



# Coefficient Estimates of Certain Subclasses of Bi-Bazilevic Functions Associated with Chebyshev Polynomials and Mittag-Leffler Function

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## Abstract

In this present investigation, the authors introduced certain subclasses of the function class  $T_{\theta}^{\alpha}(\lambda, \beta, t)$  of bi-Bazilevic univalent functions defined in the open unit disk  $U$ , which are associated with Chebyshev polynomials and Mittag-Leffler function. We establish the Taylor Maclaurin coefficients  $|a_2|$ ,  $|a_3|$  and  $|a_4|$  for functions in the new subclass introduced and the Fekete-Szego problem is solved.

## 1 Introduction

Let  $A$  denote the class of analytic function  $f(z)$  in the open unit disk  $U = \{z : |z| < 1\}$  and  $H$  denote the subclass of  $A$  which are normalized by the condition  $f(0) = 0$  and  $f'(0) - 1 = 0$  with the representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1)$$

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Familiar subclasses of starlike and convex functions of order  $\alpha$  ( $0 \leq \alpha < 1$ ) for which the quantity written respectively as  $\operatorname{Re}\left(zf'(z)/f(z)\right) > \alpha$  and  $\operatorname{Re}\left(1 + zf''(z)/f'(z)\right) > \alpha$  and the two functions are denoted as  $S^*(\alpha)$  and  $C(\alpha)$  respectively. For a constant  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ , a function  $f$  is univalent in  $U$  and satisfies the function  $\operatorname{Re}\left(e^{i\theta} \frac{zf'(z)}{f(z)}\right) \geq \alpha$ , denote the class by  $TS^*$ .

A function  $f$  in  $H$  has an inverse  $f^{-1}$  defined as

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

$$f(f^{-1}(\omega)) = \omega, \quad (|\omega| < r_o(f), \quad r_o(f) \geq \frac{1}{4})$$

hence, the inverse function can be written as

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots \quad (2)$$

A function  $f \in S$  which is starlike is bi-univalent in  $U$  if both  $f$  and  $f^{-1}$  are both univalent in  $U$ . Let the class of all bi-univalent functions  $f$  in  $U$  be given by (1) denoted by  $\Sigma$ .

Lewin [12] introduced the concept of bi-univalent analytic functions in 1967 and showed that second coefficient  $|a_2| < 1.51$ . Many authors like Netanyahu [15], Brannan and Clunie [8], Murugurusundaramoorthy [14], Awolere and Ibrahim-Tiamiyu [6], Amol and Uday [5] have worked on bi-univalent analytic functions with interesting results. In recent time, several researchers have studied families of analytic functions involving special functions such as Sigmoid function, error function, Bessel function defined in  $A$  to find different conditions such that members of such functions have certain geometric properties like univalence, starlikeness and convexity, see [5, 6, 9, 10, 11, 16]. It was observed in [3, 4] that Mittag-Leffler function denoted by  $E_{\lambda, \beta}$  does not belong to the family of  $A$ . Originally the Mittag-Leffler function is given as

$$E_{\lambda}(z) = \sum_{k=2}^{\infty} \frac{z^k}{\Gamma(\lambda n + 1)}, \quad (z \in C, R(\lambda) > 0) \quad (3)$$

which was introduced by Mittag-Leffler [13].

A more general form of (3) denoted by  $E_{\lambda,\beta}(z)$  is defined as

$$E_{\lambda,\beta}(z) = \sum_{k=2}^{\infty} \frac{z^k}{\Gamma(\lambda k + \beta)}, \quad (z, \lambda, \beta \in C, R(\lambda) > 0). \tag{4}$$

The function in (3) arises naturally in the solution of fractional order differential and integral equations, most especially in the study of fractional generalization of kinetic equation, random walks levy flights, super diffusive transport and also in the study of complex systems. These polynomials interpolate between a purely exponential law and power law like behaviour of phenomena governed by ordinary kinetic equations and their fractional counterparts [3, 4].

By convolution and normalization, we obtain from (1) and (4)

$$E_{\lambda,\beta}f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\lambda(k-1) + \beta)} a_k z^k. \tag{5}$$

The Chebyshev polynomials are a sequence of orthogonal polynomials which are practically related to De Moivre’s formula which play an important role in Numerical Analysis. Recently, the classical Chebyshev polynomials of degree  $n$  of the first and second kinds have generated a great deal of interest because of the major roles played in Applied Mathematics and Approximation Theory. Chebyshev polynomials have been studied by several authors as in [1, 2, 7, 10, 11].

**Lemma 1** (See 9). *If  $\omega(z) = b_1z + b_2z^2 + b_3z^3 + \dots, b_1 \neq 0$  is analytic and satisfies  $|\omega(z)| < 1$  on the unit disk  $U$ , then for each  $0 < r < 1$ ,  $|\omega'(0)| < 1$  and  $|\omega(re^{i\theta})| < 1$  unless  $\omega(z) = e^{i\theta}z$  for some real number  $\theta$ .*

**Definition 1** (See 16). The Chebyshev polynomials of the first kind are defined by the following three terms recurrence relation

$$\begin{aligned} T_0(t) &= 1 \\ T_1(t) &= t \\ T_{n+1}(t) &= 2tT_n(t) - T_{n-1}(t). \end{aligned}$$

The first few terms of the Chebyshev polynomials of the first kind are

$$T_2(t) = 2t^2 - 1, \quad T_3(t) = 4t^3 - 3t, \quad T_4(t) = 8t^4 - 8t^2 + 1, \dots$$

The generating function for Chebyshev polynomials of the first kind,  $T_n(t)$  is given by

$$F(z, t) = \frac{1 - zt}{1 - 2tz + z^2} = \sum_{n=0}^{\infty} T_n(t)z^n, \quad (z \in U)$$

**Definition 2** (See 16). The Chebyshev polynomials of the second kind are defined by the following three terms recurrence relation

$$\begin{aligned} U_0(t) &= 1 \\ U_1(t) &= 2t \\ U_{n+1}(t) &= 2tU_n(t) - U_{n-1}(t). \end{aligned}$$

The first few terms of the Chebyshev polynomials of the second kind,  $U_n(t)$  is given by

$$U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, \dots$$

The generating function for Chebyshev polynomials of the second kind,  $U_n(t)$  is given by

$$H(z, t) = \frac{1}{1 - 2tz + z^2} = \sum_{n=0}^{\infty} U_n(t)z^n, \quad (z \in U).$$

The Chebyshev polynomial of first and second kinds are connected by the following relations

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tT_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

Given two functions  $f, g \in A$ . The function  $f(z)$  is said to be subordinate to  $g(z)$  in  $U$ , written  $f(z) \prec g(z)$ , if there exists a Schwarz function  $\omega(z)$ , analytic in  $U$ , with  $\omega(0) = 0$  and  $|\omega(z)| < 1$  for all  $z \in U$  such that  $f(z) = g(\omega(z))$ .

**Definition 3.** A function  $f(z) \in \Sigma$  given by (1) is said to be in the class  $T_{\theta}^{\alpha}(\lambda, \beta, t)$  if the following conditions are satisfied

$$e^{i\theta} \left( \frac{z^{1-\alpha}(E_{\lambda, \beta} f(z))'}{(E_{\lambda, \beta} f(z))^{1-\alpha}} \right) \prec H(z, t) \cos \theta + i \sin \theta \quad (6)$$

and

$$e^{i\theta} \left( \frac{\omega^{1-\alpha} (E_{\lambda,\beta} g(\omega))'}{(E_{\lambda,\beta} g(\omega))^{1-\alpha}} \right) \prec H(\omega, t) \cos\theta + i \sin\theta \tag{7}$$

where  $z, \omega \in U, \alpha \geq 0$  and the function  $g \equiv f^{-1}$ .

## 2 Main Result

**Theorem 1.** Let  $f \in T_{\theta}^{\alpha}(\lambda, \beta, t)$ , since  $\beta, \lambda \in C, A_k$  is complex number. Then

$$|a_2| \leq \frac{2t\sqrt{2t}\cos\theta}{\sqrt{|2t^2 [(\alpha^2 + \alpha - 2)A_2^2 + 2(\alpha + 2)A_3] \cos\theta - 2(4t^2 - 1)e^{i\theta}(\alpha^2 + 2\alpha + 1)A_2^2|}} \tag{8}$$

$$|a_3| \leq \left( \frac{2t\cos\theta}{(\alpha + 1)^2 A_2} \right)^2 + \frac{2t\cos\theta}{(\alpha + 2)A_3} \tag{9}$$

$$|a_4| \leq \frac{10t^2\cos^2\theta}{(\alpha^2 + 3\alpha + 2)A_2A_3} + \frac{20t^3\cos^3\theta}{(\alpha^3 + 3\alpha^2 + 3\alpha + 1)A_2^3} - \frac{4Mt^3\cos^3\theta}{(\alpha^3 + 3\alpha^2 + 3\alpha + 1)(\alpha + 3)A_2^3A_4} + \frac{2(4t^3 + 4t^2 - t - 1)\cos\theta}{(\alpha + 3)A_4} \tag{10}$$

where  $M = 5(\alpha + 3)A_4 + 2(\alpha - 1)(\alpha + 3)A_2A_3 + \frac{(\alpha-1)(\alpha-2)(\alpha+3)A_2^3}{3}$  and  $A_k = \frac{\Gamma(\beta)}{\Gamma(\lambda(k-1)+\beta)}, k \geq 2$

*Proof.* From Definition 3

$$e^{i\theta} \left( \frac{z^{1-\alpha} (E_{\lambda,\beta} f(z))'}{(E_{\lambda,\beta} f(z))^{1-\alpha}} \right) \prec H(z, t) \cos\theta + i \sin\theta \tag{11}$$

and

$$e^{i\theta} \left( \frac{\omega^{1-\alpha} (E_{\lambda,\beta} g(\omega))'}{(E_{\lambda,\beta} g(\omega))^{1-\alpha}} \right) \prec H(\omega, t) \cos\theta + i \sin\theta \tag{12}$$

where  $H(z, t)$  and  $H(\omega, t)$  are Chebyshev polynomials with the following power series

$$H(z, t) = 1 + c_1u_1(t)z + (c_2u_1(t) + c_1^2u_2(t))z^2 + \dots \quad (z \in U)$$

$$H(\omega, t) = 1 + d_1 u_1(t)\omega + (d_2 u_1(t) + d_1^2 u_2(t))\omega^2 + \dots \quad (\omega \in U).$$

For some analytic  $\varphi, \psi$  such that  $\varphi(0) = \psi(0) = 0$  and  $|\varphi(z)| = |c_1 z + c_2 z^2 + \dots| < 1$ ,  $|\psi(\omega)| = |d_1 \omega + d_2 \omega^2 + \dots| < 1$ , for all  $z, \omega \in U$ . It is well known that  $|c_k| \leq 1, |d_k| \leq 1$ ; for every  $k \in N$ .

Now, by (11) and (12), we have

$$(\alpha + 1)A_2 a_2 = c_1 u_1(t) e^{-i\theta} \cos\theta \quad (13)$$

$$\frac{(\alpha - 1)(\alpha + 2)}{2} A_2^2 a_2^2 + (\alpha + 2)A_3 a_3 = c_2 u_1(t) e^{-i\theta} \cos\theta + c_1^2 u_2(t) e^{-i\theta} \cos\theta \quad (14)$$

$$\begin{aligned} & (\alpha + 3)A_4 a_4 + (\alpha - 1)(\alpha + 3)A_2 A_3 a_2 a_3 + \frac{(\alpha - 1)(\alpha - 2)(\alpha + 3)}{6} A_2^3 a_2^3 \\ &= c_3 u_1(t) e^{-i\theta} \cos\theta + 2c_1 c_2 u_2(t) e^{-i\theta} \cos\theta + c_1^3 u_3(t) e^{-i\theta} \cos\theta - (\alpha + 1)A_2 a_2 \\ &= d_1 u_1(t) e^{-i\theta} \cos\theta \end{aligned} \quad (15)$$

$$\begin{aligned} & \left[ 2(\alpha + 2)A_3 + \frac{(\alpha - 1)(\alpha + 2)}{2} A_2^2 \right] a_2^2 - (\alpha + 2)A_3 a_3 \\ &= d_2 u_1(t) e^{-i\theta} \cos\theta + d_1^2 u_2(t) e^{-i\theta} \cos\theta \end{aligned} \quad (16)$$

and

$$\begin{aligned} & [5(\alpha + 3)A_4 + (\alpha - 1)(\alpha + 3)A_2 A_3] a_2 a_3 \\ & - \left[ 5(\alpha + 3)A_4 + 2(\alpha - 1)(\alpha + 3)A_2 A_3 + \frac{(\alpha - 1)(\alpha - 2)(\alpha + 3)}{6} A_2^3 \right] a_2^3 \\ & - (\alpha + 3)A_4 a_4 \\ &= d_3 u_1(t) e^{-i\theta} \cos\theta + 2d_1 d_2 u_2(t) e^{-i\theta} \cos\theta + d_1^3 u_3(t) e^{-i\theta} \cos\theta. \end{aligned} \quad (17)$$

From (13) and (15), we have

$$a_2 = \frac{c_1 u_1(t) e^{-i\theta} \cos\theta}{(\alpha + 1)A_2} = -\frac{d_1 u_1(t) e^{-i\theta} \cos\theta}{(\alpha + 1)A_2} \quad (18)$$

which implies that

$$c_1 = -d_1 \quad (19)$$

and

$$2e^{2i\theta}(\alpha + 1)^2 A_2^2 a_2^2 = (c_1^2 + d_1^2)\cos^2\theta + u_1^2(t). \tag{20}$$

By equations (14) and (16), we have

$$\frac{e^{i\theta}}{\cos\theta} [(\alpha - 1)(\alpha + 2)A_2^2 + 2(\alpha + 2)A_3] a_2^2 = (c_2 + d_2)u_1(t) + (c_1^2 + d_1^2)u_2(t). \tag{21}$$

Now, by (20) and (21)

$$a_2^2 = \frac{u_1^3(t)(c_2 + d_2)e^{-i\theta}\cos^2\theta}{u_1^2(t)[(\alpha^2 + \alpha - 2)A_2^2 + 2(\alpha + 2)A_3]\cos\theta - 2e^{i\theta}u_2(t)(\alpha^2 + 2\alpha + 1)A_2^2}. \tag{22}$$

Since by definition,  $u_1(t) = 2t$ ,  $u_2(t) = 4t^2 - 1$  and  $u_3(t) = 8t^3 - 4t$ .

Applying Lemma 1 for the coefficients  $c_2$  and  $d_2$ , we immediately have

$$|a_2| \leq \frac{2t\sqrt{2t\cos\theta}}{\sqrt{|2t^2[(\alpha^2 + \alpha - 2)A_2^2 + 2(\alpha + 2)A_3]\cos\theta - 2(4t^2 - 1)e^{i\theta}(\alpha^2 + 2\alpha + 1)A_2^2|}}. \tag{23}$$

Next, to find the bound on  $|a_3|$ , we subtract (16) from (14) to get

$$\frac{e^{i\theta}}{\cos\theta} [2(\alpha + 2)A_3a_3 - 2(\alpha + 2)A_3a_2^2] = u_1(t)(c_2 - d_2) + u_2(t)(c_1^2 - d_1^2). \tag{24}$$

Thus, it follows from (19), (20) and (24) that

$$a_3 = \frac{u_1^2(t)(c_1^2 + d_1^2)e^{-2i\theta}\cos^2\theta}{2(\alpha + 1)^2 A_2^2} + \frac{u_1(t)(c_2 - d_2)e^{-i\theta}\cos\theta}{2(\alpha + 2)A_3}. \tag{25}$$

By Lemma 1, the coefficients  $c_1, d_1, c_2$  and  $d_2$  yield

$$|a_3| \leq \left( \frac{2t\cos\theta}{(\alpha + 1)^2 A_2} \right)^2 + \frac{2t\cos\theta}{(\alpha + 2)A_3}. \tag{26}$$

Furthermore, by (13), (15), (17) and (25), we have

$$\begin{aligned} a_4 = & \frac{5u_1^2(t)c_1(c_2 - d_2)e^{-2i\theta}\cos^2\theta}{4(\alpha + 1)(\alpha + 2)A_2A_3} + \frac{5u_1^3(t)c_1(c_1^2 + d_1^2)e^{-3i\theta}\cos^3\theta}{4(\alpha + 1)^3 A_2^3} \\ & - \frac{Mu_1^3(t)c_1^3e^{-3i\theta}\cos^3\theta}{2(\alpha + 1)^3(\alpha + 3)A_2^3A_4} + \frac{2u_2(t)(c_1c_2 - d_1d_2)e^{-i\theta}\cos\theta}{2(\alpha + 3)A_4} \\ & + \frac{u_1(t)(c_3 - d_3)e^{-i\theta}\cos\theta}{2(\alpha + 3)A_4} + \frac{u_3(t)(c_1^3 - d_1^3)e^{-i\theta}\cos\theta}{2(\alpha + 3)A_4}. \end{aligned} \tag{27}$$

On application of Lemma 1 for coefficients  $c_1, d_1, c_2, d_2, c_3$  and  $d_3$ , then

$$|a_4| \leq \frac{10t^2 \cos^2 \theta}{(\alpha^2 + 3\alpha + 2)A_2 A_3} + \frac{20t^3 \cos^3 \theta}{(\alpha^3 + 3\alpha^2 + 3\alpha + 1)A_2^3} - \frac{4Mt^3 \cos^3 \theta}{(\alpha^3 + 3\alpha^2 + 3\alpha + 1)(\alpha + 3)A_2^3 A_4} + \frac{2(4t^3 + 4t^2 - t - 1)\cos \theta}{(\alpha + 3)A_4}, \quad (28)$$

where

$$M = 5(\alpha + 3)A_4 + 2(\alpha - 1)(\alpha + 3)A_2 A_3 + \frac{(\alpha - 1)(\alpha - 2)(\alpha + 3)A_2^3}{3}$$

which yield the required result, hence the proof completes.  $\square$

**Corollary 1.** Let  $f \in T_0^\alpha(\lambda, \beta, t)$ , since  $\beta, \lambda \in C, A_k$  is complex number. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|2t^2 [(\alpha^2 + \alpha - 2)A_2^2 + 2(\alpha + 2)A_3] - 2(4t^2 - 1)(\alpha^2 + 2\alpha + 1)A_2^2|}} \quad (29)$$

$$|a_3| \leq \frac{2t}{(\alpha + 1)^2 A_2} + \frac{2t}{(\alpha + 2)A_3} \quad (30)$$

$$|a_4| \leq \frac{10t^2}{(\alpha^2 + 3\alpha + 2)A_2 A_3} + \frac{20t^3}{(\alpha^3 + 3\alpha^2 + 3\alpha + 1)A_2^3} - \frac{4Mt^3}{(\alpha^3 + 3\alpha^2 + 3\alpha + 1)(\alpha + 3)A_2^3 A_4} + \frac{2(4t^3 + 4t^2 - t - 1)}{(\alpha + 3)A_4}. \quad (31)$$

**Corollary 2.** Let  $f \in T_0^0(\lambda, \beta, t)$ , since  $\beta, \lambda \in C, A_k$  is complex number. Then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|2A_3 + (1 - 6t^2)A_2^2|}} \quad (32)$$

$$|a_3| \leq \frac{2t}{A_2} + \frac{t}{A_3} \quad (33)$$

$$|a_4| \leq \frac{5t^2}{A_2 A_3} + \frac{20t^3}{A_2^3} - \frac{4Mt^3}{3A_2^3 A_4} + \frac{2(4t^3 + 4t^2 - t - 1)}{3A_4}. \quad (34)$$



**Corollary 3.** *Let  $f \in T_\theta^\alpha(\lambda, \beta, 1)$ , since  $\beta, \lambda \in C, A_k$  is complex number. Then*

$$|a_2| \leq \frac{\sqrt{2}\cos\theta}{\sqrt{|\cos\theta [(\alpha^2 + \alpha - 2)A_2^2 + 2(\alpha + 2)A_3] - 3e^{i\theta}(\alpha^2 + 2\alpha + 1)A_2^2|}} \quad (35)$$

$$|a_3| \leq \left(\frac{2\cos\theta}{(\alpha + 1)^2A_2}\right)^2 + \frac{2\cos\theta}{(\alpha + 2)A_3} \quad (36)$$

$$|a_4| \leq \frac{10\cos^2\theta}{(\alpha^2 + 3\alpha + 2)A_2A_3} + \frac{20\cos^3\theta}{(\alpha^3 + 3\alpha^2 + 3\alpha + 1)A_2^3} - \frac{4M\cos^3\theta}{(\alpha^3 + 3\alpha^2 + 3\alpha + 1)(\alpha + 3)A_2^3A_4} + \frac{12\cos\theta}{(\alpha + 3)A_4}. \quad (37)$$

**Corollary 4.** *Let  $f \in T_0^\alpha(\lambda, \beta, 1)$ , since  $\beta, \lambda \in C, A_k$  is complex number. Then*

$$|a_2| \leq \frac{\sqrt{2}}{\sqrt{|\cos\theta [(\alpha^2 + \alpha - 2)A_2^2 + 2(\alpha + 2)A_3] - 3(\alpha^2 + 2\alpha + 1)A_2^2|}} \quad (38)$$

$$|a_3| \leq \left(\frac{2}{(\alpha + 1)^2A_2}\right)^2 + \frac{2}{(\alpha + 2)A_3} \quad (39)$$

$$|a_4| \leq \frac{10}{(\alpha^2 + 3\alpha + 2)A_2A_3} + \frac{20}{(\alpha^3 + 3\alpha^2 + 3\alpha + 1)A_2^3} - \frac{4M}{(\alpha^3 + 3\alpha^2 + 3\alpha + 1)(\alpha + 3)A_2^3A_4} + \frac{12}{(\alpha + 3)A_4}. \quad (40)$$

**Corollary 5.** *Let  $f \in T_0^0(\lambda, \beta, 1)$ , since  $\beta, \lambda \in C, A_k$  is complex number. Then*

$$|a_2| \leq \frac{\sqrt{2}}{\sqrt{|4A_3 - 5A_2^2|}} \quad (41)$$

$$|a_3| \leq \frac{4}{A_2^2} + \frac{1}{A_3} \quad (42)$$

$$|a_4| \leq \frac{5}{A_2A_3} + \frac{20}{A_2^3} - \frac{4M}{3A_2^3A_4} + \frac{4}{A_4}. \quad (43)$$

**Theorem 2.** *Let  $f \in T_\theta^\alpha(\lambda, \beta, t)$  and  $\varphi \in R$ , since  $\beta, \lambda \in C, A_k$  is complex*

number. Then

$$\begin{aligned}
 & |a_3 - \varphi a_2^2| \\
 \leq & \begin{cases} \frac{2t \cos \theta}{(\alpha+2)A_3}, & |\varphi - 1| \leq M \\ \frac{8|1-\varphi|t^3 \cos^2 \theta}{|2t^2[(\alpha^2+\alpha-2)A_2^2+2(\alpha+2)A_3] \cos \theta - (4t^2-1)e^{i\theta}(\alpha^2+2\alpha+1)A_2^2|}, & |\varphi - 1| \geq M \end{cases} \quad (44)
 \end{aligned}$$

where

$$M = \frac{|2t^2 [(\alpha^2 + \alpha - 2)A_2^2 + 2(\alpha + 2)A_3] \cos \theta - (4t^2 - 1)e^{i\theta}(\alpha^2 + 2\alpha + 1)A_2^2|}{4t^2 \cos \theta (\alpha + 2)A_3},$$

Proof.

$$\begin{aligned}
 a_3 - \varphi a_2^2 &= \frac{(1 - \varphi)u_1^3(t)(c_2 + d_2)e^{-i\theta} \cos^2 \theta}{u_1^2(t) [(\alpha^2 + \alpha - 2)A_2^2 + 2(\alpha + 2)A_3] \cos \theta - 2e^{i\theta}u_2(t)(\alpha^2 + 2\alpha + 1)A_2^2} \\
 &\quad + \frac{u_1(t)(c_2 - d_2)e^{-i\theta} \cos \theta}{2(\alpha + 2)A_3} \\
 &= u_1(t)e^{-i\theta} \cos \theta \left[ \left( h(\varphi) + \frac{1}{2(\alpha + 2)A_3} \right) c_2 + \left( h(\varphi) - \frac{1}{2(\alpha + 2)A_3} \right) d_2 \right] \quad (45)
 \end{aligned}$$

where

$$h(\varphi) = \frac{(1 - \varphi)u_1^2(t) \cos \theta}{u_1^2(t) [(\alpha^2 + \alpha - 2)A_2^2 + 2(\alpha + 2)A_3] \cos \theta - 2e^{i\theta}u_2(t)(\alpha^2 + 2\alpha + 1)A_2^2}$$

since  $u_1(t) = 2t$ ,  $u_2(t) = 4t^2 - 1$  and  $u_3(t) = 8t^3 - 4t$ , then we have

$$|a_3 - \varphi a_2^2| \leq \begin{cases} \frac{2t \cos \theta}{(\alpha+2)A_3}, & 0 \leq |h(\varphi)| \leq \frac{1}{2(\alpha+2)A_3} \\ 4t |h(\varphi)| \cos \theta, & |h(\varphi)| \geq \frac{1}{2(\alpha+2)A_3} \end{cases} \quad (46)$$

□

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