The Interpolative Berinde Weak Mapping Theorem in 
$\eta$-Cone Pentagonal Metric Space

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Abstract

In this paper we introduce a concept of $\eta$-cone pentagonal metric space, which combines the notions of cone pentagonal metric space [1], and $\eta$-cone metric space [2]. Moreover, a variant of the interpolative Berinde weak mapping theorem obtained in [3] is proved in this setting.

1 Introduction and Preliminaries

Definition 1.1. [4] Let $E$ be a real Banach space with norm $\| \cdot \|$ and $P$ be a subset of $E$. Then $P$ is called a cone if and only if

(a) $P$ is closed, nonempty, and $P \neq \{ \theta \}$, where $\theta$ is the zero vector in $E$;

(b) for any nonnegative real numbers $a$ and $b$, and $x, y \in P$, we have $ax + by \in P$;

(c) for $x \in P$, if $-x \in P$, then $x = \theta$.

Definition 1.2. [4] Given a cone $P$ in a Banach space $E$, we define on $E$ a partial order $\preceq$ with respect to $P$ by

$$x \preceq y \iff y - x \in \text{int}(P).$$

We shall write $x \prec y$ whenever $x \preceq y$ and $x \neq y$. While $x \ll y$ will stand for $y - x \in \text{Int}(P)$, where $\text{Int}(P)$ designates the interior of $P$.
Definition 1.3. The cone $P$ is said to be normal if there is a real number $C > 0$ such that for all $x, y \in E$, we have
\[ \theta \preceq x \preceq y \implies \|x\| \leq C\|y\|. \]
The least positive number satisfying the above inequality is called the normal constant of $P$. In particular, we will say that $P$ is a $K$-normal cone to indicate the fact that the normal constant is $K$.

Definition 1.4. Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \mapsto E$ satisfies
\begin{itemize}
  \item[(a)] $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$;
  \item[(b)] $d(x, y) = d(y, x)$ for all $x, y \in X$;
  \item[(c)] $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, y)$ for all $x, y \in X$ and for all distinct points $z, w, u \in X - \{x, y\}$ [pentagonal property].
\end{itemize}
Then $d$ is called a cone pentagonal metric on $X$, and $(X, d)$ is called a cone pentagonal metric space.

Example 1.5. Let $X = \{r, s, t, u, v\}$, $E = \mathbb{R}^2$, and $P = \{(x, y) : x, y \geq 0\}$ be a cone in $E$. Define $d : X \times X \mapsto E$ as follows
\begin{align*}
  d(x, x) &= 0, \text{ for all } x \in X, \\
  d(r, s) &= d(s, r) = (4, 8), \\
  d(r, t) &= d(t, r) = d(t, u) = d(u, t) = d(s, t) \\
  &= d(t, s) = d(s, u) = d(u, s) = d(r, u) = d(u, r) = (1, 2), \\
  d(r, v) &= d(v, r) = d(s, v) = d(v, s) = d(t, v) = d(v, t) = d(u, v) = d(v, u) = (3, 6).
\end{align*}
Then $(X, d)$ is a complete cone pentagonal metric space, but not a complete cone rectangular metric space, as it lacks the triangular property:
\[ (4, 8) = d(r, s) > d(r, t) + d(t, u) + d(u, s) \\
= (1, 2) + (1, 2) + (1, 2) = (3, 6) \]
as $(4, 8) - (3, 6) = (1, 2) \in P$. 

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Definition 1.6. [2] Let $X$ be a nonempty set and $\eta : X \times X \mapsto [1, \infty)$ be a map. A function $d_\eta : X \times X \mapsto E$ will be called an $\eta$-cone metric on $X$ if

(a) $\theta \leq d_\eta(x, y)$ for all $x \in X$ and $d_\eta(x, y) = \theta$ iff $x = y$;

(b) $d_\eta(x, y) = d_\eta(y, x)$ for all $x, y \in X$;

(c) $d_\eta(x, z) \leq \eta(x, z)[d_\eta(x, y) + d_\eta(y, z)]$ for all $x, y, z \in X$.

Moreover the pair $(X, d_\eta)$ is called an $\eta$-cone metric space.

Remark 1.7. [2] If for all $x, y \in X$

(a) $\eta(x, y) = 1$, then we obtain the definition of cone metric space [4].

(b) $\eta(x, y) = L$, where $L \geq 1$, then we obtain the definition of cone metric type space [6].

(c) $\eta(x, y) = C$, where $C \geq 1$, $E = \mathbb{R}$ and $P = [0, \infty)$, then we obtain the definition of metric type space [7].

Example 1.8. [2] Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subseteq \mathbb{R}^2$ and $X = \{1, 2, 3\}$. Let $\alpha \geq 0$ be a constant and define $\eta : X \times X \mapsto [1, \infty)$ and $d_\eta : X \times X \mapsto E$ by

\[
\eta(x, y) = 1 + x + y,
\]

\[
d_\eta(1, 1) = d_\eta(2, 2) = d_\eta(3, 3) = (0, 0),
\]

\[
d_\eta(1, 2) = d_\eta(2, 1) = 80(1, \alpha),
\]

\[
d_\eta(1, 3) = d_\eta(3, 1) = 1000(1, \alpha),
\]

\[
d_\eta(2, 3) = d_\eta(3, 2) = 600(1, \alpha).
\]

Then $(X, d_\eta)$ is an $\eta$-cone metric space.

Now we introduce the following

Definition 1.9. Let $X$ be a nonempty set and $\eta : X \times X \mapsto [1, \infty)$ be a map. A function $d_\eta : X \times X \mapsto E$ will be called an $\eta$-cone pentagonal metric on $X$ if
(a) \( \theta \leq d_\eta(x, y) \) for all \( x, y \in X \) and \( d_\eta(x, y) = \theta \) iff \( x = y \);

(b) \( d_\eta(x, y) = d_\eta(y, x) \) for all \( x, y \in X \);

(c) \( d_\eta(x, y) \leq \eta(x, y)[d_\eta(x, w) + d_\eta(w, z) + d_\eta(z, v) + d_\eta(v, y)] \) for all \( x, y \in X \) and for all distinct points \( w, z, v \in X - \{x, y\} \).

Moreover the pair \((X, d_\eta)\) will be called an \( \eta \)-cone pentagonal metric space.

**Example 1.10.** Let \( X = \{r, s, t, u, v\} \), \( E = \mathbb{R}^2 \), and \( P = \{(x, y) : x, y \geq 0\} \) be a cone in \( E \). Define \( d_\eta : X \times X \mapsto E \) as follows

\[
\begin{align*}
d_\eta(x, x) &= (0, 0), \text{ for all } x \in X, \\
d_\eta(r, s) &= d_\eta(s, r) = (4, 8), \\
d_\eta(r, t) &= d_\eta(t, r) = d_\eta(t, u) = d_\eta(u, t) = d_\eta(t, s) = d_\eta(s, u) = d_\eta(u, s) \\
&= d_\eta(r, u) = d_\eta(u, r) = (1, 2), \\
d_\eta(r, v) &= d_\eta(v, r) = d_\eta(s, v) = d_\eta(v, s) = d_\eta(t, v) = d_\eta(v, t) \\
&= d_\eta(u, v) = d_\eta(v, u) = (3, 6).
\end{align*}
\]

Also define \( \eta : X \times X \mapsto [1, \infty) \) as follows

\[
\begin{align*}
\eta(x, x) &= 0, \text{ for all } x \in X, \\
\eta(r, s) &= \eta(s, r) = \frac{1}{2}, \\
\eta(r, t) &= \eta(t, r) = \eta(t, u) = \eta(u, t) = \eta(s, t) = \eta(t, s) = \eta(s, u) = \eta(u, s) \\
&= \eta(r, u) = \eta(u, r) = \frac{1}{11}, \\
\eta(r, v) &= \eta(v, r) = \eta(s, v) = \eta(v, s) = \eta(t, v) = \eta(v, t) = \eta(u, v) \\
&= \eta(v, u) = \frac{1}{3}.
\end{align*}
\]

By definition of \( d_\eta \), it is trivial to check Definition 1.9(a), and Definition 1.9(b).

Now we check Definition 1.9(c). Obviously we have the following cases
Case 1: \((x, x)\) for all \(x \in X\)

In this case it is enough to check

\[
d_\eta(r, r) \leq \eta(r, r) [d_\eta(r, s) + d_\eta(s, t) + d_\eta(t, u) + d_\eta(u, r)].
\]

Since \(d_\eta(r, r) = (0, 0)\) and \(\eta(r, r) = 0\), we have equality. In particular, we have

\[
(0, 0) = d_\eta(r, r) \leq \eta(r, r) [d_\eta(r, s) + d_\eta(s, t) + d_\eta(t, u) + d_\eta(u, r)]
= 0[d_\eta(r, s) + d_\eta(s, t) + d_\eta(t, u) + d_\eta(u, r)] = (0, 0).
\]

Case 2: \((r, s) = (s, r)\)

In this case it is enough to check

\[
d_\eta(r, s) \leq \eta(r, s) [d_\eta(r, t) + d_\eta(s, u) + d_\eta(v, u) + d_\eta(v, s)].
\]

In particular, we have

\[
(4, 8) = d_\eta(r, s)
\leq \eta(r, s) [d_\eta(r, t) + d_\eta(s, u) + d_\eta(u, v) + d_\eta(v, s)]
= \frac{1}{2}[(1, 2) + (1, 2) + (3, 6) + (3, 6)]
= \frac{1}{2}(8, 16) = (4, 8).
\]

Case 3: \((r, t) = (t, r) = (t, u) = (u, t) = (s, t) = (t, s) = (s, u) = (u, s) = (r, u) = (u, r)\)

In this case it is enough to check

\[
d_\eta(r, t) \leq \eta(r, t) [d_\eta(r, s) + d_\eta(s, u) + d_\eta(u, v) + d_\eta(v, t)].
\]
In particular, we have
\[\begin{align*}
(1, 2) &= d_\eta(r, t) \\
&\leq \eta(r, t)[d_\eta(r, s) + d_\eta(s, u) + d_\eta(u, v) + d_\eta(v, t)] \\
&= \frac{1}{11}(4, 8) + (1, 2) + (3, 6) + (3, 6) = \frac{1}{11}(11, 22) = (1, 2).
\end{align*}\]

Case 4: \((r, v) = (v, r) = (s, v) = (t, v) = (u, v) = (v, u)\)

In this case it is enough to check that
\[d_\eta(r, v) \leq \eta(r, v)[d_\eta(r, s) + d_\eta(s, t) + d_\eta(t, u) + d_\eta(u, v)].\]

In particular, we have
\[\begin{align*}
(3, 6) &= d_\eta(r, v) \\
&\leq \eta(r, v)[d_\eta(r, s) + d_\eta(s, t) + d_\eta(t, u) + d_\eta(u, v)] \\
&= \frac{1}{3}(4, 8) + (1, 2) + (1, 2) + (3, 6)] = \frac{1}{3}(9, 18) = (3, 6).
\end{align*}\]

It follows that \((X, d_\eta)\) is an \(\eta\)-cone pentagonal metric space. Note that \((X, d_\eta)\) is not an \(\eta\)-cone rectangular metric space, as it lacks Definition 1.9(c) \[8\]. In particular,
\[\begin{align*}
(4, 8) &= d_\eta(r, s) > \eta(r, s)[d_\eta(r, r) + d_\eta(r, t) + d_\eta(t, s)] \\
&= \frac{1}{2}[(0, 0) + (1, 2) + (1, 2)] \\
&= \frac{1}{2}(2, 4) = (1, 2)
\end{align*}\]
as \((4, 8) - (1, 2) = (3, 6) \in P.\]

**Definition 1.11.** Let \((X, d_\eta)\) be a \(\eta\)-cone pentagonal metric space. Let \(\{x_n\}\) be a sequence in \((X, d_\eta)\) and \(x \in X\). If for every \(c \in E\) with \(0 \ll c\), there exists \(n_0 \in \mathbb{N}\) such that for all \(n > n_0\), \(d_\eta(x_n, c) \ll c\), then we say \(\{x_n\}\) is convergent, and \(\{x_n\}\) converges to \(x\). Moreover, \(x\) will be called the limit of \(\{x_n\}\). We sometimes write \(\lim_{n \to \infty} x_n = x\), or \(x_n \to x\) as \(n \to \infty\).
Definition 1.12. Let \((X,d)\) be a \(\eta\)-cone pentagonal metric space. If for every \(c \in E\), with \(0 \ll c\), there exists \(n_0 \in \mathbb{N}\) such that for all \(n > n_0\), \(d(x_n, x) \ll c\), then we say \(\{x_n\}\) is a Cauchy sequence in \((X,d)\).

Definition 1.13. Let \((X,d)\) be a \(\eta\)-cone pentagonal metric space. If every Cauchy sequence is convergent in \((X,d)\), then \((X,d)\) will be called a complete \(\eta\)-cone pentagonal metric space.

Notation 1.14. Let \(P\) be a cone as defined in this paper. \(\Phi\) will denote the set of all nondecreasing continuous functions \(\varphi : P \to P\) satisfying

(a) \(0 < \varphi(t) < t\) for all \(t \in P\)\(\backslash\{0\}\);

(b) the series \(\sum_{n \geq 0} \varphi^n(t)\) converges for all \(t \in P\)\(\backslash\{0\}\).

Note that from (a), we have \(\varphi(0) = 0\), and from (b) we have \(\lim_{n \to \infty} \varphi^n(t) = 0\) for all \(t \in P\)\(\backslash\{0\}\).

Definition 1.15. Let \((X,d)\) be a \(\eta\)-cone pentagonal metric space. A map \(T : X \to X\) will be called an (alternate) Interpolative Berinde Weak operator if it satisfies

\[d(Tx, Ty) \leq \lambda \frac{1}{\sqrt{2}} d(x, y) \sqrt{d(x, Tx)}\]

where \(\lambda \in (0,1)\), for all \(x, y \in X\), \(x, y \notin \text{Fix}(T)\), where \(\text{Fix}(T) = \{x \in X : Tx = x\}\).

2 Main Results

Our main result is as follows

Theorem 2.1. Let \((X,d)\) be a \(\eta\)-cone pentagonal metric space. Suppose \(T : X \to X\) satisfies

\[d(Tx, Ty) \leq \varphi \left( \frac{1}{\sqrt{2}} d(x, y) \sqrt{d(x, Tx)} \right)\]

for all \(x, y \in X\), \(x, y \notin \text{Fix}(T)\), and \(\varphi \in \Phi\). If \((X,d)\) is complete and

\[\lim_{n,m \to \infty} \eta(x_0, x_m)\]

exists and is finite, then the fixed point of \(T\) exists.
Proof. Let $x_0$ be an arbitrary point in $X$. Define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$ for all $n = 0, 1, 2, \ldots$. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Now observe from the inequality of the theorem, we have

$$d_\eta(x_n, x_{n+1}) = d_\eta(Tx_n, Tx_{n+1})$$

$$\leq \varphi\left( \frac{1}{\sqrt{2}} d_\eta(x_{n-1}, x_n)^{\frac{1}{2}} d_\eta(x_{n-1}, Tx_n)^{\frac{1}{2}} \right)$$

$$= \varphi\left( \frac{1}{\sqrt{2}} d_\eta(x_{n-1}, x_n)^{\frac{1}{2}} d_\eta(x_{n-1}, x_n)^{\frac{1}{2}} \right)$$

$$< \varphi\left( d_\eta(x_{n-1}, x_n) \right)$$

$$= \varphi\left( d_\eta(Tx_{n-2}, Tx_{n-1}) \right)$$

$$\leq \varphi\left( \varphi\left( d_\eta(x_{n-2}, x_{n-1}) \right) \right)$$

$$= \varphi^2\left( d_\eta(x_{n-2}, x_{n-1}) \right)$$

$$\vdots$$

$$\leq \varphi^n\left( d_\eta(x_0, x_1) \right).$$

Now observe we have the following

$$d_\eta(x_n, x_{n+2}) = d_\eta(Tx_n, Tx_{n+1})$$

$$\leq \varphi\left( \frac{1}{\sqrt{2}} d_\eta(x_{n-1}, x_{n+1})^{\frac{1}{2}} d_\eta(x_{n-1}, Tx_{n-1})^{\frac{1}{2}} \right)$$

$$= \varphi\left( \frac{1}{\sqrt{2}} d_\eta(x_{n-1}, x_{n+1})^{\frac{1}{2}} d_\eta(x_{n-1}, x_{n+1})^{\frac{1}{2}} \right)$$

$$\leq \varphi\left( \frac{1}{\sqrt{2}} d_\eta(x_{n-1}, x_{n+1})^{\frac{1}{2}} 2^{\frac{1}{2}} d_\eta(x_{n-1}, x_{n+1})^{\frac{1}{2}} \right)$$

$$= \varphi\left( d_\eta(x_{n-1}, x_{n+1}) \right)$$

$$= \varphi\left( d_\eta(Tx_{n-2}, Tx_{n-1}) \right)$$

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\[ \leq \varphi \left( \varphi \left( d_{\eta}(x_{n-2}, x_n) \right) \right) \]
\[ = \varphi^2 \left( d_{\eta}(x_{n-2}, x_n) \right) \]
\[ \vdots \]
\[ \leq \varphi^n \left( d_{\eta}(x_0, x_2) \right). \]

Further we deduce the following

\[
d_{\eta}(x_n, x_{n+3}) = d_{\eta}(Tx_{n-1}, Tx_{n+2}) \leq \varphi \left( \frac{1}{\sqrt{2}} d_{\eta}(x_{n-1}, x_{n+2}) \right)^{\frac{1}{2}} d_{\eta}(x_{n-1}, Tx_{n-1})^{\frac{1}{2}} \]
\[ = \varphi \left( \frac{1}{\sqrt{2}} d_{\eta}(x_{n-1}, x_{n+2}) \right)^{\frac{1}{2}} d_{\eta}(x_{n-1}, x_n)^{\frac{1}{2}} \]
\[ \leq \varphi \left( \frac{1}{\sqrt{2}} d_{\eta}(x_{n-1}, x_{n+2}) \right)^{\frac{1}{2}} 2^{\frac{1}{2}} d_{\eta}(x_{n-1}, x_{n+2})^{\frac{1}{2}} \]
\[ = \varphi \left( d_{\eta}(x_{n-1}, x_{n+2}) \right) \]
\[ = \varphi \left( d_{\eta}(Tx_{n-2}, Tx_{n+1}) \right) \]
\[ \leq \varphi \left( \varphi \left( d_{\eta}(x_{n-2}, x_{n+1}) \right) \right) \]
\[ = \varphi^2 \left( d_{\eta}(x_{n-2}, x_{n+1}) \right) \]
\[ \vdots \]
\[ \leq \varphi^n \left( d_{\eta}(x_0, x_3) \right). \]

Now for \( k = 1, 2, 3, \cdots \) we have

\[
d_{\eta}(x_n, x_{n+3k+1}) \leq \varphi^n \left( d_{\eta}(x_0, x_{3k+1}) \right)\]
\[d_\eta(x_n, x_{n+3k+2}) \leq \varphi^n(d_\eta(x_0, x_{3k+2}))\]

\[d_\eta(x_n, x_{n+3k+3}) \leq \varphi^n(d_\eta(x_0, x_{3k+3})).\]

Since \(d_\eta(x_n, x_{n+1}) \leq \varphi^n\left(d_\eta(x_0, x_1)\right)\), we deduce the following using the \(\eta\)-cone pentagonal property

\[d_\eta(x_0, x_4) \leq \eta(x_0, x_4)\left[d_\eta(x_0, x_1) + d_\eta(x_1, x_2) + d_\eta(x_2, x_3) + d_\eta(x_3, x_4)\right]\]

\[\leq \eta(x_0, x_4) \left[d_\eta(x_0, x_1) + \varphi\left(d_\eta(x_0, x_1)\right) + \varphi^2\left(d_\eta(x_0, x_1)\right) + \varphi^3\left(d_\eta(x_0, x_1)\right)\right]\]

\[\leq \eta(x_0, x_4) \left[\sum_{i=0}^{3} \varphi^i\left(d_\eta(x_0, x_1)\right)\right].\]

Similarly, we can show the following

\[d_\eta(x_0, x_7) \leq \eta(x_0, x_7) \left[\sum_{i=0}^{6} \varphi^i\left(d_\eta(x_0, x_1)\right)\right].\]

By induction for \(k = 1, 2, 3, \ldots\), we have

\[d_\eta(x_0, x_{3k+1}) \leq \eta(x_0, x_{3k+1}) \left[\sum_{i=0}^{3k} \varphi^i\left(d_\eta(x_0, x_1)\right)\right].\]

Since \(d_\eta(x_n, x_{n+1}) \leq \varphi^n\left(d_\eta(x_0, x_1)\right)\) and \(d_\eta(x_n, x_{n+2}) \leq \varphi^n\left(d_\eta(x_0, x_2)\right)\), we deduce the following using the \(\eta\)-cone pentagonal property

\[d_\eta(x_0, x_5) \leq \eta(x_0, x_5)\left[d_\eta(x_0, x_1) + d_\eta(x_1, x_2) + d_\eta(x_2, x_3) + d_\eta(x_3, x_5)\right]\]

\[\leq \eta(x_0, x_5) \left[d_\eta(x_0, x_1) + \varphi\left(d_\eta(x_0, x_1)\right) + \varphi^2\left(d_\eta(x_0, x_1)\right) + \varphi^3\left(d_\eta(x_0, x_2)\right)\right]\]

\[\leq \eta(x_0, x_5) \left[\sum_{i=0}^{2} \varphi^i\left(d_\eta(x_0, x_1)\right) + \varphi^3\left(d_\eta(x_0, x_2)\right)\right].\]
Similarly, we can show the following

\[
d\eta(x_0, x_8) \leq \eta(x_0, x_8) \left[ \sum_{i=0}^{5} \varphi^i \left( d\eta(x_0, x_1) \right) + \varphi^6 \left( d\eta(x_0, x_2) \right) \right].
\]

By induction for \( k = 1, 2, 3, \cdots \), we have

\[
d\eta(x_0, x_{3k+2}) \leq \eta(x_0, x_{3k+2}) \left[ \sum_{i=0}^{3k-1} \varphi^i \left( d\eta(x_0, x_1) \right) + \varphi^{3k} \left( d\eta(x_0, x_2) \right) \right].
\]

Since \( d\eta(x_n, x_{n+1}) \leq \varphi^n \left( d\eta(x_0, x_1) \right) \) and \( d\eta(x_n, x_{n+3}) \leq \varphi^n \left( d\eta(x_0, x_3) \right) \), we deduce the following using the \( \eta \)-cone pentagonal property

\[
d\eta(x_0, x_6) \leq \eta(x_0, x_6) \left[ \sum_{i=0}^{2} \varphi^i \left( d\eta(x_0, x_1) \right) + \varphi^{3} \left( d\eta(x_0, x_3) \right) \right].
\]

Similarly, we can show the following

\[
d\eta(x_0, x_9) \leq \eta(x_0, x_9) \left[ \sum_{i=0}^{5} \varphi^i \left( d\eta(x_0, x_1) \right) + \varphi^6 \left( d\eta(x_0, x_3) \right) \right].
\]

By induction for \( k = 1, 2, 3, \cdots \), we have

\[
d\eta(x_0, x_{3k+3}) \leq \eta(x_0, x_{3k+3}) \left[ \sum_{i=0}^{3k-1} \varphi^i \left( d\eta(x_0, x_1) \right) + \varphi^{3k} \left( d\eta(x_0, x_3) \right) \right].
\]

Since

\[
d\eta(x_n, x_{n+3k+1}) \leq \varphi^n \left( d\eta(x_0, x_{3k+1}) \right)
\]

and

\[
d\eta(x_0, x_{3k+1}) \leq \eta(x_0, x_{3k+1}) \left[ \sum_{i=0}^{3k} \varphi^i \left( d\eta(x_0, x_1) \right) \right],
\]
we deduce the following

\[ d_\eta(x_n, x_{n+3k+1}) \leq \varphi^n(d_\eta(x_0, x_{3k+1})) \]

\[ \leq \varphi^n\left( \eta(x_0, x_{3k+1}) \left[ \sum_{i=0}^{3k-1} \varphi^i\left( d_\eta(x_0, x_1) \right) \right] \right) \]

\[ \leq \varphi^n\left( \eta(x_0, x_{3k+1}) \left[ \sum_{i=0}^{3k-1} \varphi^i\left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \right] \right) \]

\[ \leq \varphi^n\left( \eta(x_0, x_{3k+1}) \left[ \sum_{i=0}^{3k-1} \varphi^i\left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \right] \right). \]

Since \( d_\eta(x_n, x_{n+3k+2}) \leq \varphi^n\left( d_\eta(x_0, x_{3k+2}) \right) \) and

\[ d_\eta(x_0, x_{3k+2}) \leq \eta(x_0, x_{3k+2}) \left[ \sum_{i=0}^{3k-1} \varphi^i\left( d_\eta(x_0, x_1) \right) + \varphi^{3k}\left( d_\eta(x_0, x_2) \right) \right], \]

we deduce the following

\[ d_\eta(x_n, x_{n+3k+2}) \leq \varphi^n\left( d_\eta(x_0, x_{3k+2}) \right) \]

\[ \leq \varphi^n\left( \eta(x_0, x_{3k+2}) \left[ \sum_{i=0}^{3k-1} \varphi^i\left( d_\eta(x_0, x_1) + \varphi^{3k}\left( d_\eta(x_0, x_2) \right) \right) \right] \right) \]

\[ \leq \varphi^n\left( \eta(x_0, x_{3k+2}) \left[ \sum_{i=0}^{3k-1} \varphi^i\left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \right] \right) \]

\[ + \varphi^{3k}\left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \]

\[ \leq \varphi^n\left( \eta(x_0, x_{3k+2}) \left[ \sum_{i=0}^{3k} \varphi^i\left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \right] \right) \]

\[ \leq \varphi^n\left( \eta(x_0, x_{3k+2}) \left[ \sum_{i=0}^{\infty} \varphi^i\left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \right] \right). \]
Since 
\[ d_\eta(x_n, x_{n+3k+3}) \leq \varphi^n \left( d_\eta(x_0, x_{3k+3}) \right) \]
and
\[ d_\eta(x_0, x_{3k+3}) \leq \eta(x_0, x_{3k+3}) \left[ \sum_{i=0}^{3k-1} \varphi^i \left( d_\eta(x_0, x_1) + \varphi^3 d_\eta(x_0, x_3) \right) \right], \]
for \( k = 1, 2, 3, \cdots \) we can show the following
\[ d_\eta(x_n, x_{n+3k+3}) \leq \varphi^n \left( \eta(x_0, x_{3k+3}) \left[ \sum_{i=0}^\infty \varphi^i \left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \right] \right). \]

Consequently it follows for each \( m \), we have the following
\[ d_\eta(x_n, x_{n+m}) \leq \varphi^n \left( \eta(x_0, x_m) \left[ \sum_{i=0}^\infty \varphi^i \left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \right] \right). \]

Since \( \sum_{i=0}^\infty \varphi^i \left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \) is convergent, where
\[ d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \in P \setminus \{0\} \]
P is closed, then \( \sum_{i=0}^\infty \varphi^i \left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \in P \setminus \{0\}. \) Since \( \lim_{n,m \to \infty} \eta(x_0, x_m) \) exists and is finite, then
\[ 0 = \lim_{n,m \to \infty} \varphi^n \left( \eta(x_0, x_m) \left[ \sum_{i=0}^\infty \varphi^i \left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \right] \right). \]

So given \( 0 \ll c \), we can find natural number \( N_1 \) such that
\[ \varphi^n \left( \eta(x_0, x_m) \left[ \sum_{i=0}^\infty \varphi^i \left( d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \right) \right] \right) \ll c \]
for all \( n \geq N_1 \). Consequently,
\[ d_\eta(x_n, x_{n+m}) \ll c \text{ for all } n \geq N_1. \]
Therefore \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there is a point \( z \in X \) such that
\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} Tx_{n-1} = z.
\]
Finally we show existence of the fixed point, that is, \( Tz = z \). Since \( 0 \ll c \), we can choose natural numbers \( N_2, N_3, N_4 \) such that \( \eta(z, x_n) \ll \frac{c}{4\eta(Tz, z)} \), for all \( n \geq N_2 \), \( \eta(x_{n+1}, x_n) \ll \frac{c}{4\eta(Tz, z)} \), for all \( n \geq N_3 \), and \( \eta(x_{n-1}, z) \ll \frac{c}{4\eta(Tz, z)} \), for all \( n \geq N_4 \).
Since \( x_n \neq x_m \) for \( n \neq m \), therefore by \( \eta \)-cone pentagonal property, we deduce the following
\[
d_\eta(Tz, z) \leq \eta(Tz, z)[d_\eta(Tz, Tx_n) + d_\eta(Tx_n, Tx_{n-1}) + d_\eta(Tx_{n-1}, Tx_{n-2}) + d_\eta(Tx_{n-2}, z)]
\leq \eta(Tz, z)[d_\eta(Tz, Tx_n) + d_\eta(x_{n+1}, x_n) + d_\eta(x_n, x_{n-1}) + d_\eta(x_{n-1}, z)].
\]
By the inequality of the theorem, we have
\[
d(Tz, Tx_n) \leq \varphi\left(\frac{1}{\sqrt{2}} d_\eta(z, x_n)^{\frac{1}{2}} d_\eta(z, Tz)^{\frac{1}{2}}\right)
\leq \varphi\left(\frac{1}{\sqrt{2}} d_\eta(z, x_n)^{\frac{1}{2}} (d_\eta(z, x_n) + d_\eta(x_n, Tz))^{\frac{1}{2}}\right)
\leq \varphi\left(\frac{1}{\sqrt{2}} d_\eta(z, x_n)^{\frac{1}{2}} (2d_\eta(z, x_n))^{\frac{1}{2}}\right)
\leq \varphi(d_\eta(z, x_n))
< d_\eta(z, x_n).
\]
Thus
\[
d_\eta(Tz, z) \leq \eta(Tz, z)[d_\eta(Tz, Tx_n) + d_\eta(Tx_n, Tx_{n-1}) + d_\eta(Tx_{n-1}, Tx_{n-2}) + d_\eta(Tx_{n-2}, z)]
\leq \eta(Tz, z)[d_\eta(Tz, Tx_n) + d_\eta(x_{n+1}, x_n) + d_\eta(x_n, x_{n-1}) + d_\eta(x_{n-1}, z)]
\leq \eta(Tz, z)[d_\eta(z, x_n) + d_\eta(x_{n+1}, x_n) + d_\eta(x_n, x_{n-1}) + d_\eta(x_{n-1}, z)].
\]
Hence
\[
d_\eta(Tz, z) \ll \eta(Tz, z)[\frac{c}{4\eta(Tz, z)}] \ll c
\]
for all \( n \geq N \), where \( N := \max\{N_2, N_3, N_4\} \). Since \( c \) is arbitrary, we have 
\[
d_\eta(Tz, z) \ll \frac{c}{m}
\]
for all \( m \in \mathbb{N} \). Since \( \frac{c}{m} \to 0 \) as \( m \to \infty \), we conclude that
\[
\frac{c}{m} - d_\eta(Tz, z) \to -d_\eta(Tz, z) \text{ as } m \to \infty.
\]
Since \( P \) is closed, \( -d_\eta(Tz, z) \in P \). Hence \( d_\eta(Tz, z) \in P \cap (-P) \). By definition of cone, we have \( d_\eta(Tz, z) = 0 \). So the fixed point exists, and the proof is finished. 

\[\square\]

3 Open Problems

To conclude this paper, we suggest some unsolved problems

**Conjecture 3.1.** Let \((X, d_\eta)\) be a \(\eta\)-cone pentagonal metric space. Suppose \(T : X \mapsto X\) satisfies the following for some positive integer \(m\)
\[
d_\eta(T^m x, T^m y) \leq \varphi \left( \frac{1}{\sqrt{2}} d_\eta(x, y) \right)^{\frac{1}{2}} d_\eta(x, T^m x) \right)^{\frac{1}{2}}
\]
for all \(x, y \in X, x, y \notin Fix(T)\), and \(\varphi \in \Phi\). If \((X, d_\eta)\) is complete and
\[
\lim_{n,m \to \infty} \eta(x_0, x_m)
\]
exists and is finite, then the fixed point of \(T\) exists.

**Conjecture 3.2.** Let \((X, d_\eta)\) be a \(\eta\)-cone pentagonal metric space. Suppose \(T : X \mapsto X\) satisfies the following
\[
d_\eta(Tx, Ty) \leq \lambda d_\eta(x, y) \right)^{\frac{1}{2}} d_\eta(x, Tx) \right)^{\frac{1}{2}}
\]
for all \(x, y \in X, x, y \notin Fix(T)\), and \(\lambda \in (0, 1)\). If \((X, d_\eta)\) is complete and
\[
\lim_{n,m \to \infty} \eta(x_0, x_m)
\]
exists and is finite, then the fixed point of \(T\) exists.

**Conjecture 3.3.** Theorem 2.1 holds in \(\eta\)-cone rectangular metric space \([8]\).
References


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