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The Interpolative Berinde Weak Mapping Theorem in η-Cone Pentagonal Metric Space

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Abstract

In this paper we introduce a concept of η-cone pentagonal metric space, which combines the notions of cone pentagonal metric space [1], and η-cone metric space [2]. Moreover, a variant of the interpolative Berinde weak mapping theorem obtained in [3] is proved in this setting.

1 Introduction and Premilinaries

Definition 1.1. [4] Let E be a real Banach space with norm $\|\cdot\|$ and P be a subset of E. Then P is called a cone if and only if

(a) P is closed, nonempty, and $P \neq \{\theta\}$, where θ is the zero vector in E;

(b) for any nonnegative real numbers a and b, and $x, y \in P$, we have $ax + by \in P$;

(c) for $x \in P$, if $-x \in P$, then $x = \theta$.

Definition 1.2. [4] Given a cone P in a Banach space E, we define on E a partial order $\preceq$ with respect to P by

$$x \preceq y \iff y - x \in int(P).$$

We shall write $x \prec y$ whenever $x \preceq y$ and $x \neq y$. While $x \ll y$ will stand for $y - x \in Int(P)$, where $Int(P)$ designates the interior of P.

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Definition 1.3. [4] The cone P is said to be normal if there is a real number $C > 0$ such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq C\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. In particular, we will say that P is a K-normal cone to indicate the fact that the normal constant is K.

Definition 1.4. [1] Let X be a nonempty set. Suppose the mapping $d : X \times X \mapsto E$ satisfies

(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, y)$ for all $x, y \in X$ and for all distinct points $z, w, u \in X - \{x, y\}$ [pentagonal property].

Then d is called a cone pentagonal metric on X, and (X, d) is called a cone pentagonal metric space.

Example 1.5. [5] Let $X = \{r, s, t, u, v\}$, $E = \mathbb{R}^2$, and $P = \{(x, y) : x, y \geq 0\}$ be a cone in E. Define $d : X \times X \mapsto E$ as follows

$$\begin{aligned}
d(x, x) &= 0, \text{ for all } x \in X,\\
d(r, s) &= d(s, r) = (4, 8),\\
d(r, t) &= d(t, r) = d(t, u) = d(u, t) = d(s, t)\\
&= d(t, s) = d(s, u) = d(u, s) = d(r, u) = d(u, r) = (1, 2),\\
d(r, v) &= d(v, r) = d(s, v) = d(v, s) = d(t, v) = d(v, t) = d(u, v) = d(v, u) = (3, 6).
\end{aligned}$$

Then (X, d) is a complete cone pentagonal metric space, but not a complete cone rectangular metric space, as it lacks the triangular property:

$$\begin{aligned}
(4, 8) = d(r, s) &> d(r, t) + d(t, u) + d(u, s)\\
&= (1, 2) + (1, 2) + (1, 2) = (3, 6)
\end{aligned}$$

as $(4, 8) - (3, 6) = (1, 2) \in P$.

Definition 1.6. [2] Let X be a nonempty set and $\eta : X \times X \mapsto [1, \infty)$ be a map. A function $d_\eta : X \times X \mapsto E$ will be called an η-cone metric on X if

(a) $\theta \preceq d_\eta(x, y)$ for all $x \in X$ and $d_\eta(x, y) = \theta$ iff $x = y$;

(b) $d_\eta(x, y) = d_\eta(y, x)$ for all $x, y \in X$;

(c) $d_\eta(x, z) \preceq \eta(x, z)[d_\eta(x, y) + d_\eta(y, z)]$ for all $x, y, z \in X$.

Moreover the pair (X, d_η) is called an η-cone metric space.

Remark 1.7. [2] If for all $x, y \in X$

(a) $\eta(x, y) = 1$, then we obtain the definition of cone metric space [4].

(b) $\eta(x, y) = L$, where $L \geq 1$, then we obtain the definition of cone metric type space [6].

(c) $\eta(x, y) = C$, where $C \geq 1$, $E = \mathbb{R}$ and $P = [0, \infty)$, then we obtain the definition of metric type space [7].

Example 1.8. [2] Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subseteq \mathbb{R}^2$ and $X = \{1, 2, 3\}$. Let $\alpha \geq 0$ be a constant and define $\eta : X \times X \mapsto [1, \infty)$ and $d_\eta : X \times X \mapsto E$ by

$$\eta(x, y) = 1 + x + y,$$

$$d_\eta(1, 1) = d_\eta(2, 2) = d_\eta(3, 3) = (0, 0),$$

$$d_\eta(1, 2) = d_\eta(2, 1) = 80(1, \alpha),$$

$$d_\eta(1, 3) = d_\eta(3, 1) = 1000(1, \alpha),$$

$$d_\eta(2, 3) = d_\eta(3, 2) = 600(1, \alpha).$$

Then (X, d_η) is an η-cone metric space.

Now we introduce the following

Definition 1.9. Let X be a nonempty set and $\eta : X \times X \mapsto [1, \infty)$ be a map. A function $d_\eta : X \times X \mapsto E$ will be called an η-cone pentagonal metric on X if

(a) $\theta \preceq d_\eta(x,y)$ for all $x, y \in X$ and $d_\eta(x,y) = \theta$ iff $x = y$;

(b) $d_\eta(x,y) = d_\eta(y,x)$ for all $x, y \in X$;

(c) $d_\eta(x,y) \leq \eta(x,y)[d_\eta(x,w) + d_\eta(w,z) + d_\eta(z,v) + d_\eta(v,y)]$ for all $x, y \in X$ and for all distinct points $w, z, v \in X - \{x, y\}$.

Moreover the pair (X, d_η) will be called an η-cone pentagonal metric space.

Example 1.10. Let $X = \{r, s, t, u, v\}$, $E = \mathbb{R}^2$, and $P = \{(x,y) : x, y \geq 0\}$ be a cone in E. Define $d_\eta : X \times X \mapsto E$ as follows

$$\begin{aligned}
d_\eta(x,x) &= (0,0), \text{ for all } x \in X,\\
d_\eta(r,s) &= d_\eta(s,r) = (4,8),\\
d_\eta(r,t) &= d_\eta(t,r) = d_\eta(t,u) = d_\eta(u,t) = d_\eta(s,t) = d_\eta(t,s) = d_\eta(s,u) = d_\eta(u,s)\\
&= d_\eta(r,u) = d_\eta(u,r) = (1,2),\\
d_\eta(r,v) &= d_\eta(v,r) = d_\eta(s,v) = d_\eta(v,s) = d_\eta(t,v) = d_\eta(v,t)\\
&= d_\eta(u,v) = d_\eta(v,u) = (3,6).
\end{aligned}$$

Also define $\eta : X \times X \mapsto [1, \infty)$ as follows

$$\begin{aligned}
\eta(x,x) &= 0, \text{ for all } x \in X,\\
\eta(r,s) &= \eta(s,r) = \frac{1}{2},\\
\eta(r,t) &= \eta(t,r) = \eta(t,u) = \eta(u,t) = \eta(s,t) = \eta(t,s) = \eta(s,u) = \eta(u,s)\\
&= \eta(r,u) = \eta(u,r) = \frac{1}{11},\\
\eta(r,v) &= \eta(v,r) = \eta(s,v) = \eta(v,s) = \eta(t,v) = \eta(v,t) = \eta(u,v)\\
&= \eta(v,u) = \frac{1}{3}.
\end{aligned}$$

By definition of d_η, it is trivial to check Definition 1.9(a), and Definition 1.9(b). Now we check Definition 1.9(c). Obviously we have the following cases

Case 1: (x,x) for all $x \in X$

In this case it is enough to check

$$d_\eta(r,r) \leq \eta(r,r)[d_\eta(r,s) + d_\eta(s,t) + d_\eta(t,u) + d_\eta(u,r)].$$

Since $d_\eta(r,r) = (0,0)$ and $\eta(r,r) = 0$, we have equality. In particular, we have

$$\begin{aligned}(0,0) = d_\eta(r,r) &\leq \eta(r,r)[d_\eta(r,s) + d_\eta(s,t) + d_\eta(t,u) + d_\eta(u,r)]\\ &= 0[d_\eta(r,s) + d_\eta(s,t) + d_\eta(t,u) + d_\eta(u,r)] = (0,0).\end{aligned}$$

Case 2: $(r,s) = (s,r)$

In this case it is enough to check

$$d_\eta(r,s) \leq \eta(r,s)[d_\eta(r,t) + d_\eta(t,u) + d_\eta(u,v) + d_\eta(v,s)].$$

In particular, we have

$$\begin{aligned}(4,8) &= d_\eta(r,s)\\ &\leq \eta(r,s)[d_\eta(r,t) + d_\eta(t,u) + d_\eta(u,v) + d_\eta(v,s)]\\ &= \frac{1}{2}[(1,2) + (1,2) + (3,6) + (3,6)]\\ &= \frac{1}{2}(8,16) = (4,8).\end{aligned}$$

Case 3: $(r,t) = (t,r) = (t,u) = (u,t) = (s,t) = (t,s) = (s,u) = (u,s) = (r,u) = (u,r)$

In this case it is enough to check

$$d_\eta(r,t) \leq \eta(r,t)[d_\eta(r,s) + d_\eta(s,u) + d_\eta(u,v) + d_\eta(v,t)].$$

In particular, we have

$$\begin{aligned}(1,2) &= d_\eta(r,t)\\ &\leq \eta(r,t)[d_\eta(r,s)+d_\eta(s,u)+d_\eta(u,v)+d_\eta(v,t)]\\ &= \frac{1}{11}[(4,8)+(1,2)+(3,6)+(3,6) = \frac{1}{11}(11,22) = (1,2).\end{aligned}$$

Case 4: $(r,v)=(v,r)=(s,v)=(v,s)=(t,v)=(v,t)=(u,v)=(v,u)$

In this case it is enough to check that

$$d_\eta(r,v) \leq \eta(r,v)[d_\eta(r,s)+d_\eta(s,t)+d_\eta(t,u)+d_\eta(u,v)].$$

In particular, we have

$$\begin{aligned}(3,6) &= d_\eta(r,v)\\ &\leq \eta(r,v)[d_\eta(r,s)+d_\eta(s,t)+d_\eta(t,u)+d_\eta(u,v)]\\ &= \frac{1}{3}[(4,8)+(1,2)+(1,2)+(3,6)] = \frac{1}{3}(9,18) = (3,6).\end{aligned}$$

It follows that (X,d_η) is an η-cone pentagonal metric space. Note that (X,d_η) is not an η-cone rectangular metric space, as it lacks Definition 1.9(c) [8]. In particular,

$$\begin{aligned}(4,8) = d_\eta(r,s) &> \eta(r,s)[d_\eta(r,r)+d_\eta(r,t)+d_\eta(t,s)]\\ &= \frac{1}{2}[(0,0)+(1,2)+(1,2)]\\ &= \frac{1}{2}(2,4) = (1,2)\end{aligned}$$

as $(4,8)-(1,2)=(3,6)\in P$.

Definition 1.11. Let (X,d_η) be a η-cone pentagonal metric space. Let $\{x_n\}$ be a sequence in (X,d_η) and $x\in X$. If for every $c\in E$ with $0\ll c$, there exists $n_0\in\mathbb{N}$ such that for all $n>n_0$, $d_\eta(x_n,c)\ll c$, then we say $\{x_n\}$ is convergent, and $\{x_n\}$ converges to x. Moreover, x will be called the limit of $\{x_n\}$. We sometimes write $\lim_{n\to\infty} x_n = x$, or $x_n\to x$ as $n\to\infty$.

Definition 1.12. Let (X, d_η) be a η-cone pentagonal metric space. If for every $c \in E$, with $0 \ll c$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_\eta(x_n, x) \ll c$, then we say $\{x_n\}$ is a Cauchy sequence in (X, d_η).

Definition 1.13. Let (X, d_η) be a η-cone pentagonal metric space. If every Cauchy sequence is convergent in (X, d_η), then (X, d_η) will be called a complete η-cone pentagonal metric space.

Notation 1.14. [1] Let P be a cone as defined in this paper. Φ will denote the set of all nondcreasing continuous functions $\varphi : P \mapsto P$ satisfying

(a) $0 < \varphi(t) < t$ for all $t \in P \backslash \{0\}$;

(b) the series $\sum_{n \geq 0} \varphi^n(t)$ converges for all $t \in P \backslash \{0\}$.

Note that from (a), we have $\varphi(0) = 0$, and from (b) we have $\lim_{n \to \infty} \varphi^n(t) = 0$ for all $t \in P \backslash \{0\}$.

Definition 1.15. Let (X, d_η) be a η-cone pentagonal metric space. A map $T : X \mapsto X$ will be called an (alternate) Interpolative Berinde Weak operator if it satisfies

$$d_\eta(Tx, Ty) \leq \lambda \ d_\eta(x, y)^{\frac{1}{2}} d_\eta(x, Tx)^{\frac{1}{2}},$$

where $\lambda \in (0, 1)$, for all $x, y \in X, \ x, y \notin Fix(T)$, where $Fix(T) = \{x \in X : Tx = x\}$.

2 Main Results

Our main result is as follows

Theorem 2.1. *Let (X, d_η) be a η-cone pentagonal metric space. Suppose $T : X \mapsto X$ satisfies*

$$d_\eta(Tx, Ty) \leq \varphi\left(\frac{1}{\sqrt{2}} \ d_\eta(x, y)^{\frac{1}{2}} \ d_\eta(x, Tx)^{\frac{1}{2}}\right)$$

for all $x, y \in X, \ x, y \notin Fix(T)$, and $\varphi \in \Phi$. If (X, d_η) is complete and

$$\lim_{n,m \to \infty} \eta(x_0, x_m)$$

exists and is finite, then the fixed point of T exists.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n = 0, 1, 2, \cdots$. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Now observe from the inequality of the theorem, we have

$$\begin{aligned}
d_\eta(x_n, x_{n+1}) &= d_\eta(Tx_{n-1}, Tx_n) \\
&\leq \varphi\left(\frac{1}{\sqrt{2}}\ d_\eta(x_{n-1}, x_n)^{\frac{1}{2}}\ d_\eta(x_{n-1}, Tx_{n-1})^{\frac{1}{2}}\right) \\
&= \varphi\left(\frac{1}{\sqrt{2}}\ d_\eta(x_{n-1}, x_n)^{\frac{1}{2}}\ d_\eta(x_{n-1}, x_n)^{\frac{1}{2}}\right) \\
&< \varphi\left(d_\eta(x_{n-1}, x_n)\right) \\
&= \varphi\left(d_\eta(Tx_{n-2}, Tx_{n-1})\right) \\
&\leq \varphi\left(\varphi\left(d_\eta(x_{n-2}, x_{n-1})\right)\right) \\
&= \varphi^2\left(\ d_\eta(x_{n-2}, x_{n-1})\right) \\
&\vdots \\
&\leq \varphi^n\left(d_\eta(x_0, x_1)\right).
\end{aligned}$$

Now observe we have the following

$$\begin{aligned}
d_\eta(x_n, x_{n+2}) &= d_\eta(Tx_{n-1}, Tx_{n+1}) \\
&\leq \varphi\left(\frac{1}{\sqrt{2}}\ d_\eta(x_{n-1}, x_{n+1})^{\frac{1}{2}}\ d_\eta(x_{n-1}, Tx_{n-1})^{\frac{1}{2}}\right) \\
&= \varphi\left(\frac{1}{\sqrt{2}}\ d_\eta(x_{n-1}, x_{n+1})^{\frac{1}{2}}\ d_\eta(x_{n-1}, x_n)^{\frac{1}{2}}\right) \\
&\leq \varphi\left(\frac{1}{\sqrt{2}}\ d_\eta(x_{n-1}, x_{n+1})^{\frac{1}{2}}\ 2^{\frac{1}{2}} d_\eta(x_{n-1}, x_{n+1})^{\frac{1}{2}}\right) \\
&= \varphi\left(d_\eta(x_{n-1}, x_{n+1})\right) \\
&= \varphi\left(d_\eta(Tx_{n-2}, Tx_n)\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \varphi\Bigg(\varphi\Bigg(d_\eta(x_{n-2}, x_n)\Bigg)\Bigg) \\
&= \varphi^2\Bigg(\ d_\eta(x_{n-2}, x_n)\Bigg) \\
&\vdots \\
&\le \varphi^n\Bigg(d_\eta(x_0, x_2)\Bigg).
\end{aligned}
$$

Further we deduce the following

$$
\begin{aligned}
d_\eta(x_n, x_{n+3}) &= d_\eta(Tx_{n-1}, Tx_{n+2}) \\
&\le \varphi\Bigg(\frac{1}{\sqrt{2}}\ d_\eta(x_{n-1}, x_{n+2})^{\frac{1}{2}}\ d_\eta(x_{n-1}, Tx_{n-1})^{\frac{1}{2}}\Bigg) \\
&= \varphi\Bigg(\frac{1}{\sqrt{2}}\ d_\eta(x_{n-1}, x_{n+2})^{\frac{1}{2}}\ d_\eta(x_{n-1}, x_n)^{\frac{1}{2}}\Bigg) \\
&\le \varphi\Bigg(\frac{1}{\sqrt{2}}\ d_\eta(x_{n-1}, x_{n+2})^{\frac{1}{2}}\ 2^{\frac{1}{2}} d_\eta(x_{n-1}, x_{n+2})^{\frac{1}{2}}\Bigg) \\
&= \varphi\Bigg(d_\eta(x_{n-1}, x_{n+2})\Bigg) \\
&= \varphi\Bigg(d_\eta(Tx_{n-2}, Tx_{n+1})\Bigg) \\
&\le \varphi\Bigg(\varphi\Bigg(d_\eta(x_{n-2}, x_{n+1})\Bigg)\Bigg) \\
&= \varphi^2\Bigg(\ d_\eta(x_{n-2}, x_{n+1})\Bigg) \\
&\vdots \\
&\le \varphi^n\Bigg(d_\eta(x_0, x_3)\Bigg).
\end{aligned}
$$

Now for $k = 1, 2, 3, \cdots$ we have

$$
d_\eta(x_n, x_{n+3k+1}) \le \varphi^n\Bigg(d_\eta(x_0, x_{3k+1})\Bigg)
$$

$$d_\eta(x_n, x_{n+3k+2}) \leq \varphi^n\Bigg(d_\eta(x_0, x_{3k+2})\Bigg)$$

$$d_\eta(x_n, x_{n+3k+3}) \leq \varphi^n\Bigg(d_\eta(x_0, x_{3k+3})\Bigg).$$

Since $d_\eta(x_n, x_{n+1}) \leq \varphi^n\Bigg(d_\eta(x_0, x_1)\Bigg)$, we deduce the following using the η-cone pentagonal property

$$\begin{aligned} d\eta(x_0, x_4) &\leq \eta(x_0, x_4)[d\eta(x_0, x_1) + d\eta(x_1, x_2) + d\eta(x_2, x_3) + d\eta(x_3, x_4)] \\ &\leq \eta(x_0, x_4)\Bigg[d\eta(x_0, x_1) + \varphi\Bigg(d_\eta(x_0, x_1)\Bigg) + \varphi^2\Bigg(d_\eta(x_0, x_1)\Bigg) + \varphi^3\Bigg(d_\eta(x_0, x_1)\Bigg)\Bigg] \\ &\leq \eta(x_0, x_4)\ \Bigg[\sum_{i=0}^{3}\varphi^i\Bigg(d_\eta(x_0, x_1)\Bigg)\Bigg]. \end{aligned}$$

Similarly, we can show the following

$$d\eta(x_0, x_7) \leq \eta(x_0, x_7)\ \Bigg[\sum_{i=0}^{6}\varphi^i\Bigg(d_\eta(x_0, x_1)\Bigg)\Bigg].$$

By induction for $k = 1, 2, 3, \cdots$, we have

$$d\eta(x_0, x_{3k+1}) \leq \eta(x_0, x_{3k+1})\ \Bigg[\sum_{i=0}^{3k}\varphi^i\Bigg(d_\eta(x_0, x_1)\Bigg)\Bigg].$$

Since $d_\eta(x_n, x_{n+1}) \leq \varphi^n\Bigg(d_\eta(x_0, x_1)\Bigg)$ and $d_\eta(x_n, x_{n+2}) \leq \varphi^n\Bigg(d_\eta(x_0, x_2)\Bigg)$, we deduce the following using the η-cone pentagonal property

$$\begin{aligned} d\eta(x_0, x_5) &\leq \eta(x_0, x_5)[d\eta(x_0, x_1) + d\eta(x_1, x_2) + d\eta(x_2, x_3) + d\eta(x_3, x_5)] \\ &\leq \eta(x_0, x_5)\Bigg[d\eta(x_0, x_1) + \varphi\Bigg(d_\eta(x_0, x_1)\Bigg) + \varphi^2\Bigg(d_\eta(x_0, x_1)\Bigg) + \varphi^3\Bigg(d_\eta(x_0, x_2)\Bigg)\Bigg] \\ &\leq \eta(x_0, x_5)\ \Bigg[\sum_{i=0}^{2}\varphi^i\Bigg(d_\eta(x_0, x_1)\Bigg) + \varphi^3\Bigg(d_\eta(x_0, x_2)\Bigg)\Bigg]. \end{aligned}$$

Similarly, we can show the following

$$d\eta(x_0, x_8) \leq \eta(x_0, x_8) \left[\sum_{i=0}^{5} \varphi^i \Big(d_\eta(x_0, x_1) \Big) + \varphi^6 \Big(d_\eta(x_0, x_2) \Big) \right].$$

By induction for $k = 1, 2, 3, \cdots$, we have

$$d\eta(x_0, x_{3k+2}) \leq \eta(x_0, x_{3k+2}) \left[\sum_{i=0}^{3k-1} \varphi^i \Big(d_\eta(x_0, x_1) \Big) + \varphi^{3k} \Big(d_\eta(x_0, x_2) \Big) \right].$$

Since $d_\eta(x_n, x_{n+1}) \leq \varphi^n \Big(d_\eta(x_0, x_1) \Big)$ and $d_\eta(x_n, x_{n+3}) \leq \varphi^n \Big(d_\eta(x_0, x_3) \Big)$, we deduce the following using the η-cone pentagonal property

$$\begin{aligned} d\eta(x_0, x_6) &\leq \eta(x_0, x_6)[d\eta(x_0, x_1) + d\eta(x_1, x_2) + d\eta(x_2, x_3) + d\eta(x_3, x_6)] \\ &\leq \eta(x_0, x_6) \left[d\eta(x_0, x_1) + \varphi \Big(d_\eta(x_0, x_1) \Big) + \varphi^2 \Big(d_\eta(x_0, x_1) \Big) + \varphi^3 \Big(d_\eta(x_0, x_3) \Big) \right] \\ &\leq \eta(x_0, x_6) \left[\sum_{i=0}^{2} \varphi^i \Big(d_\eta(x_0, x_1) \Big) + \varphi^3 \Big(d_\eta(x_0, x_3) \Big) \right]. \end{aligned}$$

Similarly, we can show the following

$$d\eta(x_0, x_9) \leq \eta(x_0, x_9) \left[\sum_{i=0}^{5} \varphi^i \Big(d_\eta(x_0, x_1) \Big) + \varphi^6 \Big(d_\eta(x_0, x_3) \Big) \right].$$

By induction for $k = 1, 2, 3, \cdots$, we have

$$d\eta(x_0, x_{3k+3}) \leq \eta(x_0, x_{3k+3}) \left[\sum_{i=0}^{3k-1} \varphi^i \Big(d_\eta(x_0, x_1) \Big) + \varphi^{3k} \Big(d_\eta(x_0, x_3) \Big) \right].$$

Since

$$d_\eta(x_n, x_{n+3k+1}) \leq \varphi^n \Big(d_\eta(x_0, x_{3k+1}) \Big)$$

and

$$d\eta(x_0, x_{3k+1}) \leq \eta(x_0, x_{3k+1}) \left[\sum_{i=0}^{3k} \varphi^i \Big(d_\eta(x_0, x_1) \Big) \right],$$

we deduce the following

$$\begin{aligned}
d_\eta(x_n, x_{n+3k+1}) &\leq \varphi^n\Big(d_\eta(x_0, x_{3k+1})\Big)\\
&\leq \varphi^n\Bigg(\eta(x_0, x_{3k+1})\ \left[\sum_{i=0}^{3k}\varphi^i\Big(d_\eta(x_0, x_1)\Big)\right]\Bigg)\\
&\leq \varphi^n\Bigg(\eta(x_0, x_{3k+1})\ \left[\sum_{i=0}^{3k}\varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)\right]\Bigg)\\
&\leq \varphi^n\Bigg(\eta(x_0, x_{3k+1})\ \left[\sum_{i=0}^{\infty}\varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)\right]\Bigg).
\end{aligned}$$

Since $d_\eta(x_n, x_{n+3k+2}) \leq \varphi^n\Big(d_\eta(x_0, x_{3k+2})\Big)$ and

$$d\eta(x_0, x_{3k+2}) \leq \eta(x_0, x_{3k+2})\ \left[\sum_{i=0}^{3k-1}\varphi^i\Big(d_\eta(x_0, x_1)\Big) + \varphi^{3k}\Big(d_\eta(x_0, x_2)\Big)\right],$$

we deduce the following

$$\begin{aligned}
d_\eta(x_n, x_{n+3k+2}) &\leq \varphi^n\Big(d_\eta(x_0, x_{3k+2})\Big)\\
&\leq \varphi^n\Bigg(\eta(x_0, x_{3k+2})\ \left[\sum_{i=0}^{3k-1}\varphi^i\Big(d_\eta(x_0, x_1)\Big) + \varphi^{3k}\Big(d_\eta(x_0, x_2)\Big)\right]\Bigg)\\
&\leq \varphi^n\Bigg(\eta(x_0, x_{3k+2})\ \left[\sum_{i=0}^{3k-1}\varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)\right.\\
&\quad \left. + \varphi^{3k}\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)\right]\Bigg)\\
&\leq \varphi^n\Bigg(\eta(x_0, x_{3k+2})\ \left[\sum_{i=0}^{3k}\varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)\right]\Bigg)\\
&\leq \varphi^n\Bigg(\eta(x_0, x_{3k+2})\ \left[\sum_{i=0}^{\infty}\varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)\right]\Bigg).
\end{aligned}$$

Since

$$d_\eta(x_n, x_{n+3k+3}) \leq \varphi^n\Big(d_\eta(x_0, x_{3k+3})\Big)$$

and

$$d\eta(x_0, x_{3k+3}) \leq \eta(x_0, x_{3k+3}) \left[\sum_{i=0}^{3k-1} \varphi^i\Big(d_\eta(x_0, x_1)\Big) + \varphi^{3k}\Big(d_\eta(x_0, x_3)\Big)\right],$$

for $k = 1, 2, 3, \cdots$ we can show the following

$$d_\eta(x_n, x_{n+3k+3}) \leq \varphi^n\left(\eta(x_0, x_{3k+3}) \left[\sum_{i=0}^{\infty} \varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)\right]\right).$$

Consequently it follows for each m, we have the following

$$d_\eta(x_n, x_{n+m}) \leq \varphi^n\left(\eta(x_0, x_m) \left[\sum_{i=0}^{\infty} \varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)\right]\right).$$

Since $\sum_{i=0}^{\infty} \varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)$ is convergent, where

$$d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3) \in P\backslash\{0\}$$

P is closed, then $\sum_{i=0}^{\infty} \varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big) \in P\backslash\{0\}$. Since $\lim_{n,m\to\infty} \eta(x_0, x_m)$ exists and is finite, then

$$0 = \lim_{n,m\to\infty} \varphi^n\left(\eta(x_0, x_m) \left[\sum_{i=0}^{\infty} \varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)\right]\right).$$

So given $0 \ll c$, we can find natural number N_1 such that

$$\varphi^n\left(\eta(x_0, x_m) \left[\sum_{i=0}^{\infty} \varphi^i\Big(d_\eta(x_0, x_1) + d_\eta(x_0, x_2) + d_\eta(x_0, x_3)\Big)\right]\right) \ll c$$

for all $n \geq N_1$. Consequently,

$$d_\eta(x_n, x_{n+m}) \ll c \text{ for all } n \geq N_1.$$

Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there is a point $z \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = z.$$

Finally we show existence of the fixed point, that is, $Tz = z$. Since $0 \ll c$, we can choose natural numbers N_2, N_3, N_4 such that $d_\eta(z, x_n) \ll \frac{c}{4\eta(Tz,z)}$, for all $n \geq N_2$, $d_\eta(x_{n+1}, x_n) \ll \frac{c}{4\eta(Tz,z)}$, for all $n \geq N_3$, and $d_\eta(x_{n-1}, z) \ll \frac{c}{4\eta(Tz,z)}$, for all $n \geq N_4$. Since $x_n \neq x_m$ for $n \neq m$, therefore by η-cone pentagonal property, we deduce the following

$$\begin{aligned} d_\eta(Tz,z) &\leq \eta(Tz,z)[d_\eta(Tz,Tx_n) + d_\eta(Tx_n,Tx_{n-1}) + d_\eta(Tx_{n-1},Tx_{n-2}) + d_\eta(Tx_{n-2},z)] \\ &\leq \eta(Tz,z)[d_\eta(Tz,Tx_n) + d_\eta(x_{n+1},x_n) + d_\eta(x_n,x_{n-1}) + d_\eta(x_{n-1},z)]. \end{aligned}$$

By the inequality of the theorem, we have

$$\begin{aligned} d(Tz,Tx_n) &\leq \varphi\left(\frac{1}{\sqrt{2}}\ d_\eta(z,x_n)^{\frac{1}{2}}\ d_\eta(z,Tz)^{\frac{1}{2}}\right) \\ &\leq \varphi\left(\frac{1}{\sqrt{2}}\ d_\eta(z,x_n)^{\frac{1}{2}}\ (d_\eta(z,x_n) + d_\eta(x_n,Tz))^{\frac{1}{2}}\right) \\ &\leq \varphi\left(\frac{1}{\sqrt{2}}\ d_\eta(z,x_n)^{\frac{1}{2}}\ (2d_\eta(z,x_n))^{\frac{1}{2}}\right) \\ &\leq \varphi(d_\eta(z,x_n)) \\ &< d_\eta(z,x_n). \end{aligned}$$

Thus

$$\begin{aligned} d_\eta(Tz,z) &\leq \eta(Tz,z)[d_\eta(Tz,Tx_n) + d_\eta(Tx_n,Tx_{n-1}) + d_\eta(Tx_{n-1},Tx_{n-2}) + d_\eta(Tx_{n-2},z)] \\ &\leq \eta(Tz,z)[d_\eta(Tz,Tx_n) + d_\eta(x_{n+1},x_n) + d_\eta(x_n,x_{n-1}) + d_\eta(x_{n-1},z)] \\ &\leq \eta(Tz,z)[d_\eta(z,x_n) + d_\eta(x_{n+1},x_n) + d_\eta(x_n,x_{n-1}) + d_\eta(x_{n-1},z)]. \end{aligned}$$

Hence

$$d_\eta(Tz,z) \ll \eta(Tz,z)[4 \cdot \frac{c}{4\eta(Tz,z)}] \ll c$$

for all $n \geq N$, where $N := \max\{N_2, N_3, N_4\}$. Since c is arbitrary, we have $d_\eta(Tz, z) \ll \frac{c}{m}$ for all $m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude that $\frac{c}{m} - d_\eta(Tz, z) \to -d_\eta(Tz, z)$ as $m \to \infty$. Since P is closed, $-d_\eta(Tz, z) \in P$. Hence $d_\eta(Tz, z) \in P \cap (-P)$. By definition of cone, we have $d_\eta(Tz, z) = 0$. So the fixed point exists, and the proof is finished. □

3 Open Problems

To conclude this paper, we suggest some unsolved problems

Conjecture 3.1. *Let (X, d_η) be a η-cone pentagonal metric space. Suppose $T : X \mapsto X$ satisfies the following for some positive integer m*

$$d_\eta(T^m x, T^m y) \leq \varphi\left(\frac{1}{\sqrt{2}}\ d_\eta(x, y)^{\frac{1}{2}}\ d_\eta(x, T^m x)^{\frac{1}{2}}\right)$$

for all $x, y \in X,\ x, y \notin Fix(T)$, and $\varphi \in \Phi$. If (X, d_η) is complete and

$$\lim_{n,m\to\infty} \eta(x_0, x_m)$$

exists and is finite, then the fixed point of T exists.

Conjecture 3.2. *Let (X, d_η) be a η-cone pentagonal metric space. Suppose $T : X \mapsto X$ satisfies the following*

$$d_\eta(Tx, Ty) \leq \lambda\ d_\eta(x, y)^{\frac{1}{2}}\ d_\eta(x, Tx)^{\frac{1}{2}}$$

for all $x, y \in X,\ x, y \notin Fix(T)$, and $\lambda \in (0, 1)$. If (X, d_η) is complete and

$$\lim_{n,m\to\infty} \eta(x_0, x_m)$$

exists and is finite, then the fixed point of T exists.

Conjecture 3.3. *Theorem 2.1 holds in η-cone rectangular metric space [8].*

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