Detour Extra Straight Lines in the Euclidean Plane

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Abstract

Using the theory of exploded numbers by the axiom-systems of real numbers and Euclidean geometry, we explode the Euclidean plane. Exploding the Euclidean straight lines we get super straight lines. The extra straight line is the window phenomenon of super straight line. In general, the extra straight lines are curves in Euclidean sense, but they have more similar properties to Euclidean straight lines. On the other hand, with respect of parallelism we find a surprising property: there are detour straight lines.

Short Introduction of the Theory of Exploded Numbers

The concept of exploded numbers had already been introduced in [1] and the explosion of k-dimensional space was discussed in the 4th part of [2]. In this article $k = 2$ is used, only.

For the sake of a better understanding we repeat the main results of the theory of exploded numbers. Denoting by $\mathbb{R}$ the set of real numbers, *we use the axioms of real numbers*: commutativity and associativity of both addition and multiplication, existence of unit elements for addition and multiplication (0 and 1, respectively) existence of additive inverse element $(-a)$ where $a \in \mathbb{R}$, and multiplicative inverse element of $a \in \mathbb{R}$ $(\frac{1}{a},$ where $a \neq 0)$, and distributivity. Summarizing, we have the field $(\mathbb{R}, +, \cdot, =)$. Moreover, we have the relation "<" and its monotonity for addition and multiplication (if $a < b,$ then $a + c < b + c$ and assuming that $c > 0$ we have that $a \cdot c < b \cdot c$), so we say that $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field. Finally, we mention that the completeness axiom. The

final collection is seen e.g. in [3]. Moreover, together with the axiom system of real numbers, *we use the following postulates and requirements.*

Postulate of extension: The set of real numbers $\mathbb{R}$ is a proper subset of the set of exploded real numbers $\breve{\mathbb{R}}$. For any real number x there exists only one exploded number $\breve{x}$ called exploded x or the exploded number of x. On the other hand x is called the compressed number of $\breve{x}$ that is

$$\underline{(\breve{x})} = x, \qquad x \in \mathbb{R},$$

where the compression is denoted by "$\underline{}$". Moreover, for any $u \in \breve{\mathbb{R}}$ there exists only one real number, which is called the compressed number of u and denoted by $\underline{u}$, such that

$$\widetilde{\left(\underline{u}\right)} = u, \qquad u \in \breve{\mathbb{R}}.$$

Postulate of unambiguity: For any pair of real numbers x and y, their exploded numbers are equal (in $\breve{\mathbb{R}}$) if and only if x is equal to y (in $\mathbb{R}$). Shortly,

$$\breve{x} \cong \breve{y} \Longleftrightarrow x = y.$$

Postulate of ordering: For any pair of real numbers x and y, the exploded x is smaller (in $\breve{\mathbb{R}}$) than exploded y if and only if x smaller than y (in $\mathbb{R}$). Shortly,

$$\breve{x} \lesssim \breve{y} \Longleftrightarrow x < y.$$

Postulate of super-addition: For any pair of real numbers x and y, the super-sum of their exploded number is the exploded of their sum. To put it by symbols:

$$\breve{x} \overline{\oplus} \breve{y} = \widetilde{x + y}.$$

Postulate of super-multiplication: For any pair of real numbers x and y, the super-product of their exploded numbers is the exploded number of their product. Expressed by symbols:

$$\breve{x} \overline{\odot} \breve{y} = \widetilde{x \cdot y}.$$

Requirement of equality for exploded real numbers: If x and y are real numbers, then x as an exploded number equals to y as an exploded number if and only if they are equal in the traditional sense. Shortly,

$$x \cong y \Longleftrightarrow x = y.$$

Requirement of ordering for exploded real numbers: If x and y are real numbers, then x as an exploded number is smaller than y as an exploded number if and only if x is

smaller than y in the traditional sense. Shortly,

$$x \lessapprox y \Leftrightarrow x < y.$$

After the requirements of equality and ordering, distinguishing between the equalities and orderings in $\breve{\mathbb{R}}$ and $\mathbb{R}$ is unnecessary, so instead of "$\cong$" we can write "$=$" and instead of "$\lessapprox$" we can write "$<$".

Requirement for zero: The exploded number of 0 is itself. Expressed by symbols

$$\breve{0} = 0.$$

Requirement for explosion: The set $\breve{\mathbb{R}} \setminus \mathbb{R}$ contains positive and negative elements, too.

After the latter requirement the following definitions are given:

The exploded number $\breve{x}$ is called positive if $0 < \breve{x}$.

(By the Postulate of ordering it is fulfilled if and only if $x > 0$.)

The exploded number $\breve{x}$ is called negative if $\breve{x} < 0$.

(By the Postulate of ordering it is fulfilled if and only if $x < 0$.)

Requirement of monotonity of super–addition: If $\breve{x}$ and $\breve{y}$ are arbitrary exploded numbers and $\breve{x}$ is smaller than $\breve{y}$, then, for any exploded number $\breve{z}$, the super-sum $\breve{x}\overline{\oplus}\breve{z}$ is smaller than super-sum $\breve{x}\overline{\oplus}\breve{z}$.

Requirement of monotonity of super-multiplication: If $\breve{x}$ and $\breve{y}$ are arbitrary exploded numbers and $\breve{x}$ is smaller than $\breve{y}$, then for any positive exploded number $\breve{z}$ the super-product $\breve{x}\overline{\odot}\breve{z}$ is smaller than super-product $y\overline{\odot}\breve{z}$.

By isomorphism

$$x \leftrightarrow \breve{x}, \qquad x \in \mathbb{R}$$

we can find that the set of exploded real number $\breve{\mathbb{R}}$ is an ordered field with respect to super-addition and super-multiplication. It is important to remark that super-operations $\overline{\oplus}$ and $\overline{\odot}$ are not extensions of traditional operations $+$ and $\cdot$, respectively.

1. A Comfortable Explosion of the Euclidean Plane

First we give a possible realisation of the explosion of real numbers belonging into

the open interval $]-1,1[$. For the sake of comfort we use the hyperbolic exploder function (see [4], (4.1) with $\sigma = 1$, page 41) and say

$$\check{x} = \tanh^{-1} x \left(= \tfrac{1}{2} ln \tfrac{1+x}{1-x}\right), \quad \text{where } -1 < x < 1. \tag{1.1}$$

If $-\infty < x \leq -1$ or $1 \leq x < \infty$, then we say that $\check{x}$ is an *invisible exploded number*. Extending the explosion for all real numbers (see [4], (3.2.2), page 24) we have a model for visibility of invisible exploded numbers, but in this article this visibility is not important. (See [4], Fig. 3.5.2, page 25.) This model satisfies the postulates and requirements mentioned in the short introduction. By the Postulate of ordering we have that the $\widetilde{-1}$ is the greatest invisible exploded number which is less than each real number, called *negative discriminator*. The $\check{1}$ is the smallest invisible exploded number which is greater than each real number, called *positive discriminator*.

By the Postulate of extension the explosion formula (1.1) yields the compression formula

$$\underline{x} = \tanh x \left(= \tfrac{e^x + e^{-x}}{e^x + e^{-x}}\right), \quad \text{where } -\infty < x < \infty. \tag{1.2}$$

The compression formula (1.2) shows that if $x \in \mathbb{R}$, then $-1 < \underline{x} < 1$.

We denote by $\mathbb{R}^2$ the Euclidean plane with the Cartesian coordinate system with right angles. The point $\mathcal{P} = (x, y)$ of the plane is exploded by each coordinate to obtain the exploded plane $\widetilde{\mathbb{R}^2}$

$$\widetilde{\mathbb{R}^2} = \{\check{\mathcal{P}} = (\check{x}, \check{y}) | \mathcal{P} \in \mathbb{R}^2\}. \tag{1.3}$$

If $\mathcal{P} = (u, v) \in \widetilde{\mathbb{R}^2} \setminus \mathbb{R}^2$, then it is invisible on the Euclidean plane. The Euclidean plane has four invisible boundaries:

$$\mathbb{L}_{lower} = \left\{\mathcal{P} = (u, v) \in \widetilde{\mathbb{R}^2} \middle| {}_{v=\widetilde{-1}}^{\widetilde{-1} \leq u \leq \check{1}}\right\}, \tag{1.4}$$

$$\mathbb{L}_{right} = \left\{\mathcal{P} = (u, v) \in \widetilde{\mathbb{R}^2} \middle| {}_{\widetilde{-1} \leq v \leq \check{1}}^{u = \check{1}}\right\}, \tag{1.5}$$

$$\mathbb{L}_{upper} = \left\{\mathcal{P} = (u, v) \in \widetilde{\mathbb{R}^2} \middle| {}_{v=\check{1}}^{\widetilde{-1} \leq u \leq \check{1}}\right\} \tag{1.6}$$

and

$$\mathbb{L}_{left} = \left\{\mathcal{P} = (u, v) \in \widetilde{\mathbb{R}^2} \middle| {}_{\widetilde{-1} \leq v \leq \check{1}}^{u = \widetilde{-1}}\right\}. \tag{1.7}$$

$\mathbb{L}_{lower} \cup \mathbb{L}_{right} \cup \mathbb{L}_{upper} \cup \mathbb{L}_{left}$ is a "very large-sized" square with the peak points determined by discriminators:

$$\mathcal{P}_{left,lower} = (\overset{\frown}{-1}, \overset{\frown}{-1}), \; \mathcal{P}_{right,lower} = (\overset{\frown}{1}, \overset{\frown}{-1}), \; \mathcal{P}_{right,upper} = (\overset{\frown}{1}, \overset{\frown}{1})$$

$$\text{and } \mathcal{P}_{left,upper} = (\overset{\frown}{1}, \overset{\frown}{-1}).$$

If the set $\mathbb{S} \subseteq \widetilde{\mathbb{R}^2}$, then it may have invisible points. For the visible points of $\mathbb{S}$ we use the concept of *window-phenomenon*.

$$\mathbb{S}_{window} = \mathbb{S} \cap \mathbb{R}^2. \tag{1.8}$$

Of course, the box-phenomenon may be the empty set. On the other hand $\widetilde{\mathbb{R}^2}_{window} = \mathbb{R}^2$.

We may consider the exploded and compressed of sets, too. More exactly,

$$\breve{\mathbb{S}} = \{\breve{\mathcal{P}} = (\breve{x}, \breve{y}) | \mathcal{P} \in \mathbb{S} \subseteq \mathbb{R}^2\} \tag{1.9}$$

and

$$\underline{\mathbb{S}} = \{\underline{\mathcal{P}} = (\underline{u}, \underline{v}) | \mathcal{P} \in \mathbb{S} \subseteq \widetilde{\mathbb{R}^2}\} \tag{1.10}$$

are called exploded and compressed $\mathbb{S}$, respectively. For example, the compressed of Euclidean plane is the open square

$$\underline{\mathbb{R}^2} = \left\{\mathcal{P} = (x, y) \in \mathbb{R}^2 | \begin{smallmatrix} -1 < y < 1 \\ -1 < y < 1 \end{smallmatrix}\right\}. \text{ (See (1.2).)} \tag{1.11}$$

2. Extra-lines in the Euclidean Plane

Let $\mathcal{P}_0 = (x_0, y_0) \in \underline{\mathbb{R}^2}$ and $E = (e_x, e_y) \in \mathbb{R}^2$ be a given point and vector, such that $\|E\| = \sqrt{e_x^2 + e_y^2} = 1$. We consider the Euclidean straight line $\mathbb{L}_{P_0,E}$, given by the vector-equation

$$P_t = P_0 + t \cdot E, \quad -\infty < t < \infty, \tag{2.1}$$

where $\mathcal{P}_t = (x_t, y_t) \in \mathbb{R}^2$. The straight line $\mathbb{L}_{P_0,E}$ is described by the equation-system

$$\mathbb{L}_{P_0,E}: \qquad \begin{smallmatrix} x_t = x_0 + t e_x \\ y_t = y_0 + t e_y \end{smallmatrix} \quad -\infty < t < \infty, \tag{2.2}$$

too. Clearly, for the distance between P_0 and P_t we have $d(P_0, P_t) = |t|$.

Exploding the straight line $\mathbb{L}_{P_0,E}$ we get the *super straight line* $\widetilde{\mathbb{L}_{P_0,E}}$ which has the

equation-system

$$\widetilde{\mathbb{L}_{P_0,E}} : \qquad \begin{matrix} u = \widetilde{x_0} \widetilde{\oplus} (\tilde{t} \widetilde{\odot} \widetilde{e_x}) \\ v = \widetilde{y_0} \widetilde{\oplus} (\tilde{t} \widetilde{\odot} \widetilde{e_y}) \end{matrix} \qquad -\infty < t < \infty, \qquad (2.3)$$

where the super-operations given by Postulates of super-addition and super-multiplication were used.

Of course, $\widetilde{\mathbb{L}_{P_0,E}} \subset \widetilde{\mathbb{R}^2}$. On the exploded Euclidean plane the super straight lines satisfy the Euclidean axiom system, collected in [5].

Finally the *extra straight line* is defined by

$$\mathbb{L}_{P_0,E}^{extra} = \left(\widetilde{\mathbb{L}_{P_0,E}} \right)_{window}, \quad \text{see (1.8)}. \qquad (2.4)$$

In other words the extra straight line is the visible open super-passage of the super straight line $\widetilde{\mathbb{L}_{P_0,E}}$. Endpoints of $\mathbb{L}_{P_0,E}^{extra}$ are invisible because they are situated on the invisible border of the plane $\widetilde{\mathbb{R}^2}$. (See (1.4)-(1.7).) Sometimes we say that $\widetilde{\mathbb{L}_{P_0,E}}$ is the holder of $\mathbb{L}_{P_0,E}^{extra}$. The compressed extra straight line cuts the border of $\underline{\mathbb{R}^2}$. The points of intersection are visible. Their exploreds are the invisible endpoints of $\mathbb{L}_{P_0,E}^{extra}$.

For the sake of understanding we give the following

Example 2.5. Let $\mathcal{P}_0 = \left(\frac{1}{2}, \frac{1}{2} \right) \in \underline{\mathbb{R}^2}$ and $E = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$ be given. By (2.2) and (2.3) we write

$$\mathbb{L}_{P_0,E} : \qquad \begin{matrix} x_t = \dfrac{1}{2} + t \cdot \dfrac{1}{\sqrt{2}} \\ y_t = \dfrac{1}{2} - t \cdot \dfrac{1}{\sqrt{2}} \end{matrix} \qquad -\infty < t < \infty$$

and

$$\widetilde{\mathbb{L}_{P_0,E}} : \qquad \begin{matrix} u = \left(\widetilde{\frac{1}{2}} \right) \widetilde{\oplus} \left(\tilde{t} \widetilde{\odot} \left(\widetilde{\frac{1}{2}} \right) \right) = \left(\widetilde{\frac{1}{2} + t \cdot \frac{1}{\sqrt{2}}} \right) \\ v = \left(\widetilde{\frac{1}{2}} \right) \widetilde{\ominus} \left(\tilde{t} \widetilde{\odot} \left(\widetilde{\frac{1}{2}} \right) \right) = \left(\widetilde{\frac{1}{2} - t \cdot \frac{1}{\sqrt{2}}} \right) \end{matrix} \qquad -\infty < t < \infty,$$

where $\check{x} \widetilde{\ominus} \check{y} =^{def} \widetilde{x - y}$, ($x$ and y are real numbers). The point $\mathcal{P} = (u, v) \in \widetilde{\mathbb{R}^2}$ if and only if
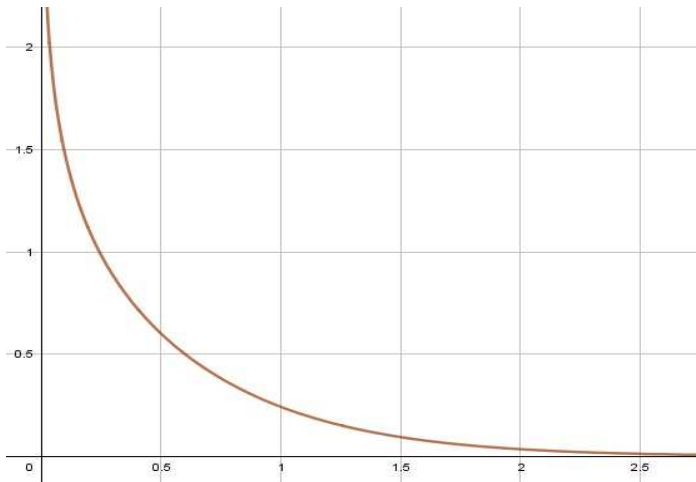
$$-1 < \frac{1}{2} + t \cdot \frac{1}{\sqrt{2}} < 1$$

and

$$-1 < \frac{1}{2} - t \cdot \frac{1}{\sqrt{2}} < 1$$

are fulfilled. (See (1.8).) Finally, (2.4) and (1.1) yield

$$\mathbb{L}_{P_0, E}^{extra}: \begin{array}{l} u = \tanh^{-1}\left(\frac{1}{2} + t \cdot \frac{1}{\sqrt{2}}\right) \\ v = \tanh^{-1}\left(\frac{1}{2} - t \cdot \frac{1}{\sqrt{2}}\right) \end{array}, \quad -\frac{1}{\sqrt{2}} < t < \frac{1}{\sqrt{2}},$$

or in the explicit form $v = \tanh^{-1}(1 - \tanh u)$ with $0 < u < \infty$ showed by the next figure

The equation of extra straight lines $\mathbb{L}_{P_0, E}^{extra}$ is $\tanh u + \tanh v = 1$, where u and v are positive numbers. The figure shows that $\mathbb{L}_{P_0, E}^{extra}$ is a Euclidean curve which has the invisible endpoints

$$\mathcal{P}_{upper} = \left(0, \breve{1}\right) \in \mathbb{L}_{upper} \text{ and } \mathcal{P}_{right} = \left(\breve{1}, 0\right) \in \mathbb{L}_{right}. \qquad \blacksquare$$

Here we repeat some properties of extra straight lines.

Property 2.6. If $\mathcal{P}_1 = (u_1, v_1)$ and $\mathcal{P}_2 = (u_2, v_2)$ are given different points in the Euclidean plane, then there exists one and only one extra straight line which contains them. (See [6], Property 1.)

Property 2.7. If $\mathcal{P}_1 = (u_1, v_1) \in \mathbb{R}^2$ and $\mathcal{P}_2 = (u_2, v_2)$ is situated in the (invisible) border of the Euclidean plane, then there exists one and only one extra straight line which contains the point $\mathcal{P}_1$ and one of its invisible endpoint is $\mathcal{P}_2$. (See [6], Property 2.)

Property 2.8. If the different points $\mathcal{P}_1 = (u_1, v_1)$ and $\mathcal{P}_2 = (u_2, v_2)$ are situated in the (invisible) border of the Euclidean plane the such that their compressed points $\underline{\mathcal{P}_1}$ and $\underline{\mathcal{P}_2}$ there are not on the same side of the closed square $\underline{\mathbb{R}^2} = \left\{ (x, y) \in \mathbb{R}^2 \big|\, {}^{-1 \leq y \leq 1}_{-1 \leq y \leq 1} \right\}$, then there exists one and only one extra straight line which has the endpoints $\mathcal{P}_1$ and $\mathcal{P}_2$. (See See [6], Property 3.)

Remark 2.9. On the figure of Example 2.5 the extra straight line $\mathbb{L}^{extra}_{P_0, E}$ seems to be one of hyperbola-branch $\mathbb{H}$ having the equation

$$uv = \left(\tanh^{-1} \frac{1}{2} \right)^2, \quad u \text{ and } v \text{ are positive numbers.}$$

The appearance deludes. As $\underline{\mathcal{P}_{upper}} = (0,1)$ and $\underline{\mathcal{P}_{right}} = (1.0)$ are not on the same side of the closed square $\underline{\mathbb{R}^2}$, so, by Property 2.8 , $\mathbb{L}^{extra}_{P_0, E}$ is unambiguously determined.

Considering the point of hyperbola $\mathcal{P}_H = \left(1, \left(\tanh^{-1} \frac{1}{2} \right)^2 \right)$ and the point of extra straight line $\mathcal{P}_L = (1, (1 - \tanh 1)\,)$ we have that $\mathcal{P}_H \neq \mathcal{P}_L$. ($\left(\tanh^{-1} \frac{1}{2} \right)^2 \approx$ 0,3017372402; $1 - \tanh 1 \approx 0,238405844$.) So $\mathbb{H} \neq \mathbb{L}^{extra}_{P_0, E}$.

3. Extra Straight Lines without Common Points in the Plane

Two straight Euclidean lines in one plane either intersect or not. The latter case is parallelism. The situation in extra geometry is more sophisticated. The concept of extra parallelism has already been introduced in [6] (See [6], part III.) This means that extra parallel extra straight lines have exactly one common (invisible) endpoint on the boundary of the plane. (Illustratively: the appointment of parallels in infinity.) The extra straight line $\mathbb{L}^{extra}_{P_0, E}$ in Example 2.5 and the "x" axis of the coordinate-system are extra parallel extra straight lines. The common endpoint is $\mathcal{P}_{right} = (\check{1}, 0)$. In the extra geometry it is possible for two extra straight lines to be neither intersecting nor extra parallel. Given the Euclidean geometry of super straight lines we distinguish two types detour.

Detour Type I. The holders of the extra straight lines beyond the boundary of the plane have a common point. (Of course this intersection is invisible on the plane.)

Detour Type II. The holders of the extra straight lines are super parallels, i.e., they have no intersection point in the exploded plane.

Example 3.1. Let

$$\mathcal{P}_0 = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \mathcal{P}_1 = \left(\frac{3}{4}, \frac{3}{4}\right), \text{ and } E = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad E_1 = \left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$$

be given points and vectors. Let us consider the following extra straight lines

$$\mathbb{L}_{\mathcal{P}_0, E}^{extra}: \quad \begin{aligned} u &= \tanh^{-1}\left(\frac{1}{2} + t \cdot \frac{1}{\sqrt{2}}\right) \\ v &= \tanh^{-1}\left(\frac{1}{2} - t \cdot \frac{1}{\sqrt{2}}\right) \end{aligned}, \quad -\frac{1}{\sqrt{2}} < t < \frac{1}{\sqrt{2}},$$

and

$$\mathbb{L}_{\mathcal{P}_1, E_1}^{extra}: \quad \begin{aligned} u &= \tanh^{-1}\left(\frac{3}{4} + \tau \cdot \frac{2}{\sqrt{5}}\right) \\ v &= \tanh^{-1}\left(\frac{3}{4} - \tau \cdot \frac{1}{\sqrt{5}}\right) \end{aligned}, \quad -\frac{\sqrt{5}}{4} < \tau < \frac{\sqrt{5}}{8}.$$
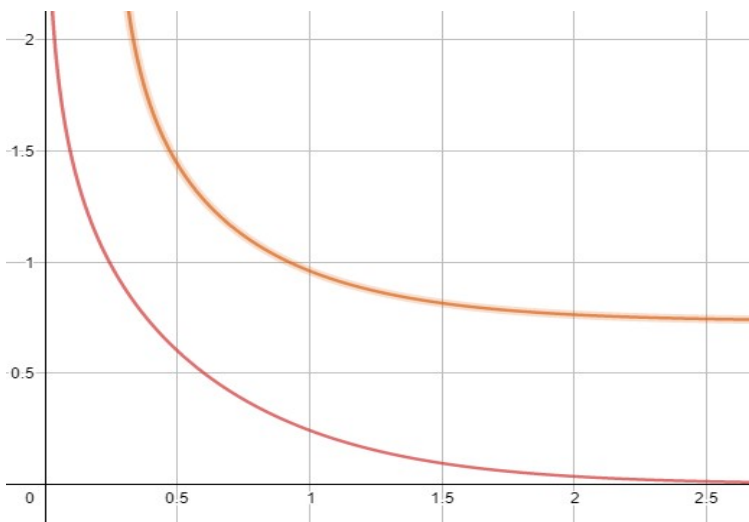
Their holders are

$$\widetilde{\mathbb{L}_{\mathcal{P}_0, E}}: \quad \begin{aligned} u &= \left(\widetilde{\frac{1}{2} + t \cdot \frac{1}{\sqrt{2}}}\right) \\ v &= \left(\widetilde{\frac{1}{2} - t \cdot \frac{1}{\sqrt{2}}}\right) \end{aligned}, \quad -\infty < t < \infty \text{ and } \widetilde{\mathbb{L}_{\mathcal{P}_1, E_1}}: \quad \begin{aligned} u &= \left(\widetilde{\frac{3}{4} + \tau \cdot \frac{2}{\sqrt{5}}}\right) \\ v &= \left(\widetilde{\frac{3}{4} - \tau \cdot \frac{1}{\sqrt{5}}}\right) \end{aligned}, \quad -\infty < \tau < \infty,$$

By the parameters $t = -\frac{3\sqrt{2}}{4}$ and $\tau = -\frac{\sqrt{5}}{2}$ we can see that $\widetilde{\mathbb{L}_{\mathcal{P}_0, E}} \cap \widetilde{\mathbb{L}_{\mathcal{P}_1, E_1}} = \left(\left(\widetilde{-\frac{1}{4}}\right), \left(\widetilde{\frac{5}{4}}\right)\right)$ exists, but invisible in the Euclidean plane. So, $\mathbb{L}_{\mathcal{P}_0, E}^{extra}$ and $\mathbb{L}_{\mathcal{P}_1, E_1}^{extra}$ show the Detour Type I.
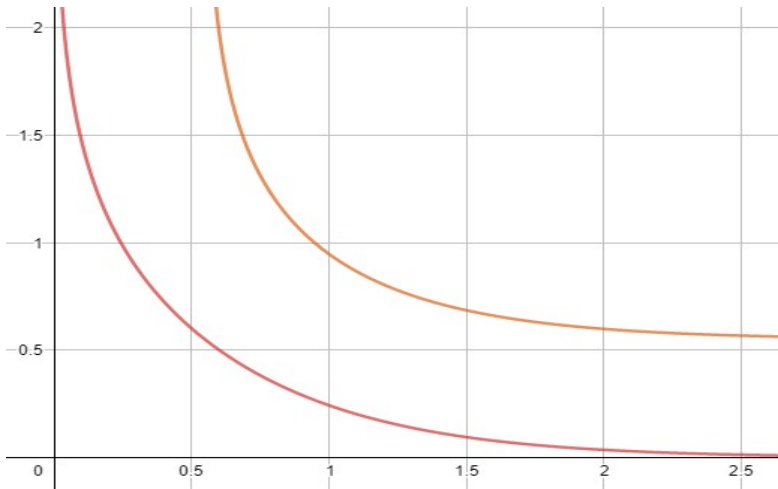
Example 3.2. Let

$$\mathcal{P}_0 = \left(\frac{1}{2}, \frac{1}{2}\right), \ \mathcal{P}_1 = \left(\frac{3}{4}, \frac{3}{4}\right), \ \text{and} \ E = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

be given points and vectors. Let us consider the following extra straight lines

$$\mathbb{L}_{P_0,E}^{extra}: \begin{aligned} u &= \tanh^{-1}\left(\frac{1}{2} + t \cdot \frac{1}{\sqrt{2}}\right) \\ v &= \tanh^{-1}\left(\frac{1}{2} - t \cdot \frac{1}{\sqrt{2}}\right) \end{aligned}, \quad -\frac{1}{\sqrt{2}} < t < \frac{1}{\sqrt{2}},$$

and

$$\mathbb{L}_{P_1,E}^{extra}: \begin{aligned} u &= \tanh^{-1}\left(\frac{3}{4} + \rho \cdot \frac{1}{\sqrt{2}}\right) \\ v &= \tanh^{-1}\left(\frac{3}{4} - \rho \cdot \frac{1}{\sqrt{2}}\right) \end{aligned}, \quad -\frac{\sqrt{2}}{4} < \rho < \frac{\sqrt{2}}{4}.$$

Their holders are the super-lines

$$\widetilde{\mathbb{L}_{P_0,E}}: \begin{aligned} u &= \overbrace{\left(\frac{1}{2} + t \cdot \frac{1}{\sqrt{2}}\right)} \\ v &= \overbrace{\left(\frac{1}{2} - t \cdot \frac{1}{\sqrt{2}}\right)} \end{aligned}, \quad -\infty < t < \infty \ \text{and} \ \widetilde{\mathbb{L}_{P_1,E}}: \begin{aligned} u &= \overbrace{\left(\frac{3}{4} + \rho \cdot \frac{1}{\sqrt{2}}\right)} \\ v &= \overbrace{\left(\frac{3}{4} - \rho \cdot \frac{1}{\sqrt{2}}\right)} \end{aligned}, \quad -\infty < \rho < \infty,$$

respectively. By Postulate of super-addition we can write

$$\widetilde{\mathbb{L}_{P_0,E}} : u \overline{\oplus} v = \breve{1} \ \text{ and } \ \widetilde{\mathbb{L}_{P_1,E}} : u \overline{\oplus} v = \left(\widetilde{\frac{3}{4}}\right).$$

Assuming that there exist a common point $\mathcal{P} = (u, v) \in \mathbb{R}^2$ of $\widetilde{\mathbb{L}_{P_0, E}}$ and $\widetilde{\mathbb{L}_{P_1, E}}$ we get the contradiction $1 = \frac{3}{4}$. (See the Postulate of unambiguity.) Hence, $\widetilde{\mathbb{L}_{P_0, E}} \cap \widetilde{\mathbb{L}_{P_1, E}} = \{\ \}$ (empty set). So, $\mathbb{L}_{P_0, E}^{extra}$ and $\mathbb{L}_{P_1, E}^{extra}$ show the Detour Type II.

Remark 3.3. On the figure of Example 3.2 the extra straight lines $\mathbb{L}_{P_0, E}^{extra}$ and $\mathbb{L}_{P_1, E}^{extra}$ seem to be shifted by the vector $\widetilde{\mathcal{P}_1} - \widetilde{\mathcal{P}_0} = \left(\tanh^{-1}\frac{3}{4} - \tanh^{-1}\frac{1}{2}, \tanh^{-1}\frac{3}{4} - \tanh^{-1}\frac{1}{2}\right)$. If it is true, then choosing $\mathcal{P}_L = (1, (1 - \tanh 1)) \in \mathbb{L}_{P_0, E}^{extra}$ the point $P_S = P_L + \left(\widetilde{\mathcal{P}_1} - \widetilde{\mathcal{P}_0}\right) \in \mathbb{L}_{P_1, E}^{extra}$. Having

$$P_S = \left(1 + \tanh^{-1}\frac{3}{4} - \tanh^{-1}\frac{1}{2},\ 1 - \tanh 1 + \tanh^{-1}\frac{3}{4} - \tanh^{-1}\frac{1}{2}\right)$$

and equation $\tanh u + \tanh v = \frac{3}{2}$, where u and v are greater than $\tanh^{-1}\frac{1}{2}$ of $\mathbb{L}_{P_1, E}^{extra}$, we get:

$$\tanh\left(1 + \tanh^{-1}\frac{3}{4} - \tanh^{-1}\frac{1}{2}\right) + \tanh\left(1 - \tanh 1 + \tanh^{-1}\frac{3}{4} - \tanh^{-1}\frac{1}{2}\right) \approx 1{,}470086907 < \frac{3}{2}.$$

Hence, $\mathcal{P}_S \notin \mathbb{L}_{P_1, E}^{extra}$, that is $\mathbb{L}_{P_0, E}^{extra}$ and $\mathbb{L}_{P_1, E}^{extra}$ are not parallel Euclidean curves.

4. Check the Detour of Extra Straight Lines by Compression

Based on (1.10) and (1.9) $\underline{\left(\widetilde{\mathbb{R}^2}\right)} = \mathbb{R}^2$, that is, the exploded plane is compressed by the plane itself. The compressed lines of the super straight lines will be Euclidean lines. The plane $\mathbb{R}^2$ is compressed into the open square (1.11). The compressed invisible boundaries of the plane will be the sides of the square. (See $\underline{\mathbb{L}_{lower}}, \underline{\mathbb{L}_{right}}, \underline{\mathbb{L}_{upper}}, \underline{\mathbb{L}_{left}}$ and (1.4)-(1.7).) With compression the extra straight lines become open passages closed by the compressions of the invisible endpoints of the extra straight lines.

Based on the last two figures (see Examples 3.1 and 3.2) the types of detour are difficult to distinguish.

***First we check the Example* 3.1.**

The extra straight lines $\mathbb{L}_{P_0, E}^{extra}$ have the equation $u \overline{\oplus} v = \check{1}$, $\left(0 < u < \check{1}\ \text{and}\ \dot{\ }0 < v < \check{1}\right)$ with the invisible endpoints $\mathcal{P}_{upper} = (0, \check{1}), \mathcal{P}_{right} = (\check{1}, 0)$ and $\mathbb{L}_{P_1, E_1}^{extra}$ has the equation $\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} u \overline{\oplus} v = \widetilde{\left(\frac{9}{8}\right)}$, $\left(\widetilde{\left(\frac{1}{4}\right)} < u < \check{1}\ \text{and}\ \dot{\ }\widetilde{\left(\frac{5}{8}\right)} < v < \check{1}\right)$ with the invisible

endpoints $\mathcal{P}^*_{upper} = \left(\left(\widetilde{\frac{1}{4}}\right), \widecheck{1}\right)$, $\mathcal{P}^*_{right} = \left(\widecheck{1}, \left(\widetilde{\frac{5}{8}}\right)\right)$, respectively. Their holders (super lines $\widetilde{\mathbb{L}_{P_0, E}}$ and $\widetilde{\mathbb{L}_{P_1, E_1}}$) give the equation system

$$u \overline{\oplus} v = \widecheck{1}$$
$$\left(\widetilde{\frac{1}{2}}\right) \overline{\odot} u \overline{\oplus} v = \left(\widetilde{\frac{9}{8}}\right) \tag{4.1}$$

without any restriction. The joint point is $\mathcal{P}_x = \left(\left(\widetilde{-\frac{1}{4}}\right), \left(\widetilde{\frac{5}{4}}\right)\right)$ is invisible. After compression instead of equation system (4.1) we get

$$\underline{u} + \underline{v} = 1$$
$$\frac{1}{2} \cdot \underline{u} + \underline{v} = \frac{9}{8} \tag{4.2}$$

The solution $\underline{u} = -\frac{1}{4}$, $\underline{v} = \frac{5}{4}$. Hence, $\underline{\mathcal{P}_x} = \left(-\frac{1}{4}, \frac{5}{4}\right)$ is visible in $\mathbb{R}^2$. Of course $\underline{\mathcal{P}_x}$ is outside $\underline{\mathbb{R}^2}$.

Second we check the Example **3.2.**

The extra straight lines $\mathbb{L}^{extra}_{P_0, E}$ have the equation $u \overline{\oplus} v = \widecheck{1}$, $\left(0 < u < \widecheck{1} \text{ and } \dot{} 0 < v < \widecheck{1}\right)$ with the invisible endpoints $\mathcal{P}_{upper} = \left(0, \widecheck{1}\right)$, $\mathcal{P}_{right} = \left(\widecheck{1}, 0\right)$ and $\mathbb{L}^{extra}_{P_1, E}$ has the equation $u \overline{\oplus} v = \left(\widetilde{\frac{3}{2}}\right)$, $\left(\left(\widetilde{\frac{1}{2}}\right) < u < \widecheck{1} \text{ and } \dot{} \left(\widetilde{\frac{2}{2}}\right) < v < \widecheck{1}\right)$ with the invisible endpoints $\mathcal{P}^{**}_{upper} = \left(\left(\widetilde{\frac{1}{2}}\right), \widecheck{1}\right)$, $\mathcal{P}^{**}_{right} = \left(\widecheck{1}, \left(\widetilde{\frac{1}{2}}\right)\right)$, respectively. Their holders (super lines $\widetilde{\mathbb{L}_{P_0, E}}$ and $\mathbb{L}^{extra}_{P_1, E}$) give the equation system

$$u \overline{\oplus} v = \widecheck{1}$$
$$u \overline{\oplus} v = \left(\widetilde{\frac{3}{2}}\right) \tag{4.3}$$

without any restriction. After compression instead of equation system (4.3) we get

$$\underline{u} + \underline{v} = 1$$
$$\underline{u} + \underline{v} = \frac{3}{2} \tag{4.4}$$

which has no solution. $\mathbb{L}_{P_0, E}$ and $\mathbb{L}_{P_1, E}$ are parallel Euclidean straight lines.

Finally, we check the intersection $\mathbb{L}^{extra}_{P_1, E}$ *and* $\mathbb{L}^{extra}_{P_1, E_1}$. Their holders produce the equation system

$$u \overline{\oplus} v = \left(\widetilde{\frac{3}{2}}\right)$$
$$\left(\widetilde{\frac{1}{2}}\right) \overline{\odot} u \overline{\oplus} v = \left(\widetilde{\frac{9}{8}}\right) \tag{4.5}$$

which gives the visible solution $\widetilde{\mathcal{P}_1} = \left(\left(\widetilde{\frac{3}{4}}\right), \left(\widetilde{\frac{3}{4}}\right)\right) \in \mathbb{R}^2$. $\left(\left(\widetilde{\frac{3}{4}}\right) \approx 0{,}9729550745.\right)$

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