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The Weibull Logistic-exponential Distribution: Its Properties and Applications

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Abstract

In this paper, a new generalized distribution known as Weibull Logistic-Exponential Distribution (WLED) is proposed using the $T\text{-}R\{Y\}$ framework. Several mathematical properties of this new distribution are studied. The maximum likelihood estimation method was used in estimating the parameters of the proposed distribution. Finally, an application of the proposed distribution to a real lifetime data set is presented and its fit was compared with the fit obtained by some comparable lifetime distributions.

1. Introduction

The logistic model is a symmetric distribution that has found application in vast areas of growth model in human populations. [11] and [9] used the logistic function in modelling data related to population. Several generalizations of the logistic distribution have been introduced in literature to improve the symmetric and tail (heavy & light) properties of the distribution. Examples of such generalization of logistic distribution are found in the works of [3], [7], [8], [14], [13], and [4].

This paper is motivated by the flexibility of the generalized distribution in terms of exhibiting and accommodating diverse kind of data sets. The proposed distribution have found an application in finance for modelling risk management and explain the return of an investment due to the strictly platykurtic nature of kurtosis $(K_S < 3)$. The distribution

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spans through left skewed, right skewed and symmetric shapes. The remaining sections of this paper are organized as follows: In Section 2, we present the density function and cumulative distribution function of the proposed distribution. General statistical properties which include the survival function, hazard rate function, moments, the relationship between WLED and Weibull distribution, quantile function, median, Rényi and Shannon entropy of WLED are obtained in Section 3. In Section 4, we estimated the maximum likelihood parameters of the proposed WLED. In Section 5, we fit the proposed distribution to a real lifetime data set and compared its fit with the fit attained by some existing related lifetime distributions and the paper concludes in Section 6.

2. CDF and PDF of the Proposed Weibull-Logistic {Exponential} Distribution (WLED)

Let T, R and Y be random variables from a known probability distribution with the cumulative distribution functions defined by $F_T(x) = P(T \leq x)$, $F_R(x) = P(R \leq x)$ and $F_T(x) = P(T \leq x)$ and probability density functions given by $f_T(x)$, $f_R(x)$ and $f_Y(x)$ respectively. Let the corresponding quantile functions be given as $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$. [2] gave a unified definition of the random variables reported in [1] by defining the cumulative distribution function of a random variable X as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = P\{T \leq Q_Y(F_R(x))\} = F_T[Q_Y(F_R(x))] \quad (1)$$

and the corresponding density function defined as

$$f_X(x) = \frac{f_R(x)}{f_Y\{Q_Y(F_R(x))\}} f_T\{Q_Y(F_R(x))\}.$$

Let R be a random variable following a logistic distribution with pdf and cdf define by

$$f_{(R)}(x) = \frac{\lambda e^{\lambda_x}}{(1+\ell^{\lambda x})^2}$$

and

$$F_R(x) = \frac{\ell^{\lambda x}}{1+\ell^{\lambda x}}.$$

Then the cdf of the proposed distribution is gotten by using the quantile function of a random variable Y defined by the relation.

$$Q_Y(P) = -\log(1 - F_R(x)).$$

By following the technique in Equation (1) we obtain the cdf of the proposed distribution.

$$F(x) = 1 - e^{-\left(\left(\frac{\log\left(1+e^{\lambda x}\right)}{k}\right)^{r}\right)}, \quad r, k, \lambda > 0, \ x \in R. \tag{2}$$

The corresponding pdf of the WLED can be obtained by finding the derivative of its cdf in (2) with respect to x, this is given by:

$$f(x) = \frac{r\lambda e^{\lambda x}\left(\frac{\log\left(1+e^{\lambda x}\right)}{k}\right)^{r-1} e^{-\left(\frac{\log\left(1+e^{\lambda x}\right)}{k}\right)^{r}}}{k\left(1+e^{\lambda x}\right)}, \quad r, k, \lambda > 0, \ x \in R. \tag{3}$$

The graphical presentation of the density function of WLED for some fixed values of the parameters is shown in Figure 1.

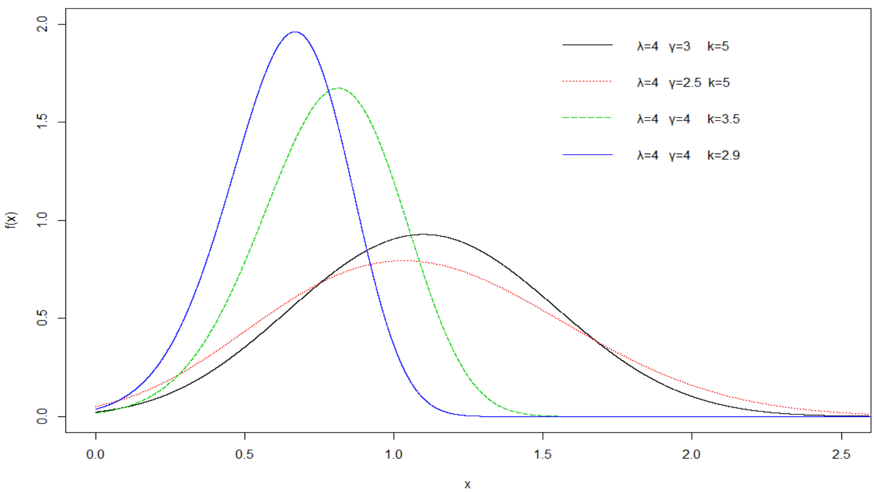

Figure 1. Probability density function of the WLED.

3. Properties of the Proposed Distribution (WLED)

3.1. Survival function

Let X be a continuous random variable with density function $f(x)$ and cumulative distribution function $F(x)$. The survival (reliability) function of the proposed WLED are defined by:

$$S(x) = 1 - F(x),$$

where $F(x)$ is the cdf of the WLED define in Equation (2).

$$S(x) = 1 - \left(1 - e^{-\left(\frac{\log(1+e^{\lambda x})}{k}\right)^r}\right)$$

$$S(x) = e^{-\left(\frac{\log(1+e^{\lambda x})}{k}\right)^r}, \quad r,\ \lambda,\ k > 0, \quad x \in R. \tag{4}$$

3.2. Hazard rate function

Let X be a continuous random variable with density function $f(x)$ and cumulative distribution function $F(x)$. The hazard rate (failure rate) function of the proposed WLED are defined by:

$$H(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)}$$

$$H(x) = \frac{\dfrac{r\lambda e^{\lambda x}\left(\dfrac{\log(1+e^{\lambda x})}{k}\right)^{r-1} e^{-\left(\frac{\log(1+e^{\lambda x})}{k}\right)^r}}{k(1+e^{\lambda x})}}{e^{-\left(\frac{\log(1+e^{\lambda x})}{k}\right)^r}}.$$

Simplify further yields

$$H(x) = \frac{r\lambda e^{\lambda x}\left(\dfrac{\log(1+e^{\lambda x})}{k}\right)^{r-1}}{k(1+e^{\lambda x})}, \quad r,\ k,\ \lambda > 0, \quad x \in R.$$

The graph of the hazard rate function of the WLED for different values of r, k and λ is given in Figure 2.

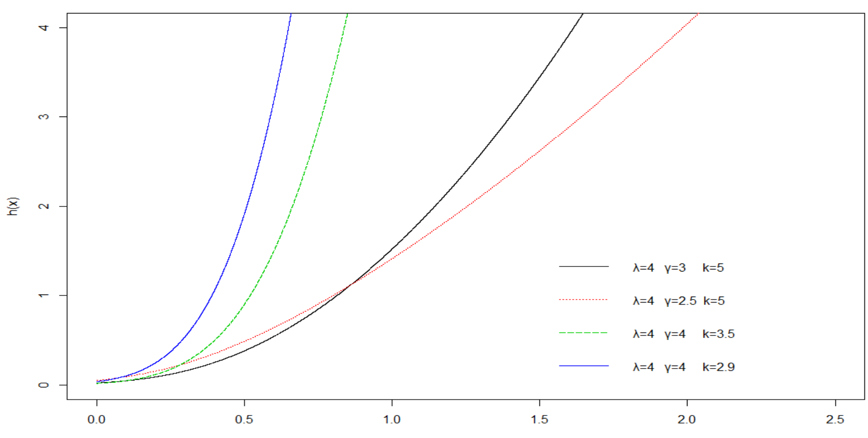

Figure 2. Hazard rate function of the WLED.

Remark. From Figure 2, it is clearly shown that the hazard rate function of the WLED at different parameters values is strictly increasing. This implies that the hazard rate function of WLED shows an increasing property.

3.3. Quantile function of the WLED

The quantile function of the WLED can be expressed in a closed form and it can be obtained by equating the cdf in (2) to P and solving for x as shown below:

$$F(x) = 1 - e^{-\left(\frac{\log(1+e^{\lambda x})}{k}\right)^{r}} = P.$$

Hence the quantile function of WLED is given by:

$$Q_x(P) = \frac{\log\left(e^{k(-\log(1-P))^{\frac{1}{r}}} - 1\right)}{\lambda}. \tag{5}$$

The median of the WLE distribution can be obtained by setting $p = 0.5$ in equation (5) which gives

$$X_{med} = Q_x(0.5) = \frac{\log\left[e^{k[-\log(0.5)]^{1/r}} - 1\right]}{\lambda}. \tag{6}$$

Table 1. Quantiles of the WLED.

P	$k=2, r=2, \lambda=0.5$	$k=1, r=2, \lambda=2$	$k=3, r=2, \lambda=4$	$k=4, r=2, \lambda=3$	$k=2, r=2, \lambda=0.1$
0.1	−5.5451	−0.3112	−0.0115	0.0438	18.5193
0.2	−0.1272	−0.2187	0.0472	0.1311	27.0309
0.3	0.1563	−0.1584	0.0884	0.1943	34.1839
0.4	0.3891	−0.1103	0.1231	0.2489	40.9019
0.5	0.6004	−0.0681	0.1555	0.3006	47.6561
0.6	0.8067	−0.0279	0.1877	0.3532	54.7927
0.7	1.0229	0.0128	0.2224	0.4105	62.8079
0.8	1.2716	0.0582	0.2631	0.4789	72.6179
0.9	1.6109	0.1175	0.3202	0.5764	86.8589

From the quantiles, it is observed that all the values falls within the support (entire real line) which shows the validity of the proposed distribution.

3.4. Moments of the Weilbull logistic exponential distribution

Let X ~ WLED, then the rth moment about the origin of X is given by

$$U'_P = \left(\frac{1}{\lambda}\right)^P \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} A_1(w_i, P, i, k, r).$$

Proof. Let X be a random variable following the Weilbull logistic exponential distribution with parameters r, k and λ, then

$$E[X^P] = U'_P = \int_{-\infty}^{\infty} x^P f(x)\, dx, \tag{7}$$

where $f(x)$ is as defined in (3) above.

Hence,

$$E[X^P] = U'_P = \int_{-\infty}^{\infty} x^P \frac{r\lambda e^{\lambda x}\left(\frac{\log(1+e^{\lambda x})}{k}\right)^{r-1} e^{-\left(\frac{\log(1+e^{\lambda x})}{k}\right)^r}}{k(1+e^{\lambda x})} dx.$$

Simplifying further yields

$$U'_P = \frac{r\lambda}{k}\int_0^{\infty} \frac{\left(\frac{\log(e^{kY}-1)}{\lambda}\right)^P (e^{kY}-1)Y^{r-1}e^{-Y^r}}{e^{kY}} \cdot \frac{ke^{kY}}{\lambda(e^{kY}-1)} dy,$$

hence

$$U'_P = \left(\frac{1}{\lambda}\right)^P \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} A_1(w_i, P, i, k, r),$$

where $A_1(w_i, P, i, k, r) = \int_0^{\infty} w^i (\log(e^{kw^{\frac{1}{r}}} - 1))^P\, dw.$

3.5. Numerical computation on the use of WLED

Here a study is carried out to obtain the mean, standard deviation, median, skewness and kurtosis of the Weilbull logistic exponential distribution for varying parameters values.

Table 2. The Mean, Standard Deviation (SD) and the Median of the WLED.

		$\lambda = 0.5$			$\lambda = 2$			$\lambda = 4$		
r	κ	Mean	S.D	Median	Mean	S.D	Median	Mean	S.D	Median
	3	4.3787	8.9554	2.8375	1.0947	2.2388	0.7094	0.5473	1.1194	0.3547
0.8	5	8.8155	13.4132	5.3164	2.2039	3.3533	1.3291	1.1019	1.6767	0.6645
	8	15.1576	20.4350	8.6975	3.7894	5.1087	2.1744	1.8947	2.5544	2.5544
	3	4.4594	3.4368	4.5073	1.1148	0.8592	1.1268	0.5574	0.4296	0.5634
2	5	8.0177	5.0854	7.8051	2.0044	1.2714	1.9513	1.0022	0.6357	0.9756
	8	13.1117	7.7563	12.5485	3.2779	1.9391	3.1371	1.6390	0.9695	1.5686
	3	5.1024	1.5839	5.3089	1.2756	0.3940	1.3272	0.6378	0.1980	0.6636
5	5	8.7867	2.4166	9.0533	2.1967	0.6042	2.2633	1.0983	0.3021	1.1317
	8	14.1338	3.7683	14.5183	3.5334	0.9421	3.6296	1.7667	0.4710	1.8148

Table 3. Skewness and Kurtosis of the WLED.

		$\lambda = 0.5$		$\lambda = 2$		$\lambda = 4$	
r	κ	Skewness (S_k)	Kurtosis (K_S)	Skewness (S_k)	Kurtosis (K_S)	Skewness (S_k)	Kurtosis (K_S)
	3	0.8213	0.6715	0.8213	0.6715	0.8213	0.6715
0.8	5	1.2279	1.1203	1.2279	1.1203	1.2279	1.1279
	8	1.4318	1.4410	1.4318	1.4410	1.4318	1.4410
	3	-0.1320	-0.2269	-0.1320	-0.2269	-0.1320	-0.2269
2	5	0.1603	-0.4626	0.1603	-0.4626	0.1603	-0.4626
	8	0.3169	-0.5454	0.3169	-0.5454	0.3169	-0.5454
	3	-0.6830	0.3476	-0.6830	0.3476	-0.6830	0.3476
5	5	-0.5324	0.0077	-0.5324	0.0077	-0.5324	0.0077
	8	-0.4620	-0.1501	-0.4620	-0.1501	-0.4620	-0.1501

From Table 2 we observed that when the shape parameter r and the rate parameter λ are held constant, the mean, standard deviation, and median increases as the scale parameter k increase.

Also from Table 3 we observed that the WLED exhibited a right-skewed $(S_k \geq 0)$, left-skewed ($S_k \leq 0$) and approximately symmetric ($S_k \approx 0$) shapes while the kurtosis are all strictly platykurtic ($K_S < 3$). This assertion clearly supports and maintains the ideal behinds the graphical illustration of the density function of the WLE distribution in Figure 1.

3.6. Moment generating function (MGF) of the WLED

Let X be a continuous random variable with density function $f(x)$, then the mgf of X following the WLED is defined by;

$$M_X(t) = E[e^{tx}] = \int_0^{rr\infty} e^{tx} f(x)\, dx.$$

Hence,

$$M_X(t) = \left(\frac{1}{\lambda}\right)^P \sum_{P=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^P}{P!} \frac{(-1)^i}{i!} \int_0^{\infty} w^i \left(\log\left(e^{kw^{\frac{1}{r}}} - 1\right)\right)^P dw. \quad (8)$$

3.7. Shannon entropy of the WLED

[12] defined the entropy of a random variable, X as a measure of variation of uncertainty denoted by $\eta_x = E[-\log f(x)]$. The Shannon entropy of the WLED is given as:

$$\eta_x = 1 + \xi\left(1 - \frac{1}{r}\right) + \log\left(\frac{k}{r}\right) - k\Gamma\left(1 + \frac{1}{r}\right) - E[\log\lambda + \lambda x] + E\left[2\log\left(1 + e^{\lambda x}\right)\right],$$

where $\xi \cong 0.5772$ is the Euler's constant.

Proof. See Appendix I.

3.8. Rényi entropy of WLED

The entropy of a random variable, X, denoted by $\tau_R(s)$ is defined as a measure of the uncertainty about the outcome of a random experiment. Let X be a random variable with pdf (x), then the Rényi entropy is defined by $\tau_R(s) = \frac{1}{1-S}\log[\int f^s(x)dx]$ for $s > 0$ and $s \neq 1$ [10].

Then the Rényi entropy of WLED is defined by:

$$\tau_R(s) = \frac{1}{1-S}\log\left[\int f^s(x)dx\right],$$

where $f(x)$ is the pdf of the WLED defined in (3) above.

Hence

$$\tau_R(s) = \frac{1}{1-S}\log\left[\int_{-\infty}^{\infty}\left(\frac{r\lambda e^{\lambda x}\left(\frac{\log\left(1+e^{\lambda x}\right)}{K}\right)^{r-1} e^{-\left(\frac{\log\left(1+e^{\lambda x}\right)}{K}\right)^{r}}}{k(1+e^{\lambda x})}\right)^{s} dx\right].$$

Simplified further yields

$$\tau_R(s) = \frac{1}{1-s}\left(\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\right)^{s} \log\int_0^{\infty}\left(w^i\right)^s dw. \qquad (9)$$

Proof. See Appendix II.

[5] provided some essential properties of Rényi entropy:

(i) The Rényi entropy can be negative;

(ii) For any $s_1 < s_2$, $R_{s_2} \leq R_{s_1}$ and equality holds if and only if X is a uniform random variable.

Numerical computations of the Rényi entropy of the WLED for varying values of parameter are shown in Table 4.

Table 4. Numerical computations of the Rényi entropy of the WLED.

S	$(k=4, \lambda=3, r=2)$	$(k=1, \lambda=5, r=3)$	$(k=2, \lambda=1, r=4)$
0.02	0.5418	3.3969	1.8192
0.05	0.3566	1.9843	1.1475
0.1	0.0518	0.5542	0.9819
0.2	-0.2953	-0.3830	0.5551
0.6	-0.4209	-0.4708	0.3896
3	-0.5247	-0.9404	0.0759
4	-0.7481	-1.0766	-0.0412
6	-0.8379	-1.1107	-0.0714
8	-0.89066	-1.1545	-0.1108
9	-0.9813	-1.1821	-0.1359

From Table 4, we clearly observe that for any two consecutive values of parameter s_i , Say (s_1 and s_2), the Rényi entropy R_{s_i} say $\left(R_{s_1}\text{ and } R_{s_2}\right)$, satisfies the condition $s_1 < s_2$ $R_{s_2} \leq R_{s_1}$ as suggested by [5].

4. Maximum Likelihood Estimates

Let $x_1, x_2, x_3, ..., x_n$ be random samples from the WLED with density function defined in equation (3), then the likelihood function is given by

$$L(x,\phi) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{\left[r\lambda\ell^{\lambda x}\right]\left[\dfrac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]^{r-1} \ell^{-\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]^{r}}}{k\left(1+\ell^{\lambda x}\right)}$$

and the log likelihood is

$$\ell(x,\phi) = \sum_{i=1}^{n} \log(f(x_i)) = \sum_{i=0}^{n} \log r + \sum_{i=0}^{n} \log\lambda + \sum_{i=0}^{n} \lambda x_i + \sum_{i=0}^{n} \log\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]^{r-1}$$

$$-\sum_{i=0}^{n}\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]^{r} - \sum_{i=0}^{n}\log k - \sum_{i=0}^{n}\log\left(1+\ell^{\lambda x}\right),$$

$$\sum_{i=1}^{n} \log(f(x_i)) = n\log r + n\log\lambda + \lambda\sum_{i=0}^{n} x_i + (r-1)\sum_{i=0}^{n}\log\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]$$

$$-\sum_{i=0}^{n}\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]^{r} - n\log k - \sum_{i=0}^{n}\log\left(1+\ell^{\lambda x}\right).$$

Taking the derivatives with respect to the various parameters, the following are obtained:

$$\frac{\partial L}{\partial r} = \frac{n}{r} + \sum_{i=1}^{n}\log\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right) - \sum_{i=1}^{n}\log\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{\ell^{\lambda x_i} x_i}{1 + \ell^{\lambda x_i}} + (r-1) \sum_{i=1}^{n} \frac{\ell^{\lambda x_i} x_i}{\left(1 + \ell^{\lambda x_i}\right) \log\left[1 + \ell^{\lambda x_i}\right]}$$

$$-\sum_{i=1}^{n} \frac{\ell^{\lambda x_i} r \left(\frac{\log\left[1 + \ell^{\lambda x_i}\right]}{k} \right)^{r-1} x_i}{\left(1 + \ell^{\lambda x_i}\right) k}$$

$$\frac{\partial L}{\partial k} = -\frac{n}{k} - \frac{n(r-1)}{k} + \sum_{i=1}^{n} \frac{r \log\left[1 + \ell^{\lambda x_i}\right] \left(\frac{\log\left[1 + \ell^{\lambda x_i}\right]}{k} \right)^{r-1}}{k^2}.$$

The maximum likelihood estimator $\hat{\varphi}$ of φ can be derived by using Newton Raphson's iterative method given by the relation:

$$\hat{\varphi} = \varphi_q - H^{-1}(\varphi_q) U(\varphi_q), \qquad \hat{\varphi} = (\hat{r}, \hat{k}, \hat{\lambda})^T,$$

where

$$H\left(\varphi_q\right) = \begin{pmatrix} \frac{\partial^2 \ell}{\partial r^2} & \frac{\partial^2 \ell}{\partial r \partial \lambda} & \frac{\partial^2 \ell}{\partial r \partial k} \\ \frac{\partial^2 \ell}{\partial \lambda \partial r} & \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial k} \\ \frac{\partial^2 \ell}{\partial k \partial r} & \frac{\partial^2 \ell}{\partial k \partial \lambda} & \frac{\partial^2 \ell}{\partial k^2} \end{pmatrix}. \tag{10}$$

Proof. See Appendix III.

5. Application and Discussion of Results

In this section, some generalized probability distribution such as Exponentiated-Weilbull Distribution, Weibull-Exponential Distribution, and Logistic-Exponential Distribution are applied in fitting real life data set and the result is being compared with the proposed Weilbull-Logistic Exponential Distribution. The comparison criterion used in the study were, Alkaike Information Criteria (AIC), and Kolmogorov-Smirnov test (K-S) and the Log-likelihood. The data set used represents the breaking stress of carbon fibers of 50 mm length in (GPa). The data was obtained from [6]. The data set is unimodal with (skewness = -0.128 and kurtosis = 0.1261208). The data is shown in Table.

Table 5. Breaking stress of carbon fibers of 50mm in length (GPA).

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11
3.56	4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15
4.90	1.57	2.67	2.93	3.22	3.39	2.81	4.20	3.33	2.55
3.31	3.31	2.85	1.25	4.38	1.84	0.39	3.68	2.48	0.85
1.61	2.79	4.70	2.03	1.89	2.88	2.82	2.05	3.65	3.75
2.43	2.95	2.97	3.39	2.96	2.35	2.55	2.59	2.03	1.61
2.12	3.15	1.08	2.56	1.80	2.53				

Source: [6].

Table 6. Comparison criterion for the data set.

Distribution	**Exponentiated Weibull**	**Weibull Exponential**	**Logistic-Exponential**	**WLED**
Parameter estimates	C =0.30952 (0.03304) $\hat{k}$ =0.8002 (0.3525) $\hat{r}$ =3.9104 (1.0666)	$\hat{u}$ = 51.99153 $\hat{b}$ = 3.069453 $\hat{c}$ = 0.07921	$\alpha = 29.3190$ (41.1783) $\beta =$ 0.01862 (0.02611)	$\hat{r}$ =3.64151 (0.4656) $\hat{\lambda}$ =1.3731 (0.92225) $\hat{k}$ =4.2485 (2.76820)
AIC	177.889	177.8337	340.2702	176.1749
Log	−85.9447	-85.91686	-167.1351	−85.5870
K-S	0.07292 (0.7809)	0.07977 (0.795)	0.62999 (p-value < $2.2e^{-16}$)	0.07866 (0.8087)

Table 6 reveals the summary statistics for the breaking stress of carbon fibers of 50mm in length data set. The parameter estimates, Log-likelihood, Akaike Information Criterion (AIC), Kolmogorov-Sminov Statistic (K-S) with their corresponding p-values of the distributions were estimated for the data set. The Table indicates that the proposed Weibull-logistics exponential distribution gives the best fit for the dataset and thus exhibits superiority over the examined lifetime distributions considered in modeling this lifetime data set. This decision was further supported by examining the probability-probability plots and the density plot distributions for the real lifetime data sets.

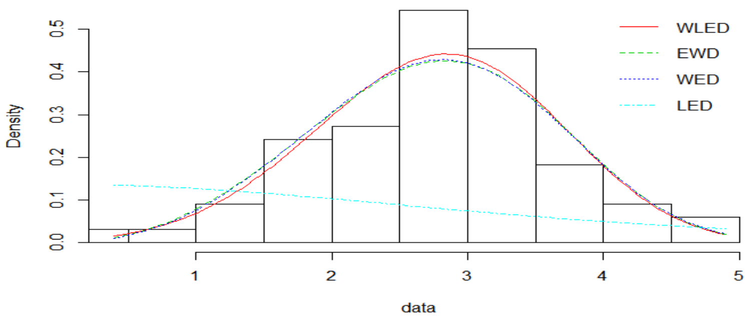

Figure 3. Density fit for the data set.

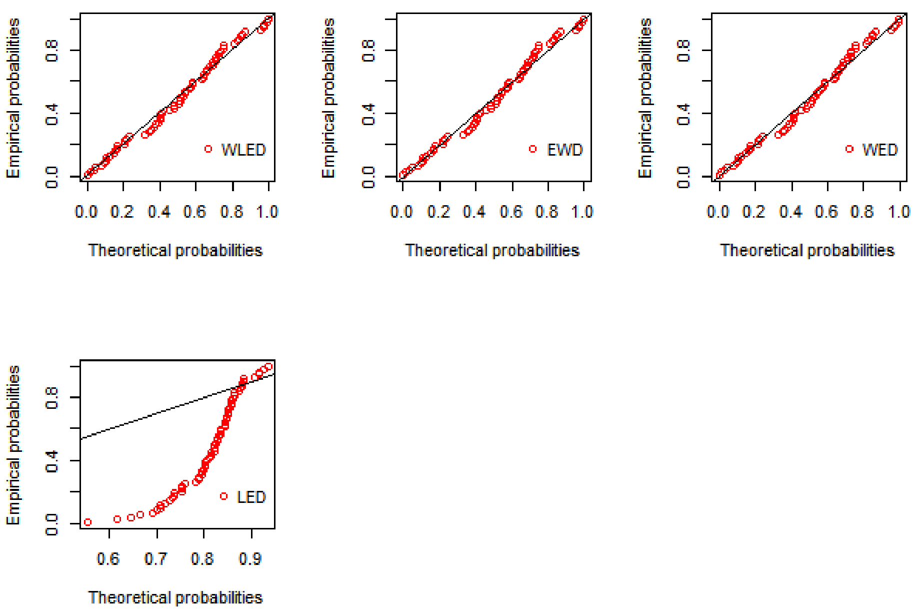

Figure 4. Fitted probability-probability plot for the data set.

6. Concluding Remark

In this paper, a new distribution known as Weibull logistic-exponential distribution is proposed. Several Mathematical properties of this new distribution are study which include the cumulative distribution function, survival function, the density function, hazard rate function, moment generating function, quantile function, Shannon and Rényi entropy are obtained, the maximum likelihood estimation method was used in estimating the parameters of the proposed distribution. The application of the proposed distribution to a real lifetime data set reveals its superiority over some well-known generalized distribution.

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Appendix I

Shannon [12] defined the entropy of a random variable X as

$$\eta_x = E[-\log f(x)].$$

However, Alzaatreh *et al.* [2] define the probability density as

$$f(x) = f_R(x).\frac{f_T\left(Q_Y\big(F_R(x)\big)\right)}{f_Y\left(Q_Y\big(F_R(x)\big)\right)}.$$

But from equation (1), $T = Q_Y\big(F_R(x)\big)$, then

$$f(x) = f_R(x).\frac{f_T(T)}{f_Y(T)} \tag{11}$$

where, $f_R(x)$, $f_T(T)$ and $f_Y(T)$ are the pdf of logistic, Weilbuland exponential distribution define by:

$$f_R(x) = \frac{\lambda e^{\lambda x}}{(1+e^{\lambda x})^2}, \qquad f_T(T) = \frac{r}{k}\left(\frac{T}{k}\right)^{r-1} e^{-\left(\frac{T}{k}\right)^r},$$

and $f_Y(T) = e^{-T}$respectively.

Taking the log of both sides of equation (11), we obtain,

$$\log f(x) = \log f_R(x) + \log f_T(T) - \log f_Y(T)$$

or

$$E[-\log f(x)] = E[-\log f_R(x)] + E[-\log f_T(T)] + E[\log f_Y(T)],$$

then

$$E[-\log f(x)] = -E\left[\log \lambda e^{\lambda x}\right] + E\left[\log\left(1 + e^{\lambda x}\right)\right]^2 + E[-\log f_T(T)]$$
$$+E[\log f_Y(T)] \qquad (12)$$

but

$$\eta_T = E[-\log f_T(T)] = 1 + \xi\left(1 - \frac{1}{r}\right) + \log\left(\frac{k}{r}\right)$$

$$E[T] = U_T = E[\log f_Y(T)] = -k\Gamma\left(1 + \frac{1}{r}\right).$$

Substituting η_T and U_T into equation (12) yields

$$\eta_x = -E\left[\log \lambda e^{\lambda x}\right] + E\left[\log\left(1 + e^{\lambda x}\right)\right]^2 + 1 + \xi\left(1 - \frac{1}{r}\right) + \log\left(\frac{k}{r}\right) - k\Gamma\left(1 + \frac{1}{r}\right)$$

$$\eta_x = 1 + \xi\left(1 - \frac{1}{r}\right) + \log\left(\frac{k}{r}\right) - k\Gamma\left(1 + \frac{1}{r}\right) - E[\log \lambda + \lambda x] + E\left[\log\left(1 + e^{\lambda x}\right)\right]^2.$$

Appendix II

Theorem 2. *Let X ~ WLED, then the Rényi entropy of the WLED is given by*

$$\tau_R(s) = \frac{1}{1-s}\left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\right)^s \log \int_0^{\infty} \left(w^i\right)^s dw.$$

Proof. The Rényi entropy of WLED is defined by

$$\tau_R(s) = \frac{1}{1-S} \log\left[\int f^s(x) dx\right],$$

where $f(x)$ is the pdf of the WLED defined in (3) above. Hence

$$\tau_R(s) = \frac{1}{1-s}\log\left[\int_{-\infty}^{\infty}\left(\frac{r\lambda e^{\lambda x}\left(\frac{\log\left(1+e^{\lambda x}\right)}{K}\right)^{r-1} e^{-}\left(\frac{\log\left(1+e^{\lambda x}\right)}{K}\right)^{r}}{k(1+e^{\lambda x})}\right)^{s} dx\right]$$

$$\tau_R(s) = \frac{1}{1-s}\left(\frac{r\lambda}{K}\right)^{s}\log\left[\int_{-\infty}^{\infty}\left(\frac{e^{\lambda x}\left(\frac{\log\left(1+e^{\lambda x}\right)}{k}\right)^{r-1} e^{-}\left(\frac{\log\left(1+e^{\lambda x}\right)}{k}\right)^{r}}{(1+e^{\lambda x})}\right)^{s} dx\right]. \quad (13)$$

Let
$$y = \frac{\log(1+\ell^{\lambda x})}{k}. \quad (14)$$

When $x = -\infty$, $y = 0$ and when $x = \infty$, $y = \infty$

$$ky = \log(1+\ell^{\lambda x})$$

$$\ell^{\lambda x} = (\ell^{ky} - 1) \quad (15)$$

$$dx = \frac{Ke^{ky}}{\lambda(e^{ky}-1)}dy. \quad (16)$$

Substituting (14), (15) and (16) into equation (13), we obtain,

$$\tau_R(s) = \frac{1}{1-s}\left(\frac{r\lambda}{k}\right)^{s}\log\int_0^{\infty}\left(\frac{\left(e^{ky}-1\right)y^{r-1}e^{-yr}}{e^{ky}}\right)^{s}\cdot\left(\frac{ke^{ky}}{\lambda(e^{ky}-1)}\right)^{s}dy.$$

Simplify further yields

$$\tau_R(s) = \frac{r^s}{1-S}\log\int_0^{\infty}(y^{r-1}e^{-yr})^s\, dy. \quad (17)$$

Let
$$w = y^r \quad (18)$$

$$y = w^{\frac{1}{r}}. \quad (19)$$

Then

$$dy = \frac{1}{r}w^{\frac{1}{r}-1}dw. \quad (20)$$

Substituting (18), (19) and (20) into (17), we obtain,

$$\tau_R(s) = \frac{r^s}{1-s}\log\int_0^\infty \left[\left(w^{\frac{1}{r}}\right)^{r-1} e^{-w}.\frac{1}{r}w^{\left(\frac{1}{r}-1\right)}\right]^s dw$$

$$\tau_R(s) = \frac{1}{1-s}\log\int_0^\infty (e^{-w})^s\, dw,$$

but $e^{-w} = \sum_{i=0}^{\infty}\frac{(-1)^i w^i}{i!}$, then

$$\tau_R(s) = \frac{1}{1-s}\log\int_0^\infty \left(\sum_{i=0}^{\infty}\frac{(-1)^i w^i}{i!}\right)^s dw$$

$$\tau_R(s) = \frac{1}{1-s}\left(\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\right)^s \log\int_0^\infty \left(w^i\right)^s dw.$$

This completes the proof.

Appendix III

The second derivative with respect to each of the parameters.

$$\frac{\partial^2 L}{\partial r^2} = -\frac{n}{r^2} + \sum_{i=1}^{n}\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^r \log\left[\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right]^2$$

$$\frac{\partial^2 L}{\partial r\partial\lambda} = \sum_{i=1}^{n}\frac{\ell^{\lambda x_i} x_i}{\left(1+\ell^{\lambda x_i}\right)\log\left[1+\ell^{\lambda x_i}\right]}$$

$$-\sum_{i=1}^{n}\left(\frac{\ell^{\lambda x_i}\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^r x_i}{\left(1+\ell^{\lambda x_i}\right)\log\left[1+\ell^{\lambda x_i}\right]} + \frac{\ell^{\lambda x_i} r\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}\log\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)x_i}{\left(1+\ell^{\lambda x_i}\right)k}\right)$$

$$\frac{\partial^2 L}{\partial r\partial k}=-\frac{n}{k}-\sum_{i=1}^{n}\left(-\frac{\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r}}{k}-\frac{r\log\left[1+\ell^{\lambda x_i}\right]\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}\log\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)}{k^2}\right)$$

$$\frac{\partial^2 L}{\partial\lambda\partial r}=\sum_{i=1}^{n}\frac{\ell^{\lambda x_i}x_i}{\left(1+\ell^{\lambda x_i}\right)\log\left[1+\ell^{\lambda x_i}\right]}$$

$$-\sum_{i=1}^{n}\left(\frac{\ell^{\lambda x_i}\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}x_i}{\left(1+\ell^{\lambda x_i}\right)k}+\frac{\ell^{\lambda x_i}r\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}\log\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)x_i}{\left(1+\ell^{\lambda x_i}\right)k}\right)$$

$$\frac{\partial^2 L}{\partial\lambda^2}=-\frac{n}{\lambda^2}-\sum_{i=1}^{n}-\left(\frac{\ell^{2\lambda x_i}x_i^2}{\left(1+\ell^{\lambda x_i}\right)^2}+\frac{\ell^{\lambda x_i}x_i^2}{1+\ell^{\lambda x_i}}\right)+(r-1)$$

$$\sum_{i=1}^{n}-\left(\frac{\ell^{2\lambda x_i}x_i^2}{\left(1+\ell^{\lambda x_i}\right)^2\log\left[1+\ell^{\lambda x_i}\right]^2}-\frac{\ell^{2\lambda x_i}x_i^2}{\left(1+\ell^{\lambda x_i}\right)^2\log\left[1+\ell^{\lambda x_i}\right]}+\frac{\ell^{\lambda x_i}x_i^2}{\left(1+\ell^{\lambda x_i}\right)\log\left[1+\ell^{\lambda x_i}\right]}\right)$$

$$-\sum_{i=1}^{n}-\left(\frac{\ell^{2\lambda x_i}x_i(r-1)r\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-2}x_i^2}{\left(1+\ell^{\lambda x_i}\right)^2k^2}+\frac{\ell^{2\lambda x_i}r\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}x_i^2}{\left(1+\ell^{\lambda x_i}\right)^2k}+\frac{\ell^{\lambda x_i}r\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}x_i^2}{\left(1+\ell^{\lambda x_i}\right)k}\right)$$

$$\frac{\partial^2 L}{\partial\lambda\partial k}=-\sum_{i=1}^{n}-\frac{\ell^{\lambda x_i}(r-1)r\log\left[1+\ell^{\lambda x_i}\right]\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-2}x_i}{\left(1+\ell^{\lambda x_i}\right)k^3}-\frac{\ell^{\lambda x_i}r\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}x_i}{\left(1+\ell^{\lambda x_i}\right)k^2}$$

$$\frac{\partial^2 L}{\partial k \partial r} = -\frac{n}{k} - \sum_{i=1}^{n} \left(-\frac{\log\left[1+\ell^{\lambda x_i}\right]\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}}{k^2} - \frac{r\log\left[1+\ell^{\lambda x_i}\right]\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}\log\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)}{k^2} \right)$$

$$\frac{\partial^2 L}{\partial k \partial \lambda} = -\sum_{i=1}^{n} \left(-\frac{\ell^{\lambda x_i}(r-1)r\log\left[1+\ell^{\lambda x_i}\right]\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-2} x_i}{\left(1+\ell^{\lambda x_i}\right)k^3} - \frac{\ell^{\lambda x_i} r\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1} x_i}{\left(1+\ell^{\lambda x_i}\right)k^2} \right)$$

$$\frac{\partial^2 L}{\partial k^2} = \frac{n}{k^2} + \frac{n(r-1)}{k^2} - \sum_{i=1}^{n} \frac{(r-1)r\log\left[1+\ell^{\lambda x_i}\right]^2\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-2}}{k^4} + \frac{2r\log\left[1+\ell^{\lambda x_i}\right]\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}}{k^3}$$