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The Fuzzy Interpolative Berinde Weak Mapping Theorem in Metric Space

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Abstract

Motivated by [1], this paper obtains a fuzzy fixed point variant of the interpolative Berinde weak mapping theorem of [2] in the setting of complete metric spaces.

1 Introduction and Preliminaries

Definition 1.1. Let X be a nonempty set. A function $d : X \times X \mapsto \mathbb{R}^+$ is called a metric if it satisfies the following conditions, for all $x, y, z \in X$,

(a) $d(x, y) = 0$ iff $x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, z) \leq d(x, y) + d(y, z)$.

Moreover, the pair (X, d) is called a metric space.

Definition 1.2. Let (X, d) be a metric space, and $\{x_n\}$ be a sequence in X. We say

(a) $\{x_n\}$ is a convergent sequence iff there exists $x \in X$ such that for all $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$ such that for all $n \geq n(\epsilon)$, we have $d(x_n, x) < \epsilon$. In particular, we write $\lim_{n\to\infty} x_n = x$.

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(*b*) $\{x_n\}$ is called a Cauchy sequence iff for all $\epsilon > 0$ there is $n(\epsilon) \in \mathbb{N}$ such that for each $m, n \geq n(\epsilon)$, we have, $d(x_n, x_m) < \epsilon$.

Definition 1.3. [3] Let (X, d) be a metric space. The Hausdorff metric on $CB(X)$ induced by d is defined as

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), sup_{y \in B} d(A, y)\}$$

for all $A, B \in CB(X)$, where $CB(X)$ denotes the family of closed and bounded subsets of X and

$$d(x, B) = \inf\{d(x, a) : a \in B\}$$

for all $x \in X$.

Definition 1.4. [4] A fuzzy set in X is a function with domain X and values in $[0, 1]$.

Notation 1.5. $P(X)$ will denote the collection of all fuzzy sets in X.

Remark 1.6. If A is a fuzzy set and $x \in X$, then the function values $A(x)$ is called the grade of membership of x in A.

Definition 1.7. The α-level set of a fuzzy set A, is denoted by $[A]_\alpha$, and is defined as

$$[A]_\alpha = \{x : A(x) \geq \alpha\}$$

where $\alpha \in (0, 1]$,

$$[A]_0 = \overline{\{x : A(x) > 0\}}.$$

Definition 1.8. Let X be a nonempty set and Y be a metric space.

(*a*) A mapping T is called a fuzzy mapping, if T is a mapping from X into $P(X)$.

(*b*) A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. $T(x)(y)$ is the grade of membership of y in Tx.

Notation 1.9. The α-level set of Tx will be denoted $[Tx]_\alpha$.

Definition 1.10. A point $x \in X$ is called an α-fuzzy fixed point of a fuzzy mapping $T : X \mapsto P(X)$ if there exists $\alpha \in (0, 1]$ such that $x \in [Tx]_\alpha$.

Lemma 1.11. [4] *Let A and B be nonempty closed and bounded subsets of a metric space (X, d). If $a \in A$, then $d(a, B) \leq H(A, B)$.*

Lemma 1.12. [4] *Let A and B be nonempty closed and bounded subsets of a metric sapce (X, d), and $0 < \alpha \in \mathbb{R}$. Then for any $a \in A$, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \alpha$.*

2 Main Result

Theorem 2.1. *Let (X, d) be a complete metric space, and $T : X \mapsto P(X)$ be a fuzzy mapping, and for $x \in X$, there exists $\alpha(x) \in (0, 1]$ satisfying the following conditions*

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq \lambda d(x, y)^{\frac{1}{2}} d(x, [Tx]_{\alpha(x)})^{\frac{1}{2}}$$

for all $x, y \in X$, where $\lambda \in (0, 1)$, $x, y \notin Fix(T) = \{x \in X : x \in [Tx]_{\alpha(x)}, \alpha(x) \in (0, 1]\}$. Then the α-fuzzy fixed point of T exists.

Proof. Let x_0 be an arbitrary point in X such that $x_1 \in [Tx_0]_{\alpha(x_0)}$. By Lemma 1.12 there exists $x_2 \in [Tx_1]_{\alpha(x_1)}$ such that

$$\begin{aligned} d(x_1, x_2) &\leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) + \lambda \\ &\leq \lambda d(x_0, x_1)^{\frac{1}{2}} d(x_0, [Tx_0]_{\alpha(x_0)})^{\frac{1}{2}} + \lambda \\ &= \lambda d(x_0, x_1)^{\frac{1}{2}} d(x_0, x_1)^{\frac{1}{2}} + \lambda \\ &= \lambda d(x_0, x_1) + \lambda. \end{aligned}$$

Similarly, there exists $x_3 \in [Tx_2]_{\alpha(x_2)}$ such that

$$\begin{aligned} d(x_2, x_3) &\le H([Tx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}) + \lambda \\ &\le \lambda d(x_1, x_2)^{\frac{1}{2}} d(x_1, [Tx_1]_{\alpha(x_1)})^{\frac{1}{2}} + \lambda \\ &= \lambda d(x_1, x_2)^{\frac{1}{2}} d(x_1, x_2)^{\frac{1}{2}} + \lambda \\ &= \lambda d(x_1, x_2) + \lambda \\ &\le \lambda(\lambda d(x_0, x_1) + \lambda) + \lambda \\ &= \lambda^2 d(x_0, x_1) + \lambda^2 + \lambda \\ &\le \lambda^2 d(x_0, x_1) + 2\lambda^2. \end{aligned}$$

Continuing the same way by induction, we obtain a sequence $\{x_n\}$ such that $x_{n-1} \in [Tx_n]_{\alpha(x_n)}$ and $x_n \in [Tx_{n+1}]_{\alpha(x_{n+1})}$, and

$$d(x_n, x_{n+1}) \le \lambda^n d(x_0, x_1) + n\lambda^n.$$

Now for any positive integer m and n with $m > n$, we have

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\le \lambda^m d(x_0, x_1) + m\lambda^m + \cdots + \lambda^{n-1} d(x_0, x_1) + (n-1)\lambda^{n-1} \\ &\le \lambda^m (1 + \lambda + \cdots + \lambda^{n-m-1}) d(x_0, x_1) + \sum_{i=m}^{n-1} i\lambda^i \\ &\le \frac{\lambda^m}{1-\lambda} d(x_0, x_1) + \sum_{i=m}^{n-1} i\lambda^i. \end{aligned}$$

Since $\lambda < 1$, by Cauchy root test $\sum i\lambda^i$ is convergent, which implies $\{x_n\}$ is a Cauchy sequence in X. As X is complete there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Finally, we show existence of the α-fuzzy fixed point. Observe we have the following

$$\begin{aligned} d(z,[Tz]_{\alpha(z)}) &\leq d(z,x_{n+1}) + d(x_{n+1},[Tz]_{\alpha(z)}) \\ &= d(z,x_{n+1}) + H([Tx_n]_{\alpha(x_n)},[Tz]_{\alpha(z)}) \\ &\leq d(z,x_{n+1}) + \lambda d(x_n,z)^{\frac{1}{2}} d(x_n,[Tx_n]_{\alpha(x_n)})^{\frac{1}{2}} \\ &= d(z,x_{n+1}) + \lambda d(x_n,z)^{\frac{1}{2}} d(x_n,x_{n+1})^{\frac{1}{2}}. \end{aligned}$$

Taking limit in the above, we get that

$$d(z,[Tz]_{\alpha(z)}) \leq 0$$

which implies

$$z \in [Tz]_{\alpha(z)}.$$

Hence, it follows that $z \in X$ is an α-fuzzy fixed point of T, and the proof is finished. □

3 Open Problem

We begin with the following

Definition 3.1. Let $S,T : X \mapsto P(X)$ be two fuzzy mappings and for $x \in X$, there exists $\alpha_S(x), \alpha_T(x) \in (0,1]$. A point x is said to be an α-fuzzy common fixed point of S and T if $x \in [Sx]_{\alpha_S(x)} \cap [Tx]_{\alpha_T(x)}$.

The main goal is to prove or disprove the following

Conjecture 3.2. *Let (X,d) be a complete metric space. Let $S,T : X \mapsto P(X)$ be two fuzzy mappings, and for $x \in X$, there exists $\alpha_S(x), \alpha_T(x) \in (0,1]$ satisfying the following conditions*

$$H([Tx]_{\alpha_T(x)},[Sy]_{\alpha_S(y)}) \leq \lambda d(x,y)^{\frac{1}{2}} d(x,[Tx]_{\alpha_T(x)})^{\frac{1}{2}}$$

for all $x,y \in X$, $x,y \notin Fix(S \cap T) = \{x \in X : x \in [Sx]_{\alpha_S(x)} \cap [Tx]_{\alpha_T(x)}\}$, where $\lambda \in (0,1)$. Then S and T have a α-fuzzy common fixed point.

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