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Properties of Generalized (r, s, t, u)-Numbers

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Abstract

In this paper, we investigate the generalized (r, s, t, u) sequence and we deal with, in detail, three special cases which we call them (r, s, t, u), Lucas (r, s, t, u) and modified (r, s, t, u) sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

1 Introduction

The sequence of Fibonacci numbers $\{F_n\}$

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \; F_1 = 1.$$

The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences. See [1, 2, 16] for some work on second-order generalization of Fibonacci numbers.

The generalized (r, s, t, u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \quad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \tag{1.1}$$

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where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r, s, t, u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [3, 6, 7, 9, 17, 18]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integers n.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t, u and initial values.

Table 1. A few special case of generalized Tetranacci sequences.

No	Sequences (Numbers)	Notation	OEIS [10]	Ref.
1	Tetranacci	$\{M_n\} = \{W_n(0,1,1,2;1,1,1,1)\}$	A000078	[12]
2	Tetranacci-Lucas	$\{R_n\} = \{W_n(4,1,3,7;1,1,1,1)\}$	A073817	[12]
3	fourth order Pell	$\{P_n^{(4)}\} = \{W_n(0,1,2,5;2,1,1,1)\}$	A103142	[13]
4	fourth order Pell-Lucas	$\{Q_n^{(4)}\} = \{W_n(4,2,6,17;2,1,1,1)\}$	A331413	[[13]
5	modified fourth order Pell	$\{E_n^{(4)}\} = \{W_n(0,1,1,3;2,1,1,1)\}$	A190139	[[13]
6	fourth order Jacobsthal	$\{J_n^{(4)}\} = \{W_n(0,1,1,1;1,1,1,2)\}$	A007909	[8]
7	fourth order Jacobsthal-Lucas	$\{j_n^{(4)}\} = \{W_n(2,1,5,10;1,1,1,2)\}$	A226309	[8]
8	modified fourth order Jacobsthal	$\{K_n^{(4)}\} = \{W_n(3,1,3,10;1,1,1,2)\}$		[8]
9	fourth-order Jacobsthal Perrin	$\{Q_n^{(4)}\} = \{W_n(3,0,2,8;1,1,1,2)\}$		[8]
10	adjusted fourth-order Jacobsthal	$\{S_n^{(4)}\} = \{W_n(0,1,1,2;1,1,1,2)\}$		[8]
11	modified fourth-order Jacobsthal-Lucas	$\{R_n^{(4)}\} = \{W_n(4,1,3,7;1,1,1,2)\}$		[8]
12	4-primes	$\{G_n\} = \{W_n(0,0,1,2;2,3,5,7)\}$		[14]
13	Lucas 4-primes	$\{H_n\} = \{W_n(4,2,10,41;2,3,5,7)\}$		[14]
14	modified 4-primes	$\{E_n\} = \{W_n(0,0,1,1;2,3,5,7)\}$		[14]

As $\{W_n\}$ is a fourth order recurrence sequence (difference equation), its characteristic equation is

$$x^4 - rx^3 - sx^2 - tx - u = 0 \tag{1.2}$$

whose roots are $\alpha, \beta, \gamma, \delta$. Note that we have the following identities

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= r, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -s, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= t, \\
\alpha\beta\gamma\delta &= -u.
\end{aligned}$$

Generalized Tetranacci numbers can be expressed, for all integers n, using Binet's formula

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} \\
&\quad + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \qquad (1.3)
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \\
p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\
p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\
p_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0.
\end{aligned}$$

Usually, it is customary to choose $\alpha, \beta, \gamma, \delta$ so that the Equ. (1.2) has at least one real (say α) solutions. Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n (see [4]).

(1.3) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n,$$

where

$$
\begin{aligned}
A_1 &= \frac{W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)},\\
A_2 &= \frac{W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)},\\
A_3 &= \frac{W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)},\\
A_4 &= \frac{W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.
\end{aligned}
$$

We have the following formula: for $n = 1, 2, 3, ...$ we have

$$W_{-n} = \frac{1}{(-u)^n}\frac{\Delta_1}{\Delta_2}W_n$$

where

$\Delta_1 = \beta^n\gamma^n\delta^n(\gamma-\delta)(\beta-\delta)(\beta-\gamma)p_1 - \alpha^n\gamma^n\delta^n(\gamma-\delta)(\alpha-\delta)(\alpha-\gamma)p_2 + \alpha^n\beta^n\delta^n(\beta-\delta)(\alpha-\delta)(\alpha-\beta)p_3 - \alpha^n\beta^n\gamma^n(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)p_4$,

$\Delta_2 = \alpha^n(\gamma-\delta)(\beta-\delta)(\beta-\gamma)p_1 - \beta^n(\gamma-\delta)(\alpha-\delta)(\alpha-\gamma)p_2 + \gamma^n(\beta-\delta)(\alpha-\delta)(\alpha-\beta)p_3 - \delta^n(\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)p_4$.

We can also give Binet's formula of the generalized (r, s, t, u) numbers (the generalized Tetranacci numbers) for the negative subscripts as follows: for $n = 1, 2, 3, ...$ we have

$$
\begin{aligned}
W_{-n} &= \frac{\alpha^3-r\alpha^2-s\alpha-t}{u(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}p_1\alpha^{1-n} + \frac{\beta^3-r\beta^2-s\beta-t}{u(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}p_2\beta^{1-n}\\
&\quad + \frac{\gamma^3-r\gamma^2-s\gamma-t}{u(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}p_3\gamma^{1-n} + \frac{\delta^3-r\delta^2-s\delta-t}{u(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}p_4\delta^{1-n}.
\end{aligned}
$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence W_n. The following lemma is a special case of a well known formula of generating functions of the generalized m-step Fibonacci numbers which can be found in the literature (see for example [15]). For completeness, we include the proof.

Lemma 1. *Suppose that $f_{W_n}(x)=\sum_{n=0}^{\infty}W_nx^n$ is the ordinary generating function of the generalized (r,s,t,u) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty}W_nx^n$ is given by*

$$\sum_{n=0}^{\infty}W_nx^n=\frac{W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2+(W_3-rW_2-sW_1-tW_0)x^3}{1-rx-sx^2-tx^3-ux^4}. \tag{1.4}$$

Proof. Using the definition of generalized (r,s,t,u) numbers, and substracting $rx\sum_{n=0}^{\infty}W_nx^n$, $sx^2\sum_{n=0}^{\infty}W_nx^n$, $tx^3\sum_{n=0}^{\infty}W_nx^n$ and $ux^4\sum_{n=0}^{\infty}W_nx^n$ from $\sum_{n=0}^{\infty}W_nx^n$, we obtain

$$\begin{aligned}
&(1-rx-sx^2-tx^3-ux^4)\sum_{n=0}^{\infty}W_nx^n\\
=&\sum_{n=0}^{\infty}W_nx^n-rx\sum_{n=0}^{\infty}W_nx^n-sx^2\sum_{n=0}^{\infty}W_nx^n\\
&-tx^3\sum_{n=0}^{\infty}W_nx^n-ux^4\sum_{n=0}^{\infty}W_nx^n\\
=&\sum_{n=0}^{\infty}W_nx^n-r\sum_{n=0}^{\infty}W_nx^{n+1}-s\sum_{n=0}^{\infty}W_nx^{n+2}\\
&-t\sum_{n=0}^{\infty}W_nx^{n+3}-u\sum_{n=0}^{\infty}W_nx^{n+4}\\
=&\sum_{n=0}^{\infty}W_nx^n-r\sum_{n=1}^{\infty}W_{n-1}x^n-s\sum_{n=2}^{\infty}W_{n-2}x^n\\
&-t\sum_{n=3}^{\infty}W_{n-3}x^n-u\sum_{n=4}^{\infty}W_{n-4}x^n\\
=&(W_0+W_1x+W_2x^2+W_3x^3)-r(W_0x+W_1x^2+W_2x^3)\\
&-s(W_0x^2+W_1x^3)-tW_0x^3\\
&+\sum_{n=4}^{\infty}(W_n-rW_{n-1}-sW_{n-2}-tW_{n-3}-uW_{n-4})x^n\\
=&W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2+(W_3-rW_2-sW_1-tW_0)x^3.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 + (W_3 - rW_2 - sW_1 - tW_0)x^3}{1 - rx - sx^2 - tx^3 - ux^4}.$$

□

We next find Binet's formula of generalized (r, s, t, u) numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 2 (Binet's formula of generalized (r, s, t, u) numbers)**.**

$$\begin{aligned} W_n &= \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} \\ &\quad + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} q_1 &= W_0\alpha^3 + (W_1 - rW_0)\alpha^2 + (W_2 - rW_1 - sW_0)\alpha + (W_3 - rW_2 - sW_1 - tW_0), \\ q_2 &= W_0\beta^3 + (W_1 - rW_0)\beta^2 + (W_2 - rW_1 - sW_0)\beta + (W_3 - rW_2 - sW_1 - tW_0), \\ q_3 &= W_0\gamma^3 + (W_1 - rW_0)\gamma^2 + (W_2 - rW_1 - sW_0)\gamma + (W_3 - rW_2 - sW_1 - tW_0), \\ q_4 &= W_0\delta^3 + (W_1 - rW_0)\delta^2 + (W_2 - rW_1 - sW_0)\delta + (W_3 - rW_2 - sW_1 - tW_0). \end{aligned}$$

Proof. Let

$$h(x) = 1 - rx - sx^2 - tx^3 - ux^4.$$

Then for some α, β, γ and δ we write

$$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)$$

i.e.,

$$1 - rx - sx^2 - tx^3 - tx^4 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x). \tag{1.6}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{r}{x} - \frac{s}{x^2} - \frac{t}{x^3} - \frac{u}{x^4} = 0.$$

This implies $x^4 - rx^3 - sx^2 - tx - u = 0$. Now, by (1.4) and (1.6), it follows that

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 + (W_3 - rW_2 - sW_1 - tW_0)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\begin{aligned} & \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 + (W_3 - rW_2 - sW_1 - tW_0)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} \\ = \; & \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)} + \frac{B_4}{(1-\delta x)}. \end{aligned} \tag{1.7}$$

So

$$\begin{aligned} & W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 + (W_3 - rW_2 - sW_1 - tW_0)x^3 \\ = \; & B_1(1-\beta x)(1-\gamma x)(1-\delta x) + B_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\ & + B_3(1-\alpha x)(1-\beta x)(1-\delta x) + B_3(1-\alpha x)(1-\beta x)(1-\gamma x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $W_0 + (W_1 - rW_0)\frac{1}{\alpha} + (W_2 - rW_1 - sW_0)\frac{1}{\alpha^2} + (W_3 - rW_2 - sW_1 - tW_0)\frac{1}{\alpha^3} = B_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})$. This gives

$$\begin{aligned} B_1 &= \frac{\alpha^3(W_0 + (W_1 - rW_0)\frac{1}{\alpha} + (W_2 - rW_1 - sW_0)\frac{1}{\alpha^2} + (W_3 - r2W_2 - sW_1 - t5W_0)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{W_0\alpha^3 + (W_1 - rW_0)\alpha^2 + (W_2 - rW_1 - sW_0)\alpha + (W_3 - rW_2 - sW_1 - tW_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{W_0\beta^3 + (W_1 - rW_0)\beta^2 + (W_2 - rW_1 - sW_0)\beta + (W_3 - rW_2 - sW_1 - tW_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{W_0\gamma^3 + (W_1 - rW_0)\gamma^2 + (W_2 - rW_1 - sW_0)\gamma + (W_3 - rW_2 - sW_1 - tW_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{W_0\delta^3 + (W_1 - rW_0)\delta^2 + (W_2 - rW_1 - sW_0)\delta + (W_3 - rW_2 - sW_1 - tW_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus (1.7) can be written as

$$\sum_{n=0}^{\infty} W_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\begin{aligned}\sum_{n=0}^{\infty} W_n x^n &= B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n + B_3 \sum_{n=0}^{\infty} \gamma^n x^n + B_4 \sum_{n=0}^{\infty} \delta^n x^n \\ &= \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n) x^n.\end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$W_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n$$

and then we get (1.5). □

Note that from (1.3) and (1.5) we have

$$\begin{aligned}& W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0 \\ = \;& W_0\alpha^3 + (W_1 - rW_0)\alpha^2 + (W_2 - rW_1 - sW_0)\alpha + (W_3 - rW_2 - sW_1 - tW_0), \\ & W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0 \\ = \;& W_0\beta^3 + (W_1 - rW_0)\beta^2 + (W_2 - rW_1 - sW_0)\beta + (W_3 - rW_2 - sW_1 - tW_0), \\ & W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0 \\ = \;& W_0\gamma^3 + (W_1 - rW_0)\gamma^2 + (W_2 - rW_1 - sW_0)\gamma + (W_3 - rW_2 - sW_1 - tW_0), \\ & W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0 \\ = \;& W_0\delta^3 + (W_1 - rW_0)\delta^2 + (W_2 - rW_1 - sW_0)\delta + (W_3 - rW_2 - sW_1 - tW_0).\end{aligned}$$

In this paper, we define and investigate, in detail, three special cases of the generalized (r, s, t, u) sequence $\{W_n\}$ which we call them (r, s, t, u), Lucas (r, s, t, u) and modified (r, s, t, u) sequences. (r, s, t, u) sequence$\{G_n\}_{n\geq 0}$, Lucas (r, s, t, u) sequence $\{H_n\}_{n\geq 0}$ and modified (r, s, t, u) sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned}
G_{n+4} &= rG_{n+3}+sG_{n+2}+tG_{n+1}+uG_n,\\
G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s,\\
H_{n+4} &= rH_{n+3}+sH_{n+2}+tH_{n+1}+uH_n,\\
H_0 &= 4, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t,\\
E_{n+4} &= rE_{n+3}+sE_{n+2}+tE_{n+1}+uE_n,\\
E_0 &= 1, E_1=r-1, E_2=-r+s+r^2, E_3=r^3-r^2+2sr-s+t.
\end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{t}{u}G_{-(n-1)}-\frac{s}{u}G_{-(n-2)}-\frac{r}{u}G_{-(n-3)}+\frac{1}{u}G_{-(n-4)}, \tag{1.8}$$
$$H_{-n} = -\frac{t}{u}H_{-(n-1)}-\frac{s}{u}H_{-(n-2)}-\frac{r}{u}H_{-(n-3)}+\frac{1}{u}H_{-(n-4)}, \tag{1.9}$$
$$E_{-n} = -\frac{t}{u}E_{-(n-1)}-\frac{s}{u}E_{-(n-2)}-\frac{r}{u}E_{-(n-3)}+\frac{1}{u}E_{-(n-4)}, \tag{1.10}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.8), (1.9) and (1.10) hold for all integers n.

Next, we present the first few values of the (r,s,t,u), Lucas (r,s,t,u) and modified (r,s,t,u) numbers with positive and negative subscripts:

Table 1. The first few values of the special fourth-order numbers with positive and negative subscripts.

n	0	1	2	3	4
G_n	0	1	r	r^2+s	$r^3+2sr+t$
G_{-n}		0	0	$\frac{1}{u}$	$-\frac{t}{u^2}$
H_n	4	r	$2s+r^2$	$r^3+3sr+3t$	$r^4+4r^2s+4tr+2s^2+4u$
H_{-n}		$-\frac{t}{u}$	$\frac{1}{u^2}(t^2-2su)$	$-\frac{1}{u^3}(t^3-3stu+3ru^2)$	$\frac{1}{u^4}(2s^2u^2-4st^2u+t^4+4rtu^2+4u^3)$
E_n	1	$r-1$	$-r+s+r^2$	$r^3-r^2+2sr-s+t$	$r^4-r^3+3r^2s-2rs+2tr+s^2-t+u$
E_{-n}		0	0	$-\frac{1}{u}$	$\frac{1}{u^2}(t+u)$

Some special cases of (r,s,t,u) sequence $\{G_n(0,1,r,r^2+s;r,s,t,u)\}$ and Lucas (r,s,t,u) sequence $\{H_n(4,r,2s+r^2,r^3+3sr+3t;r,s,t,u)\}$ are as follows:

1. $G_n(0,1,1,2;1,1,1,1) = M_n$, Tetranacci sequence,
2. $H_n(4,1,3,7;1,1,1,1) = R_n$, Tetranacci-Lucas sequence,
3. $G_n(0,1,2,5;2,1,1,1) = P_n$, fourth-order Pell sequence,
4. $H_n(4,2,6,17;2,1,1,1) = Q_n$, fourth-order Pell-Lucas sequence,
5. $G_n(0,1,1,2;1,1,1,2) = S_n$, adjusted fourth-order Jacobsthal sequence,
6. $H_n(4,1,3,7;1,1,1,2) = R_n$, modified fourth-order Jacobsthal-Lucas sequence.

For all integers n, (r,s,t,u), Lucas (r,s,t,u) and modified (r,s,t,u) numbers (using initial conditions in (1.3) or (1.5)) can be expressed using Binet's formulas as

$$\begin{aligned}
G_n &= \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} \\
&\quad + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \\
H_n &= \alpha^n + \beta^n + \gamma^n + \delta^n, \\
E_n &= \frac{(\alpha-1)\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{(\beta-1)\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{(\gamma-1)\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} \\
&\quad + \frac{(\delta-1)\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},
\end{aligned}$$

respectively.

Note that for all n we have

$$E_n = G_{n+1} - G_n,$$

and

$$\begin{aligned}
G_{-n} &= \frac{\Lambda_1}{\Lambda_2} G_n, \quad n \geq 1, \\
E_{-n} &= \frac{\Omega_1}{\Omega_2} E_n, \quad n \geq 1,
\end{aligned}$$

where

$\Lambda_1 = (\delta-\gamma)(\beta-\delta)(\beta-\gamma)\alpha^{2-n} + (\gamma-\delta)(\alpha-\delta)(\alpha-\gamma)\beta^{2-n} + (\delta-\beta)(\alpha-\delta)(\alpha-\beta)\gamma^{2-n} + (\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)\delta^{2-n}$,

$\Lambda_2 = (\delta-\gamma)(\beta-\delta)(\beta-\gamma)\alpha^{n+2} + (\gamma-\delta)(\alpha-\delta)(\alpha-\gamma)\beta^{n+2} + (\delta-\beta)(\alpha-\delta)(\alpha-\beta)\gamma^{n+2} + (\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)\delta^{n+2}$,

$\Omega_1 = (\gamma-\delta)(\beta-\delta)(\beta-\gamma)(\alpha-1)\alpha^{2-n} + (\delta-\gamma)(\alpha-\delta)(\alpha-\gamma)(\beta-1)\beta^{2-n} + (\beta-\delta)(\alpha-\delta)(\alpha-\beta)(\gamma-1)\gamma^{2-n} + (\gamma-\beta)(\alpha-\gamma)(\alpha-\beta)(\delta-1)\delta^{2-n}$,

$\Omega_2 = (\gamma-\delta)(\beta-\delta)(\beta-\gamma)(\alpha-1)\alpha^{n+2} + (\delta-\gamma)(\alpha-\delta)(\alpha-\gamma)(\beta-1)\beta^{n+2} + (\beta-\delta)(\alpha-\delta)(\alpha-\beta)(\gamma-1)\gamma^{n+2} + (\gamma-\beta)(\alpha-\gamma)(\alpha-\beta)(\delta-1)\delta^{n+2}$.

Lemma 1 gives the following results as particular examples (generating functions of (r, s, t, u), Lucas (r, s, t, u) and modified (r, s, t, u) numbers).

Corollary 3. *Generating functions of (r, s, t, u), Lucas (r, s, t, u) and modified (r, s, t, u) numbers are*

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1-rx-sx^2-tx^3-ux^4}, \\
\sum_{n=0}^{\infty} H_n x^n &= \frac{4-3rx-2sx^2-tx^3}{1-rx-sx^2-tx^3-ux^4}, \\
\sum_{n=0}^{\infty} E_n x^n &= \frac{1-x}{1-rx-sx^2-tx^3-ux^4},
\end{aligned}$$

respectively.

Proof. In Lemma 1, take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r, G_3 = r^2 + s$, $W_n = H_n$ with $H_0 = 4, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t$, and $W_n = E_n$ with $E_0 = 1, E_1 = r - 1, E_2 = -r + s + r^2, E_3 = r^3 - r^2 + 2sr - s + t$, respectively. □

2 Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized (r, s, t, u) sequence $\{W_n\}_{n\geq 0}$.

Theorem 4 (Simson Formula of Generalized (r, s, t, u) Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \quad (2.1)$$

Proof. (2.1) is given in Soykan [11]. □

The previous theorem gives the following results as particular examples.

Corollary 5. *For all integers n, Simson formula of (r, s, t, u), Lucas (r, s, t, u) and* *modified (r, s, t, u) numbers are given as, respectively,*

$$\begin{vmatrix} G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} \end{vmatrix} = (-1)^{n+1}u^{n-1},$$

$$\begin{vmatrix} H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} \end{vmatrix} = (-1)^{n}u^{n-3}g(r,s,t,u),$$

$$\begin{vmatrix} E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} \end{vmatrix} = (-1)^{n}u^{n-1}(r+s+t+u-1),$$

where

$g(r,s,t,u) = 27r^4u^2 - 18r^3stu + 4r^3t^3 + 4r^2s^3u - r^2s^2t^2 + 144r^2su^2 - 6r^2t^2u - 80rs^2tu + 18rst^3 + 192rtu^2 + 16s^4u - 4s^3t^2 + 128s^2u^2 - 144st^2u + 27t^4 + 256u^3.$

3 Some Identities

In this section, we obtain some identities of (r, s, t, u), Lucas (r, s, t, u) and modified (r, s, t, u) numbers. Firstly, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 6. *The following equalities are true:*

(a) $u^3H_n = (-t^3 + 3stu - 3ru^2)G_{n+4} + (rt^3 - 2su^2 + t^2u + 3r^2u^2 - 3rstu)G_{n+3} + (-3s^2tu + st^3 + 5rsu^2 - rt^2u - tu^2)G_{n+2} + (2s^2u^2 + t^4 + 4u^3 + 4rtu^2 - 4st^2u)G_{n+1}.$

(b) $u^2H_n = (-2su + t^2)G_{n+3} - (rt^2 + tu - 2rsu)G_{n+2} + (-st^2 + 2s^2u + 4u^2 + rtu)G_{n+1} - (3ru^2 + t^3 - 3stu)G_n.$

(c) $uH_n = -tG_{n+2} + (4u + rt)G_{n+1} + (-3ru + st)G_n + (-2su + t^2)G_{n-1}$.

(d) $H_n = 4G_{n+1} - 3rG_n - 2sG_{n-1} - tG_{n-2}$.

(e) $(16s^4u+4r^3t^3-4s^3t^2+27r^4u^2+128s^2u^2+27t^4+256u^3-r^2s^2t^2+18rst^3+192rtu^2-144st^2u+144r^2su^2+4r^2s^3u-6r^2t^2u-80rs^2tu-18r^3stu)G_n = (-6st^2+16s^2u-2r^2t^2+64u^2+6r^2su+8rtu)H_{n+3}+(-6r^3su+2r^3t^2-5r^2tu-20rs^2u+7rst^2-16ru^2-32stu+9t^3)H_{n+2}+(-3r^3tu-2r^2s^2u+r^2st^2-12r^2u^2-4rstu-3rt^3-8s^3u+4s^2t^2-32su^2-12t^2u)H_{n+1}+(-9r^3u^2+r^2stu-32rsu^2-3rt^2u+4s^2tu-48tu^2)H_n$.

(f) $(16s^4u+4r^3t^3-4s^3t^2+27r^4u^2+128s^2u^2+27t^4+256u^3-r^2s^2t^2+18rst^3+192rtu^2-144st^2u+144r^2su^2+4r^2s^3u-6r^2t^2u-80rs^2tu-18r^3stu)G_n = (48ru^2+9t^3+rst^2-4rs^2u+3r^2tu-32stu)H_{n+2}+(-3r^3tu+4r^2s^2u-r^2st^2-12r^2u^2+4rstu-3rt^3+8s^3u-2s^2t^2+32su^2-12t^2u)H_{n+1}+(-9r^3u^2+7r^2stu-2r^2t^3-32rsu^2+5rt^2u+20s^2tu-6st^3+16tu^2)H_n+(16s^2u^2+64u^3+8rtu^2-6st^2u+6r^2su^2-2r^2t^2u)H_{n-1}$.

(g) $(16s^4u+4r^3t^3-4s^3t^2+27r^4u^2+128s^2u^2+27t^4+256u^3-r^2s^2t^2+18rst^3+192rtu^2-144st^2u+144r^2su^2+4r^2s^3u-6r^2t^2u-80rs^2tu-18r^3stu)G_n = (6rt^3+32su^2+8s^3u-12t^2u+36r^2u^2-2s^2t^2-28rstu)H_{n+1}+(3st^3+16tu^2-2r^2t^3-9r^3u^2+16rsu^2-4rs^3u+5rt^2u-12s^2tu+rs^2t^2+10r^2stu)H_n+(16s^2u^2+9t^4+64u^3+rst^3+56rtu^2-38st^2u+6r^2su^2+r^2t^2u-4rs^2tu)H_{n-1}+(48ru^3+9t^3u-32stu^2-4rs^2u^2+3r^2tu^2+rst^2u)H_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$H_n = a \times G_{n+4} + b \times G_{n+3} + c \times G_{n+2} + d \times G_{n+1}$$

and solving the system of equations

$$\begin{aligned}
H_0 &= a \times G_4 + b \times G_3 + c \times G_2 + d \times G_1 \\
H_1 &= a \times G_5 + b \times G_4 + c \times G_3 + d \times G_2 \\
H_2 &= a \times G_6 + b \times G_5 + c \times G_4 + d \times G_3 \\
H_3 &= a \times G_7 + b \times G_6 + c \times G_5 + d \times G_4
\end{aligned}$$

we find that

$$\begin{aligned} a &= \frac{1}{u^3}(-t^3 + 3stu - 3ru^2), \\ b &= \frac{1}{u^3}(rt^3 - 2su^2 + t^2u + 3r^2u^2 - 3rstu), \\ c &= \frac{1}{u^3}(-3s^2tu + st^3 + 5rsu^2 - rt^2u - tu^2), \\ d &= \frac{1}{u^3}(2s^2u^2 + t^4 + 4u^3 + 4rtu^2 - 4st^2u). \end{aligned}$$

The other equalities can be proved similarly. □

Secondly, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

Lemma 7. *The following equalities are true:*

(a) $uE_n = -G_{n+4} + rG_{n+3} + sG_{n+2} + (t+u)G_{n+1}$.

(b) $E_n = G_{n+1} - G_n$.

(c) $(r+s+t+u-1)G_n = E_{n+1} - (r-1)E_n + (t+u)E_{n-1} + uE_{n-2}$.

Note that all the identities in the above lemma can be proved by induction as well.

Thirdly, we give a few basic relations between $\{H_n\}$ and $\{E_n\}$.

Lemma 8. *The following equalities are true:*

(a) $(r+s+t+u-1)H_n = -(3r+2s+t-4)E_{n+1} + (-3r+2s+3t+4u+2rs+rt+3r^2)E_n + (-2s+3t+4u+2rs-2rt-3ru+st+2s^2)E_{n-1} + (-t+4u+rt-3ru+st-2su+t^2)E_{n-2}$.

(b) $(16s^4u + 4r^3t^3 - 4s^3t^2 + 27r^4u^2 + 128s^2u^2 + 27t^4 + 256u^3 - r^2s^2t^2 + 18rst^3 + 192rtu^2 - 144st^2u + 144r^2su^2 + 4r^2s^3u - 6r^2t^2u - 80rs^2tu - 18r^3stu)E_n = (48ru^2+6st^2-16s^2u+2r^2t^2+9t^3-64u^2+rst^2-4rs^2u-6r^2su+3r^2tu-8rtu-32s tu)H_{n+3} + (-3rt^3+16ru^2+32su^2+8s^3u-12t^2u-2r^3t^2-12r^2u^2-2s^2t^2-9t^3-7rst^2+20rs^2u+6r^3su+5r^2tu-3r^3tu-r^2st^2+4r^2s^2u+32stu+4rstu)H_{n+2}+$

$(3rt^3-6st^3+32su^2+8s^3u+16tu^2+12t^2u-2r^2t^3+12r^2u^2-4s^2t^2-9r^3u^2-32rs u^2+5rt^2u+3r^3tu+20s^2tu-r^2st^2+2r^2s^2u+7r^2stu+4rstu)H_{n+1}+u(3rt^2-6st^2-4s^2t+9r^3u+16s^2u-2r^2t^2+48tu+64u^2-r^2st+6r^2su+32rsu+8rtu)H_n.$

Next, we give a few basic relations between $\{G_n\}$ and $\{W_n\}$.

Lemma 9. *The following equalities are true:*

(a) $u^2W_n=(-tW_3+(u+rt)W_2+(st-ru)W_1+(t^2-su)W_0)G_{n+3}+((u+rt)W_3-r(2u+rt)W_2+(r^2u-rst-su)W_1+(rsu-tu-rt^2)W_0)G_{n+2}+((st-ru)W_3+(r^2u-su-rst)W_2+s(2ru-st)W_1+(rtu+u^2+s^2u-st^2)W_0)G_{n+1}+((t^2-su)W_3+(rsu-tu-rt^2)W_2+(rtu+u^2-st^2+s^2u)W_1+(2stu-ru^2-t^3)W_0)G_n.$

(b) $uW_n=(W_3-rW_2-sW_1-tW_0)G_{n+2}+(-rW_3+r^2W_2+rsW_1+(u+rt)W_0)G_{n+1}+(-sW_3+rsW_2+(u+s^2)W_1+(st-ru)W_0)G_n+(-tW_3+(u+rt)W_2+(st-ru)W_1+(t^2-su)W_0)G_{n-1}.$

(c) $W_n=W_0G_{n+1}+(W_1-rW_0)G_n+(W_2-rW_1-sW_0)G_{n-1}+(W_3-rW_2-sW_1-tW_0)G_{n-2}.$

Now, we present a basic relation between $\{H_n\}$ and $\{W_n\}$.

Lemma 10. *The following equality is true:*

$(16s^4u+4r^3t^3-4s^3t^2+27r^4u^2+128s^2u^2+27t^4+256u^3-r^2s^2t^2+18rst^3+192rtu^2-144st^2u+144r^2su^2+4r^2s^3u-6r^2t^2u-80rs^2tu-18r^3stu)W_n=X_1H_{n+3}+X_2H_{n+2}+X_3H_{n+1}+X_4H_n$

where

$X_1=2(3r^3t-6r^2u-r^2s^2-16su-4s^3+18t^2+14rst)W_3+(8rs^3-39rt^2-6r^4t+4s^2t+3r^3u+2r^3s^2-48tu-27r^2st)W_2+(-42st^2+9r^4u+48s^2u+2r^2s^3+r^2t^2+8s^4+64u^2-32rs^2t-7r^3st+50r^2su+56rtu)W_1+(-16ru^2+4s^3t-4r^3t^2-27t^3-18rst^2+12rs^2u+3r^3su+7r^2tu+r^2s^2t+48stu)W_0,$

$X_2=(8rs^3-39rt^2-6r^4t+4s^2t+3r^3u+2r^3s^2-48tu-27r^2st)W_3+2(-3st^2+3r^5t+3r^4u+8s^2u-4r^2s^3+20r^2t^2-r^4s^2+32u^2-4rs^2t+13r^3st+19r^2su+52rtu)W_2+(-8rs^4-80ru^2-4s^3t-9r^5u-2r^3s^3+r^3t^2+9t^3+52rst^2-36rs^2u+$

$7r^4st - 47r^3su - 61r^2tu + 31r^2s^2t + 16stu)W_1 + (-3r^4su + 4r^4t^2 - r^3s^2t - r^3tu - 14r^2s^2u + 18r^2st^2 + 4r^2u^2 - 4rs^3t - 20rstu + 27rt^3 - 8s^3u - 32su^2 + 36t^2u)W_0,$

$X_3 = (-6r^4tu - r^3s^2u + 4r^3st^2 + 3r^3u^2 - r^2s^3t - 34r^2stu - 4rs^3u + 18rs^2t^2 + 16rsu^2 - 39rt^2u - 4s^4t - 44s^2tu + 27st^3 - 48tu^2)W_0 + 2(-6r^4su - r^4t^2 + 4r^3s^2t + r^3tu - r^2s^4 - 32r^2s^2u - 5r^2st^2 + 2r^2u^2 + 18rs^3t - 38rstu - 6rt^3 - 4s^5 - 28s^3u + 23s^2t^2 - 48su^2 - 6t^2u)W_1 + (-8rs^4 - 80ru^2 - 4s^3t - 9r^5u - 2r^3s^3 + r^3t^2 + 9t^3 + 52rst^2 - 36rs^2u + 7r^4st - 47r^3su - 61r^2tu + 31r^2s^2t + 16stu)W_2 + (-42st^2 + 9r^4u + 48s^2u + 2r^2s^3 + r^2t^2 + 8s^4 + 64u^2 - 32rs^2t - 7r^3st + 50r^2su + 56rtu)W_3,$

$X_4 = (-16ru^2 + 4s^3t - 4r^3t^2 - 27t^3 - 18rst^2 + 12rs^2u + 3r^3su + 7r^2tu + r^2s^2t + 48stu)W_3 + (-3r^4su + 4r^4t^2 - r^3s^2t - r^3tu - 14r^2s^2u + 18r^2st^2 + 4r^2u^2 - 4rs^3t - 20rstu + 27rt^3 - 8s^3u - 32su^2 + 36t^2u)W_2 + (-6r^4tu - r^3s^2u + 4r^3st^2 + 3r^3u^2 - r^2s^3t - 34r^2stu - 4rs^3u + 18rs^2t^2 + 16rsu^2 - 39rt^2u - 4s^4t - 44s^2tu + 27st^3 - 48tu^2)W_1 + (8s^4u + 4r^3t^3 - 4s^3t^2 + 9r^4u^2 + 48s^2u^2 + 27t^4 + 64u^3 - r^2s^2t^2 + 18rst^3 + 72rtu^2 - 90st^2u + 50r^2su^2 + 2r^2s^3u - 6r^2t^2u - 44rs^2tu - 10r^3stu)W_0.$

Next, we give a basic relation between $\{E_n\}$ and $\{W_n\}$.

Lemma 11. *The following equality is true:*

$(r+s+t+u-1)W_n = (W_3 + (1-r)W_2 + (1-r-s)W_1 + (1-r-s-t)W_0)E_{n+1} + ((1-r)W_3 + (r-1)^2W_2 + (1-2r-s+rs+r^2)W_1 + (u-r+rs+rt+r^2)W_0)E_n + ((1-r-s)W_3 + (1-2r-s+rs+r^2)W_2 + (u-r-s+t+2rs+r^2+s^2)W_1 + (u-s+rs-ru+st+s^2)W_0)E_{n-1} + ((1-r-s-t)W_3 + (u-r+rt+rs+r^2)W_2 + (u-s+rs-ru+st+s^2)W_1 + (u-t+rt-ru+st-su+t^2)W_0)E_{n-2}$

We now present a few special identities for the modified (r,s,t,u) sequence $\{E_n\}$.

Theorem 12 (Catalan's identity)**.** *For all integers n and m, the following identity holds*

$$\begin{aligned} E_{n+m}E_{n-m} - E_n^2 &= G_{m+n+1}\left(G_{-m+n+1} - G_{-m+n}\right) + G_{m+n}\left(G_{-m+n} - G_{-m+n+1}\right) \\ &\quad - \left(G_{n+1} - G_n\right)^2. \end{aligned}$$

Proof. We use the identity

$$E_n = G_{n+1} - G_n.$$

□

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the modified (r, s, t, u) sequence.

Corollary 13 (Cassini's identity)**.** *For all integers numbers n and m, the following identity holds*

$$E_{n+1}E_{n-1} - E_n^2 = (G_{n+2} - G_{n+1})(G_n - G_{n-1}) - (G_{n+1} - G_n)^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham's identities can also be obtained by using $E_n = G_{n+1} - G_n$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham's identities of modified (r, s, t, u) sequence $\{E_n\}$.

Theorem 14. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$\begin{aligned} E_{m+1}E_n - E_mE_{n+1} &= G_{m+2}\left(G_{n+1} - G_n\right) + G_{m+1}\left(G_n - G_{n+2}\right) \\ &\quad + G_m\left(G_{n+2} - G_{n+1}\right). \end{aligned}$$

(b) *(Gelin-Cesàro's identity)*

$$\begin{aligned} & E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 \\ = \; & (G_{n+3} - G_{n+2})(G_{n+2} - G_{n+1})(G_n - G_{n-1})(G_{n-1} - G_{n-2}) - (G_{n+1} - G_n)^4. \end{aligned}$$

(c) *(Melham's identity)*

$$\begin{aligned} E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 &= (G_{n+2} - G_{n+1})(G_{n+3} - G_{n+2})(G_{n+7} - G_{n+6}) \\ &\quad - (G_{n+4} - G_{n+3})^3. \end{aligned}$$

Proof. Use the identity $E_n = G_{n+1} - G_n$. □

4 Sum Formulas

The following theorem presents sum formulas of generalized (r, s, t, u) numbers (generalized Tetranacci numbers).

Theorem 15. *For all integers m and j, if $(H_m^2 - H_{2m} - 2H_m + 2(1 - H_{-m})(-u)^m + 2) \neq 0$, then we have*

$$\sum_{k=0}^{n} W_{mk+j} = \frac{\Delta}{\frac{1}{2}(H_m^2 - H_{2m} - 2H_m + 2(1 - H_{-m})(-u)^m + 2)} \tag{4.1}$$

where

$\Delta = -W_{mn+2m+j} + (H_m - 1)W_{mn+m+j} + (-u)^m (1 - H_{-m}) W_{mn+j} + (-u)^m W_{mn-m+j} - (-u)^m W_{-m+j} + W_{2m+j} - (H_m - 1)W_{m+j} - (H_m + \frac{1}{2}(H_{2m} - H_m^2) - 1)W_j$.

Proof. Note that

$$\begin{aligned}
\sum_{k=0}^{n} W_{mk+j} &= W_{mn+j} + \sum_{k=0}^{n-1} W_{mk+j} \\
&= W_{mn+j} + \sum_{k=0}^{n-1} (A_1\alpha^{mk+j} + A_2\beta^{mk+j} + A_3\gamma^{mk+j} + A_4\delta^{mk+j}) \\
&= W_{mn+j} + A_1\alpha^j \left(\frac{\alpha^{mn} - 1}{\alpha^m - 1}\right) + A_2\beta^j \left(\frac{\beta^{mn} - 1}{\beta^m - 1}\right) \\
&\quad + A_3\gamma^j \left(\frac{\gamma^{mn} - 1}{\gamma^m - 1}\right) + A_4\delta^j \left(\frac{\delta^{mn} - 1}{\delta^m - 1}\right).
\end{aligned}$$

Simplifying the last equalities in the last two expression imply (4.1) as required. □

Note that (4.1) can be written in the following form:

$$\sum_{k=1}^{n} W_{mk+j} = \frac{\Omega}{\frac{1}{2}(H_m^2 - H_{2m} - 2H_m + 2(1 - H_{-m})(-u)^m + 2)}$$

where

$\Omega = -W_{mn+2m+j} + (H_m - 1)W_{mn+m+j} + (H_m + \frac{1}{2}(H_{2m} - H_m^2) - 1)W_{mn+j} + (-u)^m W_{mn-m+j} - (-u)^m W_{-m+j} + W_{2m+j} - (H_m - 1)W_{m+j} + (-u)^m (H_{-m} - 1)W_j$.

As special cases of the above theorem, we have the following corollaries. Firstly, as special cases of the above theorem, we have the following corollary for the generalized Tetranacci numbers.

Corollary 16. *The following identities hold:*

1. $m = 1$, $j = 0$.
   - **(a)** $\sum_{k=0}^{n} M_k = \frac{1}{3}(M_{n+2} + 2M_n + M_{n-1} - 1)$.
   - **(b)** $\sum_{k=0}^{n} R_k = \frac{1}{3}(R_{n+2} + 2R_n + R_{n-1} + 2)$.
2. $m = -1$, $j = 0$.
   - **(a)** $\sum_{k=0}^{n} M_{-k} = \frac{1}{3}(-M_{-n+1} - 2M_{-n-1} - M_{-n-2} + 1)$.
   - **(b)** $\sum_{k=0}^{n} R_{-k} = \frac{1}{3}(-R_{-n+1} - 2R_{-n-1} - R_{-n-2} + 10)$.
3. $m = 4$, $j = -6$.
   - **(a)** $\sum_{k=0}^{n} M_{4k-6} = \frac{1}{3}(M_{4n+2} - 14M_{4n-2} + 6M_{4n-6} - M_{4n-10} - 1)$.
   - **(b)** $\sum_{k=0}^{n} R_{4k-6} = \frac{1}{3}(R_{4n+2} - 14R_{4n-2} + 6R_{4n-6} - R_{4n-10} - 10)$.
4. $m = -3$, $j = 2$.
   - **(a)** $\sum_{k=0}^{n} M_{-3k+2} = \frac{1}{9}(-M_{-3n+5} + 6M_{-3n+2} - 2M_{-3n-1} - M_{-3n-4} + 10)$.
   - **(b)** $\sum_{k=0}^{n} R_{-3k+2} = \frac{1}{9}(-R_{-3n+5} + 6R_{-3n+2} - 2R_{-3n-1} - R_{-3n-4} + 40)$.

Secondly, as special cases of the above theorem, we have the following corollary for the generalized fourth-order Pell numbers.

Corollary 17. *The following identities hold:*

1. $m = 1$, $j = 0$.

(a) $\sum_{k=0}^{n} P_k = \frac{1}{4}(P_{n+2} - P_{n+1} + 2P_n + P_{n-1} - 1)$.

(b) $\sum_{k=0}^{n} Q_k = \frac{1}{4}(Q_{n+2} - Q_{n+1} + 2Q_n + Q_{n-1} + 5)$.

(c) $\sum_{k=0}^{n} E_k = \frac{1}{4}(E_{n+2} - E_{n+1} + 2E_n + E_{n-1})$.

2. $m = -1$, $j = 0$.

(a) $\sum_{k=0}^{n} P_{-k} = \frac{1}{4}(-P_{-n+1} + P_{-n} - 2P_{-n-1} - P_{-n-2} + 1)$.

(b) $\sum_{k=0}^{n} Q_{-k} = \frac{1}{4}(-Q_{-n+1} + Q_{-n} - 2Q_{-n-1} - Q_{-n-2} + 11)$.

(c) $\sum_{k=0}^{n} E_{-k} = \frac{1}{4}(-E_{-n+1} + E_{-n} - 2E_{-n-1} - E_{-n-2})$.

3. $m = 4$, $j = -6$.

(a) $\sum_{k=0}^{n} P_{4k-6} = \frac{1}{16}(P_{4n+2} - 45P_{4n-2} + 10P_{4n-6} - P_{4n-10} - 15)$.

(b) $\sum_{k=0}^{n} Q_{4k-6} = \frac{1}{16}(Q_{4n+2} - 45Q_{4n-2} + 10Q_{4n-6} - Q_{4n-10} - 5)$.

(c) $\sum_{k=0}^{n} E_{4k-6} = \frac{1}{16}(E_{4n+2} - 45E_{4n-2} + 10E_{4n-6} - E_{4n-10} - 100)$.

4. $m = -3$, $j = 2$.

(a) $\sum_{k=0}^{n} P_{-3k+2} = \frac{1}{28}(16P_{-3n+2} - 5P_{-3n-1} - P_{-3n-4} - P_{-3n+5} + 57)$.

(b) $\sum_{k=0}^{n} Q_{-3k+2} = \frac{1}{28}(16Q_{-3n+2} - 5Q_{-3n-1} - Q_{-3n-4} - Q_{-3n+5} + 195)$.

(c) $\sum_{k=0}^{n} E_{-3k+2} = \frac{1}{28}(16E_{-3n+2} - 5E_{-3n-1} - E_{-3n-4} - E_{-3n+5} + 32)$.

Thirdly, as special cases of the above theorem, we have the following corollary for the generalized fourth-order Jacobsthal numbers.

Corollary 18. *The following identities hold:*

1. $m = 1$, $j = 0$.

(a) $\sum_{k=0}^{n} J_k = \frac{1}{4}(J_{n+2} + 3J_n + 2J_{n-1})$.

(b) $\sum_{k=0}^{n} j_k = \frac{1}{4}(j_{n+2} + 3j_n + 2j_{n-1} - 5)$.

(c) $\sum_{k=0}^{n} K_k = \frac{1}{4}(K_{n+2} + 3K_n + 2K_{n-1} - 3)$.

(d) $\sum_{k=0}^{n} Q_k = \frac{1}{4}(Q_{n+2} + 3Q_n + 2Q_{n-1} - 2)$.

(e) $\sum_{k=0}^{n} S_k = \frac{1}{4}(S_{n+2} + 3S_n + 2S_{n-1} - 1)$.

(f) $\sum_{k=0}^{n} R_k = \frac{1}{4}(R_{n+2} + 3R_n + 2R_{n-1} + 2)$.

2. $m = -1$, $j = 0$.

(a) $\sum_{k=0}^{n} J_{-k} = \frac{1}{4}(-J_{-n+1} - 3J_{-n-1} - 2J_{-n-2})$.

(b) $\sum_{k=0}^{n} j_{-k} = \frac{1}{4}(-j_{-n+1} - 3j_{-n-1} - 2j_{-n-2} + 13)$.

(c) $\sum_{k=0}^{n} K_{-k} = \frac{1}{4}(-K_{-n+1} - 3K_{-n-1} - 2K_{-n-2} + 15)$.

(d) $\sum_{k=0}^{n} Q_{-k} = \frac{1}{4}(-Q_{-n+1} - 3Q_{-n-1} - 2Q_{-n-2} + 14)$.

(e) $\sum_{k=0}^{n} S_{-k} = \frac{1}{4}(-S_{-n+1} - 3S_{-n-1} - 2S_{-n-2} + 1)$.

(f) $\sum_{k=0}^{n} R_{-k} = \frac{1}{4}(-R_{-n+1} - 3R_{-n-1} - 2R_{-n-2} + 14)$.

3. $m = 3$, $j = -6$.

(a) $\sum_{k=0}^{n} J_{3k-6} = \frac{1}{448}(16J_{3n} - 96J_{3n-3} + 240J_{3n-6} + 128J_{3n-9} + 179)$.

(b) $\sum_{k=0}^{n} j_{3k-6} = \frac{1}{224}(8j_{3n} - 48j_{3n-3} + 120j_{3n-6} + 64j_{3n-9} - 79)$.

(c) $\sum_{k=0}^{n} K_{3k-6} = \frac{1}{448}(16K_{3n} - 96K_{3n-3} + 240K_{3n-6} + 128K_{3n-9} - 591)$.

(d) $\sum_{k=0}^{n} Q_{3k-6} = \frac{1}{448}(16Q_{3n} - 96Q_{3n-3} + 240Q_{3n-6} + 128Q_{3n-9} - 643)$.

(e) $\sum_{k=0}^{n} S_{3k-6} = \frac{1}{112}(4S_{3n} - 24S_{3n-3} + 60S_{3n-6} + 32S_{3n-9} + 13)$.

(f) $\sum_{k=0}^{n} R_{3k-6} = \frac{1}{448}(16R_{3n} - 96R_{3n-3} + 240R_{3n-6} + 128R_{3n-9} - 225)$.

4. $m = -3$, $j = 2$.

(a) $\sum_{k=0}^{n} J_{-3k+2} = \frac{1}{28}(-J_{-3n+5} + 6J_{-3n+2} - 15J_{-3n-1} - 8J_{-3n-4} + 20)$.

(b) $\sum_{k=0}^{n} j_{-3k+2} = \frac{1}{28}(-j_{-3n+5} + 6j_{-3n+2} - 15j_{-3n-1} - 8j_{-3n-4} + 169)$.

(c) $\sum_{k=0}^{n} K_{-3k+2} = \frac{1}{28}(-K_{-3n+5} + 6K_{-3n+2} - 15K_{-3n-1} - 8K_{-3n-4} + 139)$.

(d) $\sum_{k=0}^{n} Q_{-3k+2} = \frac{1}{28}(-Q_{-3n+5} + 6Q_{-3n+2} - 15Q_{-3n-1} - 8Q_{-3n-4} + 110)$.

(e) $\sum_{k=0}^{n} S_{-3k+2} = \frac{1}{28}(-S_{-3n+5} + 6S_{-3n+2} - 15S_{-3n-1} - 8S_{-3n-4} + 29)$.

(f) $\sum_{k=0}^{n} R_{-3k+2} = \frac{1}{28}(-R_{-3n+5} + 6R_{-3n+2} - 15R_{-3n-1} - 8R_{-3n-4} + 114)$.

5 Matrices Related with Generalized (r, s, t, u) Numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{5.1}$$

For matrix formulation (5.1), see [5]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ r & s & t & u \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix}.$$

We define the square matrix A of order 4 as:

$$A = A_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -u$. From (1.1) we have

$$\begin{pmatrix} W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix}. \tag{5.2}$$

and from (5.1) (or using (5.2) and induction) we have

$$\begin{pmatrix} W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W_n = G_n$ in (5.2) we have

$$\begin{pmatrix} G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+2} \\ G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{5.3}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} & tW_n + uW_{n-1} & uW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} & tW_{n-1} + uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} & tW_{n-2} + uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} & tW_{n-3} + uW_{n-4} & uW_{n-3} \end{pmatrix}.$$

Theorem 19. *For all integers $m, n \geq 0$, we have*

(a) $B_n = A^n$.

(b) $C_1 A^n = A^n C_1$.

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (5.3) to 4-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

□

Some properties of matrix A^n can be given as

$$A^n = rA^{n-1} + sA^{n-2} + tA^{n-3} + uA^{n-4}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = (-u)^n$$

for all integers m and n.

Theorem 20. *For $m, n \geq 0$, we have*

$$\begin{aligned} W_{n+m} &= W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) \\ &\quad + uW_{n-3}G_m. \end{aligned} \tag{5.4}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof. □

Remark 21. *By induction, it can be proved that for all integers $m, n \leq 0$, (5.4) holds. So for all integers m, n, (5.4) is true.*

Corollary 22. *For all integers m, n, we have*

$$\begin{aligned} G_{n+m} &= G_n G_{m+1} + G_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + G_{n-2}(tG_m + uG_{m-1}) \\ &\quad + uG_{n-3}G_m, \qquad (5.5) \\ H_{n+m} &= H_n G_{m+1} + H_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + H_{n-2}(tG_m + uG_{m-1}) \\ &\quad + uH_{n-3}G_m, \qquad (5.6) \\ E_{n+m} &= E_n G_{m+1} + E_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + E_{n-2}(tG_m + uG_{m-1}) \\ &\quad + uE_{n-3}G_m. \qquad (5.7) \end{aligned}$$

6 Special Matrix Formulas

In this section, we present some specific matrix relations of fourth-order numbers (generalized (r, s, t, u) numbers).

Firstly, we present some formulas for the generalized Tetranacci numbers.

Corollary 23. *For all integers n, we have the following formulas for the generalized Tetranacci numbers.*

(a) *Tetranacci Numbers.*

$$A_{1111}^n = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} M_{n+1} & M_n + M_{n-1} + M_{n-2} & M_n + M_{n-1} & M_n \\ M_n & M_{n-1} + M_{n-2} + M_{n-3} & M_{n-1} + M_{n-2} & M_{n-1} \\ M_{n-1} & M_{n-2} + M_{n-3} + M_{n-4} & M_{n-2} + M_{n-3} & M_{n-2} \\ M_{n-2} & M_{n-3} + M_{n-4} + M_{n-5} & M_{n-3} + M_{n-4} & M_{n-3} \end{pmatrix}.$$

(b) *Tetranacci-Lucas Numbers.*

$$A_{1111}^n = \frac{1}{563} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

where

$$\begin{array}{l} a_{11} = 86R_{n+4} - 61R_{n+3} - 71R_{n+2} - 87R_{n+1} \\ a_{21} = 86R_{n+3} - 61R_{n+2} - 71R_{n+1} - 87R_{n} \\ a_{31} = 86R_{n+2} - 61R_{n+1} - 71R_{n} - 87R_{n-1} \\ a_{41} = 86R_{n+1} - 61R_{n} - 71R_{n-1} - 87R_{n-2} \end{array}, \quad \begin{array}{l} a_{12} = 15R_{n+3} + 9R_{n+2} + 112R_{n+1} - 61R_{n} \\ a_{22} = 15R_{n+2} + 9R_{n+1} + 112R_{n} - 61R_{n-1} \\ a_{32} = 15R_{n+1} + 9R_{n} + 112R_{n-1} - 61R_{n-2} \\ a_{42} = 15R_{n} + 9R_{n-1} + 112R_{n-2} - 61R_{n-3} \end{array}$$

$$\begin{array}{l} a_{13} = -R_{n+3} + 112R_{n+2} - 45R_{n+1} - 71R_{n} \\ a_{23} = -R_{n+2} + 112R_{n+1} - 45R_{n} - 71R_{n-1} \\ a_{33} = -R_{n+1} + 112R_{n} - 45R_{n-1} - 71R_{n-2} \\ a_{43} = -R_{n} + 112R_{n-1} - 45R_{n-2} - 71R_{n-3} \end{array}, \quad \begin{array}{l} a_{14} = 86R_{n+3} - 61R_{n+2} - 71R_{n+1} - 87R_{n} \\ a_{24} = 86R_{n+2} - 61R_{n+1} - 71R_{n} - 87R_{n-1} \\ a_{34} = 86R_{n+1} - 61R_{n} - 71R_{n-1} - 87R_{n-2} \\ a_{44} = 86R_{n} - 61R_{n-1} - 71R_{n-2} - 87R_{n-3} \end{array}$$

Proof. Take $r = 1, s = 1, t = 1, u = 1$ in Theorem 19 (a). Then in this case, $G_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$.

(a) In Theorem 19 (a), we take $G_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$.

(b) Take $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$. Writing

$$M_n = a \times R_{n+3} + b \times R_{n+2} + c \times R_{n+1} + d \times R_n$$

and solving the system of equations

$$\begin{aligned} M_0 &= a \times R_3 + b \times R_2 + c \times R_1 + d \times R_0 \\ M_1 &= a \times R_4 + b \times R_3 + c \times R_2 + d \times R_1 \\ M_2 &= a \times R_5 + b \times R_4 + c \times R_3 + d \times R_2 \\ M_3 &= a \times R_6 + b \times R_5 + c \times R_4 + d \times R_3 \end{aligned}$$

we find that $a = \frac{86}{563}, b = -\frac{61}{563}, c = -\frac{71}{563}, d = -\frac{87}{563}$ and so we get

$$563M_n = 86R_{n+3} - 61R_{n+2} - 71R_{n+1} - 87R_n.$$

Using the last equation and Theorem 19 (a), we get required result. □

Secondly, we present some formulas for the generalized fourth order Pell numbers.

Corollary 24. *For all integers n, we have the following formulas for the generalized fourth order Pell numbers.*

(a) *Fourth-Order Pell Numbers.*

$$A_{2111}^{n}=\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n}=\begin{pmatrix} P_{n+1} & P_{n}+P_{n-1}+P_{n-2} & P_{n}+P_{n-1} & P_{n} \\ P_{n} & P_{n-1}+P_{n-2}+P_{n-3} & P_{n-1}+P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2}+P_{n-3}+P_{n-4} & P_{n-2}+P_{n-3} & P_{n-2} \\ P_{n-2} & P_{n-3}+P_{n-4}+P_{n-5} & P_{n-3}+P_{n-4} & P_{n-3} \end{pmatrix}.$$

(b) *Fourth-Order Pell-Lucas Numbers.*

$$A_{2111}^{n}=\frac{1}{1423}\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

where

$$\begin{array}{c} b_{11}=106Q_{n+4}-133Q_{n+3}-138Q_{n+2}-182Q_{n+1} \\ b_{21}=106Q_{n+3}-133Q_{n+2}-138Q_{n+1}-182Q_{n} \\ b_{31}=106Q_{n+2}-133Q_{n+1}-138Q_{n}-182Q_{n-1} \\ b_{41}=106Q_{n+1}-133Q_{n}-138Q_{n-1}-182Q_{n-2} \end{array},\quad \begin{array}{c} b_{12}=-32Q_{n+3}+67Q_{n+2}+337Q_{n+1}-133Q_{n} \\ b_{22}=-32Q_{n+2}+67Q_{n+1}+337Q_{n}-133Q_{n-1} \\ b_{32}=-32Q_{n+1}+67Q_{n}+337Q_{n-1}-133Q_{n-2} \\ b_{42}=-32Q_{n}+67Q_{n-1}+337Q_{n-2}-133Q_{n-3} \end{array}$$

$$\begin{array}{c} b_{13}=-76Q_{n+3}+337Q_{n+2}-89Q_{n+1}-138Q_{n} \\ b_{23}=-76Q_{n+2}+337Q_{n+1}-89Q_{n}-138Q_{n-1} \\ b_{33}=-76Q_{n+1}+337Q_{n}-89Q_{n-1}-138Q_{n-2} \\ b_{43}=-76Q_{n}+337Q_{n-1}-89Q_{n-2}-138Q_{n-3} \end{array},\quad \begin{array}{c} b_{14}=106Q_{n+3}-133Q_{n+2}-138Q_{n+1}-182Q_{n} \\ b_{24}=106Q_{n+2}-133Q_{n+1}-138Q_{n}-182Q_{n-1} \\ b_{34}=106Q_{n+1}-133Q_{n}-138Q_{n-1}-182Q_{n-2} \\ b_{44}=106Q_{n}-133Q_{n-1}-138Q_{n-2}-182Q_{n-3} \end{array}$$

(c) *Modified Fourth-Order Pell Numbers.*

$$A_{2111}^{n}=\frac{1}{4}\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}$$

where

$$\begin{array}{cc} c_{11}=E_{n+4}-E_{n+3}-2E_{n+2}+E_{n+1} & c_{12}=3E_{n+2}-3E_{n+1}-2E_{n}-E_{n-1} \\ c_{21}=E_{n+3}-E_{n+2}-2E_{n+1}+E_{n} & c_{22}=3E_{n+1}-3E_{n}-2E_{n-1}-E_{n-2} \\ c_{31}=E_{n+2}-E_{n+1}-2E_{n}+E_{n-1} & c_{32}=3E_{n}-3E_{n-1}-2E_{n-2}-E_{n-3} \\ c_{41}=E_{n+1}-E_{n}-2E_{n-1}+E_{n-2} & c_{42}=3E_{n-1}-3E_{n-2}-2E_{n-3}-E_{n-4} \end{array},$$

$$\begin{array}{cc} c_{13}=E_{n+3}-3E_{n+1}-E_{n}+E_{n-1} & c_{14}=E_{n+3}-E_{n+2}-2E_{n+1}+E_{n} \\ c_{23}=E_{n+2}-3E_{n}-E_{n-1}+E_{n-2} & c_{24}=E_{n+2}-E_{n+1}-2E_{n}+E_{n-1} \\ c_{33}=E_{n+1}-3E_{n-1}-E_{n-2}+E_{n-3} & c_{34}=E_{n+1}-E_{n}-2E_{n-1}+E_{n-2} \\ c_{43}=E_{n}-3E_{n-2}-E_{n-3}+E_{n-4} & c_{44}=E_{n}-E_{n-1}-2E_{n-2}+E_{n-3} \end{array},$$

Proof. Take $r=2,s=1,t=1,u=1$ in Theorem 19 (a). Then in this case, $G_n=P_n$ with $P_0=0,P_1=1,P_2=2,P_3=5$.

(a) In Theorem 19 (a), we take $G_n=P_n$ with $P_0=0,P_1=1,P_2=2,P_3=5$.

(b) Take $W_n=Q_n$ with $Q_0=4,Q_1=2,Q_2=6,Q_2=17$. Writing

$$P_n=a\times Q_{n+3}+b\times Q_{n+2}+c\times Q_{n+1}+d\times Q_n$$

and solving the system of equations

$$\begin{aligned} P_0 &= a\times Q_3+b\times Q_2+c\times Q_1+d\times Q_0 \\ P_1 &= a\times Q_4+b\times Q_3+c\times Q_2+d\times Q_1 \\ P_2 &= a\times Q_5+b\times Q_4+c\times Q_3+d\times Q_2 \\ P_3 &= a\times Q_6+b\times Q_5+c\times Q_4+d\times Q_3 \end{aligned}$$

we find that $a=\frac{106}{1423},b=-\frac{133}{1423},c=-\frac{138}{1423},d=-\frac{182}{1423}$ and so we get

$$1423P_n=106Q_{n+3}-133Q_{n+2}-138Q_{n+1}-182Q_n.$$

Using the last equation and Theorem 19 (a), we get required result.

(c) Take $W_n=E_n$ with $E_0=0,E_1=1,E_2=1,E_2=3$. Writing

$$P_n=a\times E_{n+3}+b\times E_{n+2}+c\times E_{n+1}+d\times E_n$$

and solving the system of equations

$$\begin{aligned} P_0 &= a \times E_3 + b \times E_2 + c \times E_1 + d \times E_0 \\ P_1 &= a \times E_4 + b \times E_3 + c \times E_2 + d \times E_1 \\ P_2 &= a \times E_5 + b \times E_4 + c \times E_3 + d \times E_2 \\ P_3 &= a \times E_6 + b \times E_5 + c \times E_4 + d \times E_3 \end{aligned}$$

we find that $a = \frac{1}{4}, b = -\frac{1}{4}, c = -\frac{1}{2}, d = \frac{1}{4}$ and so we get

$$4P_n = E_{n+3} - E_{n+2} - 2E_{n+1} + E_n.$$

Using the last equation and Theorem 19 (a), we get required result. □

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