



# Cusa-Huygens, Wilker and Huygens Type Inequalities for Generalized Hyperbolic Functions

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## Abstract

In this paper, we establish Cusa-Huygens, Wilker and Huygens type inequalities for certain generalizations of the hyperbolic functions. From the established results, we recover some previous results as particular cases.

## 1 Introduction

The inequality

$$\frac{\sin z}{z} < \frac{\cos z + 2}{3}, \quad 0 < z < \frac{\pi}{2}, \quad (1)$$

is known in the literature as Cusa-Huygens inequality. Its hyperbolic counterpart, which is given as

$$\frac{\sinh z}{z} < \frac{\cosh z + 2}{3}, \quad x > 0, \quad (2)$$

was established by Neuman and Sandor [14]. The inequality

$$\left(\frac{\sin z}{z}\right)^2 + \frac{\tan z}{z} > 2, \quad 0 < z < \frac{\pi}{2}, \quad (3)$$

which is known as Wilker inequality was first proposed in the classic work [21, p.55] and subsequently attracted the attention of other researchers. In [23], Wu

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and Srivastava proved the Wilker-type inequality

$$\left(\frac{z}{\sin z}\right)^2 + \frac{z}{\tan z} > 2, \quad 0 < z < \frac{\pi}{2}. \quad (4)$$

The hyperbolic counterpart of (3) was established by Zu [24] as

$$\left(\frac{\sinh z}{z}\right)^2 + \frac{\tanh z}{z} > 2, \quad z \in \mathbb{R} \setminus \{0\}. \quad (5)$$

Also, the hyperbolic counterpart of (4) was established by Wu and Debnath [22] as

$$\left(\frac{z}{\sinh z}\right)^2 + \frac{z}{\tanh z} > 2, \quad z \in \mathbb{R} \setminus \{0\}. \quad (6)$$

Another inequality of interest is the Huygens inequality which is given as [17]

$$2\frac{\sin z}{z} + \frac{\tan z}{z} > 3, \quad 0 < z < \frac{\pi}{2}, \quad (7)$$

and its hyperbolic counterpart given as [14]

$$2\frac{\sinh z}{z} + \frac{\tanh z}{z} > 3, \quad z \in \mathbb{R} \setminus \{0\}. \quad (8)$$

Due to their usefulness, these elegant inequalities have been studied extensively and in diverse ways by several researchers. See for example [2], [3], [4], [5], [6], [7], [8], [12], [13], [15], [16], [18], [19], [20], [22], [24], [25], [26] and the related references therein.

Also, in a recent work, the Huygens-type inequality

$$\frac{2}{\cosh z} + \cosh z > \frac{\sinh z}{z} + 2\frac{\tanh z}{z} > \frac{1}{\cosh z} + 2, \quad z \in \mathbb{R} \setminus \{0\}, \quad (9)$$

was established among other things by Bagul and Chesneau [1].

Motivated by the results (2), (5), (6), (8) and (9), the objective of this paper is to establish analogous inequalities concerning certain generalizations of the hyperbolic functions. The established results serve as generalizations of the previous results.

## 2 Preliminary Definitions

In a bid to generalize a previous work [9], the authors of [10] gave the following generalizations of the hyperbolic functions.

**Definition 2.1.** The generalized hyperbolic cosine, hyperbolic sine and hyperbolic tangent functions are respectively defined as [10]

$$\cosh_a(z) = \frac{a^z + a^{-z}}{2}, \quad (10)$$

$$\sinh_a(z) = \frac{a^z - a^{-z}}{2}, \quad (11)$$

$$\tanh_a(z) = \frac{\sinh_a(z)}{\cosh_a(z)} = \frac{a^z - a^{-z}}{a^z + a^{-z}} = 1 - \frac{2}{1 + a^{2z}}, \quad (12)$$

where  $a > 1$  and  $z \in \mathbb{R}$ .

These generalized functions satisfy the following identities.

$$\cosh_a(z) + \sinh_a(z) = a^z, \quad (13)$$

$$\cosh_a(z) - \sinh_a(z) = a^{-z}, \quad (14)$$

$$(\cosh_a(z))' = (\ln a) \sinh_a(z), \quad (15)$$

$$(\sinh_a(z))' = (\ln a) \cosh_a(z), \quad (16)$$

$$(\tanh_a(z))' = \frac{\ln a}{\cosh_a^2(z)}, \quad (17)$$

$$(\cosh_a(z))'' + (\sinh_a(z))'' = (\ln a)^2 a^z, \quad (18)$$

$$(\cosh_a(z))'' - (\sinh_a(z))'' = (\ln a)^2 a^{-z}, \quad (19)$$

$$\cosh_a^2(z) + \sinh_a^2(z) = \cosh_a(2z), \quad (20)$$

$$\cosh_a^2(z) - \sinh_a^2(z) = 1, \quad (21)$$

$$2 \sinh_a(z) \cosh_a(z) = \sinh_a(2z), \quad (22)$$

$$\cosh_a^2(z) = \frac{\cosh_a(2z) + 1}{2}, \quad (23)$$

$$\sinh_a^2(z) = \frac{\cosh_a(2z) - 1}{2}. \quad (24)$$

The generalized hyperbolic secant, hyperbolic cosecant and hyperbolic cotangent functions are respectively defined as

$$\operatorname{sech}_a(z) = \frac{1}{\cosh_a(z)}, \quad \operatorname{cosech}_a(z) = \frac{1}{\sinh_a(z)}, \quad \operatorname{coth}_a(z) = \frac{1}{\tanh_a(z)}. \quad (25)$$

As pointed out in [10], several other identities can be derived from (10), (11) and (12). When  $a = e$ , where  $e = 2.71828\dots$  is the Euler's number, then the above definitions and identities reduce to their ordinary counterparts.

### 3 Results and Discussion

**Lemma 3.1.** *The inequality*

$$\cosh_a(z) < \left( \frac{\sinh_a(z)}{z} \right)^3, \quad (26)$$

holds for  $z \in \mathbb{R} \setminus \{0\}$ .

*Proof.* Inequality (26) has been proved in [11] for  $z > 0$ . Now let  $z < 0$  so that  $-z > 0$ . Then

$$\cosh_a(z) = \cosh_a(-z) < \left( \frac{\sinh_a(-z)}{-z} \right)^3 = \left( \frac{\sinh_a(z)}{z} \right)^3,$$

which completes the proof.  $\square$

Since the function  $\frac{\sinh_a(z)}{z}$  is increasing for  $z > 0$  and decreasing for  $z < 0$ , then Lemma 3.1 implies the following generalized result.

**Lemma 3.2.** *The inequality*

$$\cosh_a(z) < \left( \frac{\sinh_a(z)}{z} \right)^v, \quad (27)$$

holds for  $z \in \mathbb{R} \setminus \{0\}$  and  $v \geq 3$ .

**Lemma 3.3** ([11]). For  $z \in \mathbb{R} \setminus \{0\}$ , the inequality

$$\frac{\ln a}{\cosh_a(z)} < \frac{\sinh_a(z)}{z} < (\ln a) \cosh_a(z), \quad (28)$$

holds.

**Lemma 3.4** (Young's Inequality). Let  $x, y \geq 0$ ,  $r, s \in (0, 1)$  such that  $r + s = 1$ .

Then,

$$x^r y^s \leq rx + sy. \quad (29)$$

**Theorem 3.5.** The inequality

$$\frac{\sinh_a(z)}{z} < \frac{2 \ln a + (\ln a) \cosh_a(z)}{3}, \quad (30)$$

holds for  $z \in \mathbb{R} \setminus \{0\}$ .

*Proof.* Since the functions in each term of the inequality are even, it suffices to prove the case for  $z > 0$ . Let  $z > 0$  and  $h$  be defined as

$$h(z) = \frac{2z + z \cosh_a(z)}{\sinh_a(z)}.$$

Then

$$\begin{aligned} h'(z) &= \frac{1}{\sinh_a^2(z)} [2 \sinh_a(z) + \cosh_a(z) \sinh_a(z) - 2z(\ln a) \cosh_a(z) - (\ln a)z] \\ &= \frac{1}{\sinh_a^2(z)} \phi(z) \end{aligned}$$

and then

$$\begin{aligned} \phi'(z) &= (\ln a) [\cosh_a(2z) - 2(\ln a)z \sinh_a(z) - 1] \\ &= 2(\ln a) \sinh_a(z) [\sinh_a(z) - (\ln a)z] > 0, \end{aligned}$$

since  $\frac{\sinh_a(z)}{z} > \ln a$  for  $z \in (0, \infty)$ . Hence  $\phi(z)$  is increasing and consequently,  $\phi(z) > \phi(0) = 0$ . Thus  $h(z)$  is increasing. Hence

$$h(z) > \lim_{z \rightarrow 0} h(z) = \frac{3}{\ln a},$$

which gives (30). □

**Remark 3.6.** When  $a = e$ , then inequality (30) reduces to the hyperbolic Cusa-Huygens inequality (2).

**Theorem 3.7.** *The inequalities*

$$\left(\frac{\sinh_a(z)}{z}\right)^2 + \frac{\tanh_a(z)}{z} > 2, \quad (31)$$

$$\left(\frac{z}{\sinh_a(z)}\right)^2 + \frac{z}{\tanh_a(z)} > \frac{1 + \ln a}{(\ln a)^2}, \quad 1 < a \leq e, \quad (32)$$

hold for  $z \in \mathbb{R} \setminus \{0\}$ .

*Proof.* Let  $z \in \mathbb{R} \setminus \{0\}$ . Then by the AM-GM inequality and Lemma 3.1, we obtain

$$\begin{aligned} \left(\frac{\sinh_a(z)}{z}\right)^2 + \frac{\tanh_a(z)}{z} &\geq 2\sqrt{\left(\frac{\sinh_a(z)}{z}\right)^2 \frac{\tanh_a(z)}{z}} \\ &= 2\sqrt{\left(\frac{\sinh_a(z)}{z}\right)^3 \frac{1}{\cosh_a(z)}} \\ &> 2, \end{aligned}$$

which gives (31). To prove (32), it suffices to prove the case for  $z > 0$ . Let  $z > 0$  and define  $\psi(z)$  by

$$\psi(z) = \left(\frac{z}{\sinh_a(z)}\right)^2 + \frac{z}{\tanh_a(z)},$$

where  $1 < a \leq e$ . Then by differentiating, applying the AM-GM inequality and Lemma 3.1, we obtain

$$\begin{aligned} &\psi'(z) \\ &= \frac{1}{\sinh_a^3(z)} \left[ \sinh_a^2(z) \cosh_a(z) + (2 - \ln a)z \sinh_a(z) - 2(\ln a)z^2 \cosh_a(z) \right] \\ &\geq \frac{1}{\sinh_a^3(z)} \left[ 2\sqrt{\sinh_a^2(z) \cosh_a(z) \cdot (2 - \ln a)z \sinh_a(z)} - 2(\ln a)z^2 \cosh_a(z) \right] \\ &= \frac{2z^2}{\sinh_a^3(z)} \cdot \sqrt{\cosh_a(z)} \left[ \sqrt{2 - \ln a} \sqrt{\left(\frac{\sinh_a(z)}{z}\right)^3} - (\ln a) \sqrt{\cosh_a(z)} \right] \\ &> 0. \end{aligned}$$

Thus  $\psi(z)$  is increasing. Hence

$$\psi(z) > \lim_{z \rightarrow 0} \psi(z) = \frac{1 + \ln a}{(\ln a)^2},$$

which gives (32). □

**Remark 3.8.** When  $a = e$ , then the inequalities (31) and (31) reduce to the hyperbolic Wilker-type inequalities (5) and (6) respectively.

**Theorem 3.9.** *The inequality*

$$\left(\frac{\sinh_a(z)}{z}\right)^2 + \frac{\tanh_a(z)}{z} > \left(\frac{z}{\sinh_a(z)}\right)^2 + \frac{z}{\tanh_a(z)}, \quad (33)$$

holds for  $z \in \mathbb{R} \setminus \{0\}$ .

*Proof.* Using the fact that  $(A^2 + B)/(1/A^2 + 1/B) = A^2B$ , together with Lemma 3.1, we obtain

$$\begin{aligned} \frac{\left(\frac{\sinh_a(z)}{z}\right)^2 + \frac{\tanh_a(z)}{z}}{\left(\frac{z}{\sinh_a(z)}\right)^2 + \frac{z}{\tanh_a(z)}} &= \left(\frac{\sinh_a(z)}{z}\right)^2 \frac{\tanh_a(z)}{z} \\ &= \left(\frac{\sinh_a(z)}{z}\right)^3 \frac{1}{\cosh_a(z)} \\ &> 1, \end{aligned}$$

which concludes the proof. □

**Theorem 3.10.** *Let  $\alpha, \beta \in (0, 1)$  such that  $\alpha + \beta = 1$ . Then the inequality*

$$\alpha \left(\frac{\sinh_a(z)}{z}\right)^2 + \beta \left(\frac{\tanh_a(z)}{z}\right) > (\ln a)^{2(\alpha-\beta)}, \quad (34)$$

holds for  $z \in \mathbb{R} \setminus \{0\}$ .

*Proof.* Let  $z \in \mathbb{R} \setminus \{0\}$ . Then Youngs inequality (29) and Lemma 3.1 imply that

$$\begin{aligned} \alpha \left( \frac{\sinh_a(z)}{z} \right)^2 + \beta \left( \frac{\tanh_a(z)}{z} \right) &\geq \left( \frac{\sinh_a(z)}{z} \right)^{2\alpha} \left( \frac{\tanh_a(z)}{z} \right)^\beta \\ &= \left( \frac{\sinh_a(z)}{z} \right)^{2\alpha+\beta} \left( \frac{1}{\cosh_a(z)} \right)^\beta \\ &> \left( \frac{\sinh_a(z)}{z} \right)^{2\alpha+\beta} \left( \frac{\sinh_a(z)}{z} \right)^{-3\beta} \\ &= \left( \frac{\sinh_a(z)}{z} \right)^{2(\alpha-\beta)} \\ &> (\ln a)^{2(\alpha-\beta)}, \end{aligned}$$

which completes the proof.  $\square$

**Remark 3.11.** If  $\alpha = \beta = \frac{1}{2}$ , then (34) reduces to (31).

**Theorem 3.12.** *The inequality*

$$2 \frac{\sinh_a(z)}{z} + \frac{\tanh_a(z)}{z} > 3 \ln a, \quad (35)$$

holds for  $z \in \mathbb{R} \setminus \{0\}$ .

*Proof.* It suffices to prove the case for  $z > 0$ . Let  $z > 0$  and let  $h$  be defined as

$$h(z) = 2 \frac{\sinh_a(z)}{z} + \frac{\tanh_a(z)}{z}.$$

Then

$$\begin{aligned} z^2 h'(z) &= 2(\ln a)z \cosh_a(z) - 2 \sinh_a(z) + (\ln a)z \operatorname{sech}_a^2(z) - \tanh_a(z) \\ &= \theta(z), \end{aligned}$$

and

$$\begin{aligned} \theta'(z) &= 2(\ln a)^2 z \sinh_a(z) - 2(\ln a)^2 z \tanh_a(z) \operatorname{sech}_a^2(z) \\ &= 2(\ln a)^2 z \sinh_a(z) \left[ 1 - \frac{1}{\cosh_a^3(z)} \right] > 0, \end{aligned}$$



which shows that  $\theta(z)$  is increasing. Hence  $\theta(z) > \theta(0) = 0$ . Thus,  $h(z)$  is increasing and consequently,

$$h(z) > \lim_{z \rightarrow 0} h(z) = 3 \ln a$$

which yields (35). □

**Remark 3.13.** When  $a = e$ , then inequality (35) reduces to (8).

**Theorem 3.14.** *The inequality*

$$2 \frac{z}{\sinh_a(z)} + \frac{z}{\tanh_a(z)} > \frac{3}{\ln a}, \quad (36)$$

holds for  $z \in \mathbb{R} \setminus \{0\}$ .

*Proof.* It suffices to prove the case for  $z > 0$ . Let  $z > 0$  and  $\delta$  be defined as

$$\delta(z) = 2 \frac{z}{\sinh_a(z)} + \frac{z}{\tanh_a(z)}.$$

Then

$$\delta'(z) = \frac{1}{\sinh_a^2(z)} [2 \sinh_a(z) - z \ln a + \cosh_a(z)(\sinh_a(z) - z \ln a)] > 0$$

since  $\sinh_a(z) > z \ln a$ . Hence  $\delta(z)$  is increasing and consequently,

$$\delta(z) > \lim_{z \rightarrow 0} \delta(z) = \frac{3}{\ln a}$$

which yields (36). □

**Theorem 3.15.** *The inequality*

$$(m+1) \frac{\sinh_a(z)}{z} + m \frac{\tanh_a(z)}{z} > m \frac{\sinh_a(z)}{z} + (m+1) \frac{\tanh_a(z)}{z}, \quad (37)$$

holds for  $z \in \mathbb{R} \setminus \{0\}$  and  $m \in \mathbb{N} \cup \{0\}$ .

*Proof.*  $z \in \mathbb{R} \setminus \{0\}$ . Then

$$\frac{\sinh_a(z)}{z} - \frac{\tanh_a(z)}{z} = \frac{\sinh_a(z)}{z} \left[ 1 - \frac{1}{\cosh_a(z)} \right] > 0,$$

since  $\cosh_a(z) > 1$  for  $z \in \mathbb{R}$ . That is

$$\frac{\sinh_a(z)}{z} > \frac{\tanh_a(z)}{z}. \quad (38)$$

Adding  $m \left( \frac{\sinh_a(z)}{z} + \frac{\tanh_a(z)}{z} \right)$  to both sides of (38) completes the proof.  $\square$

**Theorem 3.16.** *The inequality*

$$\frac{2 \ln a}{\cosh_a(z)} + (\ln a) \cosh_a(z) > \frac{\sinh_a(z)}{z} + 2 \frac{\tanh_a(z)}{z} > \frac{\ln a}{\cosh_a(z)} + 2 \ln a, \quad (39)$$

holds for  $z \in \mathbb{R} \setminus \{0\}$ .

*Proof.* It suffices to prove the case for  $z > 0$ . Let  $z > 0$  and  $f$  be defined as

$$f(z) = 2(\ln a)z + (\ln a)z \cosh_a^2(z) - \sinh_a(z) \cosh_a(z) - 2 \sinh_a(z).$$

Then

$$f'(z) = 2(\ln a) + 2(\ln a)^2 z \cosh_a(z) \sinh_a(z) - (\ln a) \sinh_a^2(z) - 2(\ln a) \cosh_a(z),$$

and then

$$f''(z) = 2(\ln a)^3 z \sinh_a^2(z) + 2(\ln a)^2 \left[ (\ln a)z \cosh_a^2(z) - \sinh_a(z) \right] > 0,$$

since  $\sinh_a(z) < (\ln a)z \cosh_a(z)$  (see Lemma 3.3) and  $\cosh_a(z) < \cosh_a^2(z)$ . Hence  $f'(z)$  is increasing and so,  $f'(z) > f'(0) = 0$ . Thus,  $f(z)$  is increasing and so  $f(z) > f(0) = 0$ . This yields the left-hand side of (39). Next, for  $z > 0$ , let  $g$  be defined as

$$g(z) = \sinh_a(z) \cosh_a(z) + 2 \sinh_a(z) - (\ln a)z - 2(\ln a)z \cosh_a(z).$$

Then

$$g'(z) = (\ln a) \cosh_a^2(z) + (\ln a) \sinh_a^2(z) - \ln a - 2(\ln a)^2 z \sinh_a(z)$$

and then

$$g''(z) = 2(\ln a)^2 \cosh_a(z) [\sinh_a(z) - (\ln a)z] + 2(\ln a)^2 \sinh_a(z) [\cosh_a(z) - 1] > 0,$$

since  $\sinh_a(z) > (\ln a)z$  and  $\cosh_a(z) > 1$ . Hence  $g'(z)$  is increasing and so,  $g'(z) > g'(0) = 0$ . Thus,  $g(z)$  is increasing and so  $g(z) > g(0) = 0$ . This yields the right-hand side of (39) and that completes the proof.  $\square$

**Remark 3.17.** When  $a = e$ , then inequality (39) reduces to (9).

**Remark 3.18.** By (37) and (39), we have

$$2 \frac{\sinh_a(z)}{z} + \frac{\tanh_a(z)}{z} > \frac{\ln a}{\cosh_a(z)} + 2 \ln a.$$

This is however weaker than inequality (35).

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