v$, and $\alpha\in\Gamma$. Then

$$\begin{aligned}\mu(x,m) &\geq \mu(x\alpha y, m\alpha v)\\ &= \sup_{(y,v),(y,v)\in M_{(x\alpha y, m\alpha v)}} T[\mathbb{1}_{M\times M}(y,v),\mu(y,v)]\\ &= \mu(y,v)\end{aligned}$$

which implies $\mu(x,m)\geq\mu(y,v)$. It now follows that μ is a T-fuzzy left ideal of $M\times M$. □

Theorem 2.4. *Let M be an ordered Γ-semiring. $M\times M$ is regular iff $A\Gamma B = A\cap B$ for any right ideal A and left ideal B of $M\times M$.*

Proof. Let A and B be right and left ideals, respectively of $M\times M$, where M is a regular ordered Γ-semiring. Obviously,

$$A\Gamma B\subseteq A\cap B.$$

Let $(x,m)\in A\cap B$. Since $M\times M$ is regular, there exists $\alpha,\beta\in\Gamma$, and $(y,v)\in M\times M$ such that $x\leq x\alpha y\beta x$ and $m\leq m\alpha v\beta m$. Since $(x\alpha y, m\alpha v)\in A$ and $(x,m)\in B$, then

$$(x\alpha y\beta x, m\alpha v\beta m)\in A\Gamma B.$$

Thus, $(x,m)\in A\Gamma B$, which implies $A\Gamma B = A\cap B$.

Conversely, suppose that $A\Gamma B = A\cap B$ for every right ideal A and left ideal

B of $M \times M$. Let $(x, m) \in M \times M$, and I be the right ideal generated by (x, m), and J be the left ideal generated by (x, m). Then $(x, m) \in I \cap J = I\Gamma J$. Thus,

$$x = x\alpha y = z\beta z$$

and

$$m = m\alpha v = z'\beta m$$

for $\alpha, \beta \in \Gamma$, $(y, v), (z, z') \in M \times M$. So

$$x = x\alpha y\Omega z\beta z$$

and

$$m = m\alpha v\Omega z'\beta m$$

for some $\Omega \in \Gamma$. Hence, $M \times M$ is regular. □

Theorem 2.5. *Let M be an ordered Γ-semiring. Then $M \times M$ is regular iff $\lambda \circ \mu = \lambda \wedge \mu$ for any T-fuzzy right ideal λ and T-fuzzy left ideal μ of $M \times M$.*

Proof. Let M be an ordered Γ-semiring, λ and μ be T-fuzzy right and T-fuzzy left ideals of $M \times M$, and $(x, m) \in M \times M$. Then

$$(\lambda \circ \mu)(x, m) = \sup_{(y,v),(z,z') \in M_{(x,m)}} T[\lambda(y, v), \mu(z, z')].$$

Since $x \leq y\alpha z$ and $m \leq v\alpha z'$,

$$\mu(x, m) \geq \mu(y\alpha z, v\alpha z') \geq \mu(z, z').$$

So

$$\lambda(x, m) \geq \lambda(y\alpha z, v\alpha z') \geq \lambda(y, v).$$

Thus

$$T[\lambda(y, v), \mu(z, z')] \leq T[\lambda(x, m), \mu(x, m)].$$

This implies

$$\begin{aligned}(\lambda \circ \mu)(x, m) &= \sup_{(y,v),(z,z') \in M_{(x,m)}} T[\lambda(y, v), \mu(z, z')] \\ &\leq T[\lambda(x, m), \mu(x, m)] \\ &= (\lambda \wedge \mu)(x, m).\end{aligned}$$

Thus $\lambda \circ \mu \subseteq \lambda \wedge \mu$. Now let $(x, m) \in M \times M$. Since $M \times M$ is regular, there exists $\alpha, \beta \in \Gamma$, $(a, b) \in M \times M$ such that $x \leq x\alpha a\beta x$ and $m \leq m\alpha b\beta m$. Suppose $(y, v), (z, z') \in M \times M$, $\Omega \in \Gamma$, $x \leq y\Omega z$ and $m \leq v\Omega z'$, then

$$\lambda(x, m) \geq \lambda(y\Omega z, v\Omega z') \geq \lambda(y, v)$$

which implies

$$\mu(x, m) \geq \mu(y\Omega z, v\Omega z') \geq \mu(z, z').$$

On the other hand

$$T[\lambda(y, v), \mu(z, z')] \leq T[\lambda(x\alpha a, m\alpha b), \mu(x, m)]$$

which implies

$$\begin{aligned}(\lambda \circ \mu)(x, m) &= \sup_{(y,v),(z,z') \in M_{(x,m)}} T[\lambda(y, v), \mu(z, z')] \\ &\geq T[\lambda(x\alpha a, m\alpha b), \mu(x, m)].\end{aligned}$$

Thus

$$\begin{aligned}(\lambda \circ \mu)(x, m) &\geq T[\lambda(x\alpha a, m\alpha b), \mu(x, m)] \\ &\geq T[\lambda(x, m), \mu(x, m)] \\ &= (\lambda \wedge \mu)(x, m).\end{aligned}$$

So $\lambda \wedge \mu \subseteq \lambda \circ \mu$, which implies $\lambda \circ \mu = \lambda \wedge \mu$. Conversely, suppose $\lambda \circ \mu = \lambda \wedge \mu$ for T-fuzzy right ideal λ and T-fuzzy left ideal μ of $M \times M$. Let A and B be right and left ideals, respectively, of $M \times M$. Then $\mathbb{1}_A$ and $\mathbb{1}_B$ are T-fuzzy right and left ideals of $M \times M$, respectively. Thus,

$$\mathbb{1}_A \circ \mathbb{1}_B = \mathbb{1}_A \wedge \mathbb{1}_B.$$

Obviously, $A\Gamma B \subseteq A \cap B$. Suppose $(x, m) \in A \cap B$, then, $\mathbb{1}_A(x, m) = \mathbb{1}_B(x, m) = 1$ implies

$$(\mathbb{1}_A \wedge \mathbb{1}_B)(x, m) = T[\mathbb{1}_A, \mathbb{1}_B] = T[1, 1] = 1$$

which implies $(\mathbb{1}_A \circ \mathbb{1}_B)(x, m) = 1$. Consequently, there exists $(a, a') \in A$ and $(b, b') \in B$ such that $x \leq a\alpha b$, $m \leq a'\alpha b'$, and $(a\alpha b, a'\alpha b') \in A\Gamma B$, which implies $(x, m) \in A\Gamma B$. Thus, $A\Gamma B = A \cap B$. It now follows that $M \times M$ is regular by the previous theorem. □

Theorem 2.6. *Let M be an ordered Γ-semiring. If μ and λ are T-fuzzy left k-ideals of $M \times M$, then $\mu \wedge \lambda$ is a T-fuzzyy left k-deal of $M \times M$.*

Proof. Let M be an ordered Γ-semiring, μ and λ be T-fuzzy left k-ideals of $M \times M$. Further, let $(x, m), (y, v) \in M \times M$, and $\alpha \in \Gamma$. Observe we have the following

$$\begin{aligned}(\mu \wedge \lambda)(x + y, m + v) &= T[\mu(x + y, m + v), \lambda(x + y, m + v)] \\ &\geq T[T[\mu(x, m), \mu(y, v)], T[\lambda(x, m), \lambda(y, v)]] \\ &= T[T[\mu(x, m), \lambda(x, m)], T[\mu(y, v), \lambda(y, v)]] \\ &= T[(\mu \wedge \lambda)(x, m), (\mu \wedge \lambda)(y, v)].\end{aligned}$$

Since μ and λ are T-fuzzy left k-deals we have $\mu(x\alpha y, m\alpha v) \geq \mu(y, v)$ and $\lambda(x\alpha y, m\alpha v) \geq \lambda(y, v)$. Thus,

$$\begin{aligned}(\mu \wedge \lambda)(x\alpha y, m\alpha v) &= T[\mu(x\alpha y, m\alpha v), \lambda(x\alpha y, m\alpha v)] \\ &\geq T[\mu(y, v), \lambda(y, v)] \\ &= (\mu \wedge \lambda)(y, v).\end{aligned}$$

Suppose $(x, m), (y, v) \in M \times M$ with $x \leq y$ and $m \leq v$. Then $\mu(x, m) \geq \mu(y, v)$ and $\lambda(x, m) \geq \lambda(y, v)$. So,

$$\begin{aligned}(\mu \wedge \lambda)(x, m) &= T[\mu(x, m), \lambda(x, m)] \\ &\geq T[\mu(y, v), \lambda(y, v)] \\ &= (\mu \wedge \lambda)(y, v).\end{aligned}$$

Thus, $\mu \wedge \lambda$ is a T-fuzzy left ideal of $M \times M$. Since μ and λ are T-fuzzy left k-ideals,

$$\mu(x, m) \geq T[\mu(x + y, m + v), \mu(y, v)]$$

and

$$\lambda(x,m) \geq T[\lambda(x+y,m+v),\lambda(y,v)]$$

for all $(x,m),(y,v) \in M \times M$. So

$$\begin{aligned}(\mu\wedge\lambda)(x,m) &= T[\mu(x,m),\lambda(x,m)]\\ &\geq T[T[\mu(x+y,m+v),\mu(y,v)],T[\lambda(x+y,m+v),\lambda(y,v)]]\\ &= T[T[\mu(x+y,m+v),\lambda(x+y,m+vv)],T[\mu(y,v),\lambda(y,v)]]\\ &= T[(\mu\wedge\lambda)(x+y,m+v),(\mu\wedge\lambda)(y,v)].\end{aligned}$$

So $\mu \wedge \lambda$ is a T-fuzzy left k-ideal of $M \times M$. □

Theorem 2.7. *Let M be an ordered Γ-semiring. A fuzzy subset μ is a T-fuzzy left k-ideal of $M \times M$ iff μ_a^T is a T-fuzzy left k-ideal of $M \times M$ provided t-norm T is a combined translation.*

Proof. Let t-norm T be a combined translation, μ be a T-fuzzy left k-ideal of $M \times M$, $(x,m),(y,v) \in M \times M$, and $\alpha \in \Gamma$. Observe we have the following

$$\begin{aligned}\mu_a^T(x+y,m+v) &= \mu(x+y,m+v)+a\\ &\geq T[\mu(x,m),\mu(y,v)]+a\\ &= T[\mu(x,m)+a,\mu(y,v)+a]\\ &= T[\mu_a^T(x,m),\mu_a^T(y,v)]\end{aligned}$$

and

$$\begin{aligned}\mu_a^T(x\alpha y,m\alpha v) &= \mu(x\alpha y,m\alpha v)+a\\ &\geq \mu(y,v)+a\\ &= \mu_a^T(y,v).\end{aligned}$$

Also

$$\begin{aligned}\mu_a^T(x,m) &= \mu(x,m)+a\\ &\geq T[\mu(x+y,m+v),\mu(y,v)]+a\\ &= T[\mu(x+y,m+v)+a,\mu(y,v)+a]\\ &= T[\mu_a^T(x+y,m+v),\mu_a^T(y,v)].\end{aligned}$$

Now suppose $(x, m), (y, v) \in M \times M$, $x \leq y$ and $m \leq v$. Then $\mu(x, m) \geq \mu(y, v)$, and thus $\mu(x, m) + a \geq \mu(y, v) + a$. So, $\mu_a^T(x, m) \geq \mu_a^T(y, v)$. Hence μ_a^T is a T-fuzzy left ideal.

Conversely, suppose that μ_a^T is a T-fuzzy left k-ideal. Then obviously μ is a T-fuzzy left ideal. Let $\mu(y, v) = t_1$ and $\mu(x + y, m + v) = t_2$, and

$$t = \min\{t_1, t_2\} \geq T(t_1, t_2).$$

Then $(y, v) \in \mu_t$, $(x + y, m + v) \in \mu_t$. Since μ_t is a k-ideal $(x, m) \in \mu_t$ which implies

$$\begin{aligned} \mu(x, m) &\geq t \\ &= \min\{t_1, t_2\} \\ &\geq T(t_1, t_2) \\ &= T[\mu(y, v), \mu(x + y, m + v)]. \end{aligned}$$

It follows that μ is a T-fuzzy left k-ideal of $M \times M$. □

Theorem 2.8. *Let M be an ordered Γ-semiring. A fuzzy subset μ is a T-fuzzy left k-ideal of $M \times M$ iff μ_b^T is a T-fuzzy left k-ideal of $M \times M$ provided t-norm T is a combined translation and $b \in [0, 1]$.*

Proof. Let M be an ordered Γ-semiring, μ be a T-fuzzy k-ideal of $M \times M$, t-norm T be a combined translation, $(x, m), (y, v) \in M \times M$, and $\alpha \in \Gamma$. Observe we have the following

$$\begin{aligned} \mu_b^M(x + y, m + v) &= b\mu(x + y, m + v) \\ &\geq bT[\mu(x, m), \mu(y, v)] \\ &= T[b\mu(x, m), b\mu(y, v)] \\ &= T[\mu_b^M(x, m), \mu_b^M(y, v)] \end{aligned}$$

and

$$\begin{aligned} \mu_b^M(x\alpha y, m\alpha v) &= b\mu(x\alpha y, m\alpha v) \\ &\geq b\mu(y, v) \\ &= \mu_b^M(y, v). \end{aligned}$$

Also

$$\begin{aligned}\mu_b^M(x,m) &= b\mu(x,m)\\ &\geq bT[\mu(x+y,m+v),\mu(y,v)]\\ &= T[b\mu(x+y,m+v),b\mu(y,v)]\\ &= T[\mu_b^M(x+y,m+v),\mu_b^M(y,v)].\end{aligned}$$

Now if $x \leq y$, $m \leq v$, then $\mu(x,m) \geq \mu(y,v)$, and thus $b\mu(x,m) \geq b\mu(y,v)$. So $\mu_b^M(x,y) \geq \mu_b^M(y,v)$. Hence μ_b^M is a T-fuzzy left k-ideal of $M \times M$. □

Theorem 2.9. *Let M be an ordered Γ-semiring. If μ is an imaginable T fuzzy left k-ideal of $M \times M$, then μ is a fuzzy left k-ideal of $M \times M$.*

Proof. Let M be an ordered Γ-semiring, μ be an imaginable T-fuzzy left k-ideal of $M \times M$, $(x,m),(y,v) \in M \times M$, and $\alpha \in \Gamma$. Since μ is imaginable,

$$\begin{aligned}\min\{\mu(x,m),\mu(y,v)\} &= T[\min\{\mu(x,m),\mu(y,v)\},\min\{\mu(x,m),\mu(y,v)\}]\\ &\leq T[\mu(x,m),\mu(y,v)]\\ &\leq \min\{\mu(x,m),\mu(y,v)\}.\end{aligned}$$

Thus

$$T[\mu(x,m),\mu(y,v)] = \min\{\mu(x,m),\mu(y,v)\}.$$

Now

$$\mu(x+y,m+v) \geq T[\mu(x,m),\mu(y,v)] = \min\{\mu(x,m),\mu(y,v)\}$$

and

$$\mu(x\alpha y, m\alpha v) \geq \mu(y,v).$$

If $x \leq y$ and $m \leq v$, then $\mu(x,m) \geq \mu(y,v)$. So

$$\mu(x,m) \geq T[\mu(x+y,m+v),\mu(y,v)] = \min\{\mu(x+y,m+v),\mu(y,v)\}.$$

Hence the result. □

Theorem 2.10. *Let M be an ordered Γ-semiring, and μ be an imaginable T-fuzzy left k-ideal of $M \times M$. If μ is a T-fuzzy left k-ideal, then $\mu \times \mu$ is an imaginable T-fuzyy left k-ideal of $M^2 \times M^2$.*

Proof. Let M be an ordered Γ-semiring, and μ be an imaginable T-fuzzy left k-ideal of $M \times M$, and $((x,m),(x,m)) \in M^2 \times M^2$. Obviously, $\mu \times \mu$ is a T fuzzy left k-ideal of $M^2 \times M^2$. Now

$$\begin{aligned} & T[(\mu\times\mu)((x,m),(x,m)),(\mu\times\mu)((x,m),(x,m))] \\ &= T[T[\mu(x,m),\mu(x,m)],T[\mu(x,m),\mu(x,m)]] \\ &= T[\mu(x,m),\mu(x,m)] \\ &= (\mu\times\mu)((x,m),(x,m)) \end{aligned}$$

which implies $\mu \times \mu$ is imaginable. On the other hand

$$\begin{aligned} & (\mu\times\mu)((x,m),(z,z')) \\ &= T[\mu(x,m),\mu(z,z')] \\ &\geq T[T(\mu(x+y,m+v),\mu(y,v)),T[\mu(z+y,z'+v),\mu(y,v)]] \\ &= T[T[\mu(x+y,m+v),\mu(z+y,z'+v)],T[\mu(y,v),\mu(y,v)]] \\ &= T[(\mu\times\mu)((x+y,m+v),(z+y,z'+v)),(\mu\times\mu)((y,v),(y,v))] \end{aligned}$$

for all $((x,m),(z,z')) \in M^2 \times M^2$ and $(y,v) \in M \times M$. Thus, $\mu \times \mu$ is an imaginable T-fuzzy left k-ideal of $M^2 \times M^2$. □

Theorem 2.11. *Let M be an ordered Γ semiring. A fuzzy subset μ is a T-fuzzy quasi ideal of $M \times M$ iff*

$$\mu(x,m) \geq T[\sup_{(y,z),(v,z')\in M_{(x,m)}} \mu(y,v), \sup_{(y,z),(v,z')\in M_{(x,m)}} \mu(z,z')]$$

for all $(x,m) \in M \times M$.

Proof. Let M be an ordered Γ-semiring, and μ be a T-fuzzy quasi ideal of $M \times M$. By definition

$$\mu(x,m) \geq \mu \circ \mathbb{1}_M \wedge \mathbb{1}_M \circ \mu(x,m)$$

iff

$$\mu(x,m) \geq T[\mu \circ \mathbb{1}_M(x,m), \mathbb{1}_M \circ \mu(x,m)]$$

iff

$$\mu(x,m) \geq T\left[\sup_{(y,z),(v,z')\in M_{(x,m)}} \left(T[\mu(y,v),\mathbb{1}_M(z,z')],T[\mathbb{1}_M(y,v),\mu(z,z')]\right)\right]$$

iff

$$\mu(x,m) \geq T\left[\sup_{(y,z),(v,z')\in M_{(x,m)}} \mu(y,v), sup_{(y,z),(v,z')\in M_{(x,m)}}\mu(z,z')\right]$$

for all $(x,m) \in M \times M$.

□

Theorem 2.12. *Let M be an ordered Γ-semiring. Every fuzzy bi-ideal of $M \times M$ is a T fuzzy bi-ideal of $M \times M$.*

Proof. Let M be an ordered Γ-semiring, μ be a fuzzy bi-ideal of $M \times M$, $(x,m),(y,v),(z,z') \in M \times M$, and $\alpha,\beta \in \Gamma$. Observe, we have the following

$$\begin{aligned}\mu(x+y,m+v) &\geq \min\{\mu(x,m),\mu(y,v)\} \\ &\geq T[\mu(x,m),\mu(y,v)].\end{aligned}$$

Also

$$\mu(x\alpha y\beta z, m\alpha v\beta z') \geq \min\{\mu(x,m),\mu(z,z') \geq T[\mu(x,m),\mu(z,z')]$$

which implies μ is a T-fuzzy bi-ideal of $M \times M$. □

Theorem 2.13. *Let M be an ordered Γ-semi ring. Every fuzzy interior ideal of $M \times M$ is a T-fuzzy interior ideal of $M \times M$.*

Proof. Let M be an ordered Γ-semiring, and μ be a fuzzy interior ideal of $M \times M$. Since

$$\min\{\mu(x,m),\mu(y,v)\} \geq T[\mu(x,m),\mu(y,v)],$$

μ is a T-fuzzy interior ideal of $M \times M$. □

3 Concluding Remarks

In the present paper we introduced notions of T-fuzzy ideal, T-fuzzy quasi ideal, T-fuzzy bi-ideal, and T-fuzzy interior ideal and obtained some related properties in coupled Γ semirings.

References

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