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Generalization of Quasi Convex Functions using Convolution

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Abstract

In this paper, an up-to-date generalization of the class $\mathtt{C}^\star$ of quasi-convex functions is given by introducing new class $\mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$. Furthermore its basic properties, its relationship with other subclasses of $\mathtt{S}$, inclusion relations and some other interesting properties are derived.

1 Introduction

Let $\mathtt{A}$ denote the class of analytic functions in the open unit disk $\mathtt{E} = \{z : |z| < 1\}$ defined by the power series

$$\ell(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa. \tag{1.1}$$

The convolution or Hadamard product of two analytic functions

$$\ell(z) = \sum_{\kappa=0}^{\infty} a_\kappa z^{\kappa+1},\ \jmath(z) = \sum_{\kappa=0}^{\infty} b_\kappa z^{\kappa+1},\ z \in \mathtt{E},\ \kappa = 0, 1, 2, \cdots \tag{1.2}$$

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is defined as

$$(\ell * \jmath)(z) = \sum_{\kappa=0}^{\infty} a_\kappa b_\kappa z^{\kappa+1}. \tag{1.3}$$

Moreover, we say that $\ell(z)$ is subordinate to $\jmath(z)$ written as $\ell(z) \prec \jmath(z)$ if there exists a function $\omega(z)$ analytic in $\mathtt{E}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, for all $z \in \mathtt{E}$, such that $\ell(z) = \jmath(\omega(z))$.

Let $\mathtt{S}$, $\mathtt{C}$, $\mathtt{S}^\star$, $\mathtt{K}$ and $\mathtt{C}^\star$ be the sub-classes of $\mathtt{A}$, which contain univalent, convex, starlike, close-to-convex and quasi-convex functions respectively. For several interesting geometric properties and deatils of these classes, one can refer to the standard books [4, 12]. It is well known [4] that $\ell(z)$ defined in (1.1) is convex if and only if $z\ell'(z) \in \mathtt{S}^\star$ and that $\ell(z)$ is quasi convex if and only if $z\ell'(z)$ is close to convex, see [11, 10].

Let $\mathcal{H}$ be the class of functions

$$p(z) = 1 + \sum_{\kappa=1}^{\infty} c_\kappa z^\kappa, \tag{1.4}$$

that are regular in $\mathtt{E}$ with $p(0) = 1$. Then $p \in \mathtt{P}[\mathfrak{a}, \mathfrak{b}], -1 \le \mathfrak{b} < \mathfrak{a} \le 1$ if and only if $p\left(z\right) \prec \frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}$, or equivalently

$$p\left(z\right) = \frac{(\mathfrak{a}+1)h(z) - (\mathfrak{a}-1)}{(\mathfrak{b}+1)h(z) - (\mathfrak{b}-1)},$$

where $h \in \mathtt{P}[1, -1] = \mathtt{P}$, the class of functions with positive real part. Also $p \in \mathtt{P}(\gamma)$ if and only if

$$\Re\{p(z)\} > \gamma, \quad 0 < \gamma < 1.$$

First time Janowski [6] considered and studied extensively this class of functions. The classes $\mathtt{C}[\mathfrak{a}, \mathfrak{b}]$, $\mathtt{S}^\star[\mathfrak{a}, \mathfrak{b}]$, $\mathtt{K}[\mathfrak{a}, \mathfrak{b}]$ and $\mathtt{C}^\star[\mathfrak{a}, \mathfrak{b}]$ were defined and discussed in [1, 13].

Let $\ell(z) \in \mathtt{A}$. Denote by $\text{Đ}^\sigma : \mathtt{A} \longrightarrow \mathtt{A}$, the operator defined by

$$\text{Đ}^\sigma \ell(z) = \frac{z}{(1-z)^{(\sigma+1)}} * \ell(z) = z + \sum_{\kappa=2}^{\infty} \digamma_\kappa(\sigma) a_\kappa z^\kappa, \ \ (\sigma > -1),$$

with

$$\digamma_\kappa(\sigma) = \frac{(\sigma+1)_{\kappa-1}}{(\kappa-1)!},$$

where $(\xi)_\kappa$ is Pochhammer symbol given as

$$(\xi)_\kappa = \begin{cases} 1, & \text{for} \quad \kappa = 0, \\ \xi(\xi+1)(\xi+2)\cdots(\xi+\kappa-1), & \text{for} \quad \kappa \in \mathbb{N}. \end{cases}$$

It is obvious that $Ð^0\ell(z) = \ell(z)$, $Ð^1\ell(z) = z\ell'(z)$ and

$$Ð^n\ell(z) = \frac{z(z^{n-1}\ell(z))^n}{n!}, \ \forall \ n = 0, 1, 2, 3\cdots.$$

The operator $Ð^\sigma\ell(z)$ is called *Ruscheweyh derivative* of $\ell(z)$, see [15].

Definition 1. Let ℓ and $\mathfrak{g} \in \mathtt{A}$, such that $(\mathfrak{g} * \ell) \neq 0$ in $\mathtt{E}$. Then ℓ is said to belong to $\mathtt{S}^\star_\mathfrak{g}[\mathfrak{a}, \mathfrak{b}]$ if and only if

$$\frac{z(\mathfrak{g} * \ell)'(z)}{(\mathfrak{g} * \ell)(z)} \in \mathtt{P}[\mathfrak{a}, \mathfrak{b}]. \tag{1.5}$$

We note that if $\mathfrak{g} = \frac{z}{(1-z)}$, then $\mathtt{S}^\star_\mathfrak{g}[\mathfrak{a}, \mathfrak{b}] = \mathtt{S}^\star[\mathfrak{a}, \mathfrak{b}]$.

Definition 2. Let $\ell \in \mathtt{A}$, $(\mathfrak{g} * \ell)' \neq 0$ in $\mathtt{E}$ for $\mathfrak{g} \in \mathtt{A}$. Then ℓ is said to belong to $\mathtt{C}_\mathfrak{g}[\mathfrak{a}, \mathfrak{b}]$ if and only if

$$\frac{(z(\mathfrak{g} * \ell)'(z))'}{(\mathfrak{g} * \ell)'(z)} \in \mathtt{P}[\mathfrak{a}, \mathfrak{b}]. \tag{1.6}$$

Note that if $\mathfrak{g} = \frac{z}{(1-z)}$, then $\mathtt{C}_\mathfrak{g}[\mathfrak{a}, \mathfrak{b}] = \mathtt{C}[\mathfrak{a}, \mathfrak{b}]$.

Definition 3. Let $\ell \in \mathtt{A}$. Then $\ell \in \mathtt{K}_\mathfrak{g}[\mathfrak{a}, \mathfrak{b}]$ if and only if there exists $\mathfrak{u} \in \mathtt{S}^\star_\mathfrak{g}[\mathfrak{a}, \mathfrak{b}]$ with $(\mathfrak{g} * \mathfrak{u}) \neq 0$ such that

$$\frac{z(\mathfrak{g} * \ell)'(z)}{(\mathfrak{g} * \mathfrak{u})(z)} \in \mathtt{P}[\mathfrak{a}, \mathfrak{b}], \quad z \in \mathtt{E}. \tag{1.7}$$

Note that if $\mathfrak{g} = \frac{z}{(1-z)}$, then $\mathtt{K}_\mathfrak{g}[\mathfrak{a}, \mathfrak{b}] = \mathtt{K}[\mathfrak{a}, \mathfrak{b}]$.

Definition 4. Let $\ell \in \mathtt{A}$. Then $\ell \in \mathtt{C}^\star_\mathfrak{g}[\mathfrak{a}, \mathfrak{b}]$ if and only if there exists $\mathfrak{u} \in \mathtt{C}_\mathfrak{g}[\mathfrak{a}, \mathfrak{b}]$ with $(\mathfrak{g} * \mathfrak{u}) \neq 0$ such that

$$\frac{(z(\mathfrak{g} * \ell)')'(z)}{(\mathfrak{g} * \mathfrak{u})'(z)} \in \mathtt{P}[\mathfrak{a}, \mathfrak{b}], \quad z \in \mathtt{E}. \tag{1.8}$$

If $\mathfrak{g} = \frac{z}{(1-z)}$, $\implies \mathtt{C}^\star_\mathfrak{g}[\mathfrak{a}, \mathfrak{b}] = \mathtt{C}^\star[\mathfrak{a}, \mathfrak{b}]$.

Now we have $\ell \in \mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b},\sigma] \iff \text{Đ}^\sigma \ell \in \mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$, where Đ^σ is the Ruscheweyh derivative operator [15].

Definition 5. Let $\ell \in \mathtt{A}$ and $\mathfrak{g} \in \mathtt{A}$. Then $\ell \in \mathtt{Q}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b};\alpha]$, $0 \leq \alpha < 1$ if and only if there exists $\mathfrak{u} \in \mathtt{C}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ with $(\mathfrak{g} * \mathfrak{u}) \neq 0$ such that

$$(1-\alpha)\frac{(\mathfrak{g}*\ell)'}{(\mathfrak{g}*\mathfrak{u})'} + \alpha\frac{(z(\mathfrak{g}*\ell)')'}{(\mathfrak{g}*\mathfrak{u})'} \in \mathtt{P}[\mathfrak{a},\mathfrak{b}], \quad z \in \mathtt{E}. \tag{1.9}$$

We note that

$$\mathtt{Q}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b};\mathtt{0}] = \mathtt{K}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$$
$$\mathtt{Q}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b};\mathtt{1}] = \mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}].$$

2 Some Preliminary Results

Lemma 2.1. [14] *$\psi \in \mathtt{C}$, $\mathfrak{g} \in \mathtt{S}^\star$ and F is analytic in $\mathtt{E}$ with $F(0) = 1$, then*

$$\frac{\psi * F\mathfrak{g}}{\psi * \mathfrak{g}} \subset \bar{C}oF(\mathtt{E}), \tag{2.1}$$

where $\bar{C}o$ is the closed convex hull.

Lemma 2.2. [8] *Let $u = u_1 + \iota u_2$ and $v = v_1 + \iota v_2$, and let Ψ be the set of functions $\Psi(u,v)$ satisfying:*
(a) *$\Psi(u,v)$ is continuous in a domain D of $\mathbb{C} \times \mathbb{C}$.*
(b) *$(1,0) \in D$ and $\Re\,\Psi(1,0) > 0$.*
(c) *$\Re\,\Psi(\iota u_2, v_1) \leq 0$ when $(\iota u_2, v_1) \in D$ and $v \leq -\frac{1}{2}(1+u_2^2)$.*
If $p(z)$, given by (1.4), is an analytic function in $\mathtt{E}$ such that $(p(z), zp'(z)) \in D$ and $\Re\,\Psi(p(z), zp'(z)) > 0$, for $z \in \mathtt{E}$, then $\Re\,p(z) > 0$.

Lemma 2.3. [7] *If $\aleph$ and Λ are regular in $\mathtt{E}$, $\aleph(0) = \Lambda(0) = 0$, $\aleph$ maps $\mathtt{E}$ onto a many sheeted region which is starlike with respect to origin, and $\frac{\aleph'}{\Lambda'} \in \mathtt{P}$, then $\frac{\aleph}{\Lambda} \in \mathtt{P}$.*

Lemma 2.4. [16] *For a real number $\sigma(\sigma > 0)$, we have*

$$z(\text{Đ}^\sigma \ell(z))' = (\sigma+1)\text{Đ}^{\sigma+1}\ell(z) - \sigma \text{Đ}^\sigma \ell(z).$$

3 Main Results

We present our main work in this section.

Theorem 3.1. *Let* $\ell \in \mathtt{S}^{\star}[\mathfrak{a}, \mathfrak{b}]$ *and* $\mathfrak{g} \in \mathtt{C}$. *Then* $\ell \in \mathtt{S}^{\star}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$.

Proof. Consider

$$\begin{aligned}\frac{z(\mathfrak{g} * \ell)'}{(\mathfrak{g} * \ell)} &= \frac{(\mathfrak{g} * z\ell')}{(\mathfrak{g} * \ell)}\\ &= \frac{\mathfrak{g} * \frac{z\ell'}{\ell}\ell}{(\mathfrak{g} * \ell)}\\ &= \frac{\mathfrak{g} * F\ell}{(\mathfrak{g} * \ell)} \subset \bar{C}oF(\mathtt{E}).\end{aligned}$$

Since $F \in \mathtt{P}[\mathfrak{a}, \mathfrak{b}]$ and $\mathfrak{g} \in \mathtt{C}$, we use Lemma 2.1 to have the required result that $\ell \in \mathtt{S}^{\star}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$. □

Theorem 3.2. *The class* $\mathtt{S}^{\star}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$ *is closed under convolution with convex functions.*

Proof. Let $\ell \in \mathtt{S}^{\star}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$. Then

$$\frac{z(\mathfrak{g} * \ell)'}{(\mathfrak{g} * \ell)} = p(z) \in \mathtt{P}[\mathfrak{a}, \mathfrak{b}].$$

Let $\psi \in \mathtt{C}$, we have

$$\begin{aligned}\frac{z[\mathfrak{g} * (\psi * \ell)]'}{[\mathfrak{g} * (\psi * \ell)]} &= \frac{\psi * z(\mathfrak{g} * \ell)'}{\psi * (\mathfrak{g} * \ell)}\\ &= \frac{\psi * \frac{z(\mathfrak{g} * \ell)'}{(\mathfrak{g} * \ell)}(\mathfrak{g} * \ell)}{\psi * (\mathfrak{g} * \ell)}\\ &= \frac{\psi * p(\mathfrak{g} * \ell)}{\psi * (\mathfrak{g} * \ell)} \in \bar{C}o\,p(\mathtt{E}).\end{aligned}$$

Since $\ell \in \mathtt{S}^{\star}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$, it follows that $p \in \mathtt{P}[\mathfrak{a}, \mathfrak{b}]$.

$$(\mathfrak{g} * \ell) \in \mathtt{S}^{\star}[\mathfrak{a}, \mathfrak{b}] \subset \mathtt{S}^{\star}.$$

We use Lemma 2.1 to conclude that $\frac{\psi * p(\mathfrak{g} * \ell)}{\psi * (\mathfrak{g} * \ell)}$ lies in the convex hull of $p(\mathtt{E})$ and therefore $\psi * \ell \in \mathtt{S}^{\star}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$. □

Theorem 3.3. *Let* $\ell \in \mathtt{C}^{\star}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ *with respect to* $\mathfrak{u} \in \mathtt{C}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ *and let* $\psi \in \mathtt{C}$. *Then* $\psi * \ell \in \mathtt{C}^{\star}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ *with respect to* $z(\mathfrak{g} * \mathfrak{u})' = \mathfrak{u}_1$.

Proof. By definition $\mathtt{C}^{\star}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ implies $\mathfrak{g} * \mathfrak{u} \in \mathtt{C}[\mathfrak{a},\mathfrak{b}]$.

Let

$$p(z) = \frac{(z(\mathfrak{g} * \ell)')'}{(\mathfrak{g} * \mathfrak{u})'}.$$

Then $p \in \mathtt{P}[\mathfrak{a},\mathfrak{b}]$ in $\mathtt{E}$. Also, it is known [4] that $\mathtt{C}[\mathfrak{a},\mathfrak{b}] \subset \mathtt{C} \subset \mathtt{S}^{\star}$.

Now,

$$\begin{aligned}
\frac{(z(\mathfrak{g} * (\psi * \ell))')'}{(\mathfrak{g} * \psi * \mathfrak{u})'} &= \frac{z(\mathfrak{g} * z(\psi * \ell)')'}{\psi * z(\mathfrak{g} * \mathfrak{u})'} \\
&= \frac{\psi * \frac{(z(\mathfrak{g}*\ell)')'}{(\mathfrak{g}*\mathfrak{u})'} z(\mathfrak{g} * \mathfrak{u})'}{\psi * z(\mathfrak{g} * \mathfrak{u})'} \\
&= \frac{\psi * pz(\mathfrak{g} * \mathfrak{u})'}{\psi * z(\mathfrak{g} * \mathfrak{u})'} \\
&= \frac{\psi * p\mathfrak{u}_1}{\psi * \mathfrak{u}_1} \subset \bar{C}o\,p(\mathtt{E}).
\end{aligned}$$

Since $\psi \in \mathtt{C}$ and $\mathfrak{u}_1 \in \mathtt{S}^{\star}$, the required result follows using Lemma 2.1. That is $(\psi * \ell) \in \mathtt{C}^{\star}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$. □

Corollary 3.3.1. *If* $\mathfrak{g} = \frac{z}{(1-z)}$, *then Theorem* 3.3 *leads that* $(\psi * \ell) \in \mathtt{C}^{\star}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}] = \mathtt{C}^{\star}[\mathfrak{a},\mathfrak{b}]$.

Theorem 3.4. $\mathtt{C}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}] \subset \mathtt{S}^{\star}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$.

Proof. Consider

$$\frac{z(\mathfrak{g} * \ell)'}{(\mathfrak{g} * \ell)} = p(z). \tag{3.1}$$

Let $\ell \in \mathtt{C}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$, it can be simply seen that p is analytic and $p(0) = 1$ for $z \in \mathtt{E}$.

By logarithmically differentiation, we have

$$[1 + \frac{(\mathfrak{g} * \ell)''}{(\mathfrak{g} * \ell)'}] = p(z) + \frac{zp'(z)}{p(z)} \in \mathtt{P}[\mathfrak{a},\mathfrak{b}].$$

Since $\ell \in \mathtt{C}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$. Now using Lemma 2.2 it follows that $p \in \mathtt{P}[\mathfrak{a},\mathfrak{b}]$ which implies $\ell \in \mathtt{S}^{\star}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$. □

Corollary 3.4.1. *If* $\mathfrak{g} = \frac{z}{(1-z)}$, *then Theorem* 3.4 *follows that* $\mathtt{C}[\mathfrak{a}, \mathfrak{b}] \subset \mathtt{S}^\star[\mathfrak{a}, \mathfrak{b}]$.

Theorem 3.5. *Let* $0 < \alpha < 1$. *Then*

$$\mathtt{Q}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}; \alpha] \subset \mathtt{K}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}].$$

Proof. From Definition 5, it follows that $\ell \in \mathtt{Q}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$ if and only if

$$\pounds(z) = (1-\alpha)\ell(z) + \alpha z\ell'(z) \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]. \tag{3.2}$$

We can express (3.2) as

$$\begin{aligned}
\ell(z) =& \frac{1}{\alpha} z^{(1-\frac{1}{\alpha})} \int_0^z \tau^{(\frac{1}{\alpha}-2)} \pounds(\tau) d\tau, \;\; \pounds \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}] \\
=& \frac{c+1}{z^c} \int_0^z \tau^{(c-1)} \pounds(\tau) d\tau; \quad (\frac{1}{\alpha} = c+1) \\
=& (\sum_{\kappa=1}^{\infty} \frac{c+1}{c+\kappa} z^\kappa) * \pounds(z), \qquad \kappa = 1, 2, 3, \cdots
\end{aligned}$$

$$\ell(z) = (\phi_c * \pounds)(z), \tag{3.3}$$

where $\sum_{\kappa=1}^{\infty} \frac{c+1}{c+\kappa} z^\kappa$ is convex, see [1] and $\pounds \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$ is preserved under convex convolution. In fact from Theorem 3.2, $\phi_c * \ell_1 \in \mathtt{S}^\star_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$ if $\ell_1 \in \mathtt{S}^\star_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$, and

$$\begin{aligned}
\frac{z(\phi_c * \mathfrak{g} * \pounds)'}{\phi_c * (\mathfrak{g} * \ell_1)} =& \frac{z(\mathfrak{g} * (\phi_c * \pounds))'}{\mathfrak{g} * (\phi_c * \ell_1)}. \\
=& \frac{z(\mathfrak{g} * \ell)'}{\mathfrak{g} * \pounds_1},
\end{aligned}$$

where $(\phi_c * \ell_1) = \pounds_1 \in \mathtt{S}^\star_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$, by Theorem 3.2 and ℓ is given by equation (3.2), consequently $\ell \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$ in $\mathtt{E}$. □

Corollary 3.5.1. *Let* $\mathfrak{g} = \frac{z}{(1-z)}$, *it follows from Theorem* 3.5 *that* $\mathtt{Q}[\mathfrak{a}, \mathfrak{b}; \alpha] \subset \mathtt{K}[\mathfrak{a}, \mathfrak{b}]$, *for* $0 < \alpha < 1$.

Theorem 3.6. *Let* $\mathfrak{g} \in \mathtt{S}^\star[\mathfrak{a}, \mathfrak{b}]$, $-1 \leq \mathfrak{b} < \mathfrak{a} \leq 1$ *and let* $\frac{z\ell'}{\mathfrak{g}} \in \mathtt{P}(\beta)$ *for* $z \in \mathtt{E}$. *Then* $\Re\frac{(z\ell')'}{\mathfrak{g}'} > \beta$, $0 \leq \beta < 1$, *for* $|\, z \,| < r_1$ *where* r_1 *is the least positive root in* $(0, 1)$ *of the equation*

$$1 - (\mathfrak{a}+2)r + (2\mathfrak{b}-1)r^2 + \mathfrak{a}r^3 = 0. \tag{3.4}$$

Proof. For $z \in \mathtt{E}$, we have

$$\frac{z\ell'}{\mathfrak{g}} = h,$$

where $\mathfrak{g} \in \mathtt{S}^\star[\mathfrak{a}, \mathfrak{b}]$ and $h \in \mathtt{P}(\beta)$.

Differentiating logarithmically, we have

$$\frac{(z\ell')'}{\mathfrak{g}'} = h + \frac{\mathfrak{g}}{\mathfrak{g}'}h',$$

it follows that

$$\Re(\frac{(z\ell')'}{\mathfrak{g}'} - \beta) \geq \Re(\mid h - \beta \mid - \mid \frac{\mathfrak{g}}{\mathfrak{g}'} \mid\mid h' \mid). \tag{3.5}$$

Using distortion results, that are given in [4] as

$$\mid \frac{\mathfrak{g}}{\mathfrak{g}'} \mid \leq \frac{r(1 - \mathfrak{b}r)}{1 - \mathfrak{a}r} \tag{3.6}$$

$$\mid h' \mid \leq \frac{2\Re[h - \beta]}{1 - r^2}. \tag{3.7}$$

Using these results in equation (3.5), we have

$$\Re[\frac{(z\ell')'}{\mathfrak{g}'} - \beta] \geq \frac{\Re(h - \beta)[1 - (\mathfrak{a} + 2)r + (2\mathfrak{b} - 1)r^2 + \mathfrak{a}r^3]}{(1 - \mathfrak{a}r)(1 - r^2)}. \tag{3.8}$$

The right hand side of equation (3.8) is positive for $\mid z \mid = r < r_1$, where r_1 is the least positive root of (3.4), which completes our proof.

□

Corollary 3.6.1. *As a special case, when* $\mathfrak{a} = 0, \mathfrak{b} = -1$ and $\beta = 0$ *we have* $\Re\frac{z\ell'}{\mathfrak{g}} > 0, \mathfrak{g} \in \mathtt{S}^\star(\frac{1}{2})$ *for* $z \in \mathtt{E}$. *In this case* $\Re\frac{(z\ell')'}{\mathfrak{g}'} > 0$ *for* $\mid z \mid < \frac{1}{3}$.

Theorem 3.7. *Let* $\ell \in \mathtt{S}^\star_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$. *Then* $\ell \in \mathtt{C}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$ *for* $\mid z \mid < r_o = 2 - \sqrt{3}$.

Proof. Since $\ell \in \mathtt{S}^\star_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$, therefore we can write

$$\frac{z(\mathfrak{g} * \ell)'}{(\mathfrak{g} * \ell)} = h,$$

where h is a Carathéodory function $h \in \mathtt{P}$, this implies $\frac{z\ell_1{}'}{\ell_1} = h$, where $\ell_1 = \mathfrak{g} * \ell$. Differentiating logarithmically, we have

$$\frac{z\ell_1{}'}{\ell_1} = h + \frac{zh'}{h}.$$

Using distortion results for the class $\mathtt{P}$, see [3, 4, 5]. We have

$$\Re\frac{z(\mathfrak{g} * \ell)'}{(\mathfrak{g} * \ell)} = \Re\frac{z\ell_1{}'}{\ell_1} \geq \frac{1-r}{1+r} - \frac{2r}{1-r^2} = \frac{1-4r+r^2}{1-r^2},$$

by using mean value theorem and with some computations we get

$$T(r) = 2 - \sqrt{3} \in (0,1), \quad T(r) = 2 - \sqrt{3} \in (0,1).$$

Thus

$$\Re\frac{z(\mathfrak{g} * \ell)'}{(\mathfrak{g} * \ell)} \geq \frac{1-4r+r^2}{1-r^2} > 0, \quad \text{for } r < r_o = 2 - \sqrt{3}.$$

□

Corollary 3.7.1. *Let* $\mathfrak{g} = \frac{z}{(1-z)}$, *so Theorem* 3.7 *follows that if* $\ell \in \mathtt{S}^\star[\mathfrak{a}, \mathfrak{b}]$, *then* $\ell \in \mathtt{C}[\mathfrak{a}, \mathfrak{b}]$ *for* $|\, z \,| < r_o = 2 - \sqrt{3}$.

A converse case of Theorem 3.5, we have following result

Theorem 3.8. *Let* $\ell \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$. *Then* $\ell \in \mathtt{Q}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}; \alpha]$ *for*

$$|\, z \,| < r_\alpha = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}. \tag{3.9}$$

Proof. Consider $\ell \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$. Then we can write

$$(1-\alpha)\ell + \alpha z\ell' = \phi_c * \ell,$$

where

$$\phi_c(z) = \sum_{\kappa=1}^{\infty}[(\kappa-1)\alpha + 1]z^\kappa.$$

It can be verified with simple computations that $\phi_c(z) \in \mathtt{C}$ for $| z |< r_\alpha$, where r_α is given by (3.9). Then, using Remark 1 that the class $\mathtt{K}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ is preserved under convex convolution, it follows that $\ell \in \mathtt{Q}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b};\alpha]$ for $| z |< r_\alpha$, r_α is given by (3.9). □

Corollary 3.8.1. *For* $\mathfrak{g} = \frac{z}{(1-z)}$, *Theorem* 3.8 *follows that if* $\ell \in \mathtt{K}[\mathfrak{a},\mathfrak{b}]$, *then* $\ell \in \mathtt{Q}[\mathfrak{a},\mathfrak{b};\alpha]$, *where as* $| z |< r_\alpha$ *and* r_α *is given by* (3.9).

Theorem 3.9. *If* $z\ell' \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$, *then* $\ell \in \mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$.

Proof. Consider $\ell \in \mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ and defined by (1.8), so

$$\Longrightarrow \frac{((\mathfrak{g} * z\ell'))'}{(\mathfrak{g} * \mathfrak{u})'} \in \mathtt{P}[\mathfrak{a},\mathfrak{b}].$$

Let $F = z\ell'$,

$$\Longrightarrow \frac{(\mathfrak{g} * F)'}{(\mathfrak{g} * \mathfrak{u})'} \in \mathtt{P}[\mathfrak{a},\mathfrak{b}].$$

This shows that $F \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$. Using Alexander type relation between $\mathtt{C}$ and $\mathtt{S}^\star$ we get

$$\frac{z(\mathfrak{g} * F)'}{(\mathfrak{g} * G)} \in \mathtt{P}[\mathfrak{a},\mathfrak{b}],$$

where $G = z\mathfrak{u}'$. This completes our proof.

□

Remark 1. Using Theorem 3.9 and alike techniques of Theorem 3.3, we can easily prove that the class $\mathtt{K}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ is preserved under convex convolution that is, if $\ell \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ and $\phi \in \mathtt{C}$, then $\phi * \ell \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ in $\mathtt{E}$.

Some Inclusion Relations

Remark 2. $\mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}] \not\subset \mathtt{S}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ and $\mathtt{S}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}] \not\subset \mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$.

Proof. Noor proved that the class $\mathtt{C}^\star$ does not properly contained in the class $\mathtt{S}^\star$ and $\mathtt{S}^\star \not\subset \mathtt{C}^\star$, see [11]. If $\mathfrak{g} = \frac{z}{(1-z)}$, then our result is obvious.

□

Theorem 3.10. $\mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}] * \mathtt{S}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}] \subset \mathtt{K}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$.

Proof. Let $\ell \in \mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ and $\mathfrak{u} \in \mathtt{S}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$.

Consider that

$$\begin{aligned}(\ell * \mathfrak{u}) =& (\ell * z\mathfrak{u}'), \quad \text{where } \mathfrak{u} \in \mathtt{C}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}].\\ =& z(\ell * \mathfrak{u})'\\ =& z\ell' * \mathfrak{u}.\end{aligned}$$

It follows from Alexander type relations that $\ell * \mathfrak{u} \in \mathtt{K}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$. □

Next theorem is given as a special case of Lemma 2.3.

Theorem 3.11. $\mathtt{C}^\star_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}] \subset \mathtt{K}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$.

Proof. Let $\aleph(z) = z(\mathfrak{g} * \ell)'(z)$ and $\Lambda(z) = (\mathfrak{g} * \mathfrak{u})(z)$ be analytic functions in $\mathtt{E}$ with $\aleph(0) = \Lambda(0) = 0$ and $\Lambda \in \mathtt{S}^\star$. Then Lemma 2.3 implies

$$\frac{\aleph'}{\Lambda'} \in \mathtt{P} \implies \frac{\aleph}{\Lambda} \in \mathtt{P}, \quad \text{for } z \in \mathtt{E},$$

which leads to our required result. □

Theorem 3.12. *Let $\sigma \geq 1$. Then*

$$\mathtt{C}^\star_{\mathfrak{g}}[\sigma+1,\mathfrak{a},\mathfrak{b}] \subset \mathtt{C}^\star_{\mathfrak{g}}[\sigma,\mathfrak{a},\mathfrak{b}].$$

Proof. For $\ell \in \mathtt{A}$. Using Lemma 2.4 we can easily derive the identity

$$z(\text{Đ}^\sigma(\mathfrak{g} * \ell))' = (\sigma+1)\text{Đ}^{(\sigma+1)}(\mathfrak{g} * \ell)(z) - \sigma \text{Đ}^\sigma(\mathfrak{g} * \ell)(z). \tag{3.10}$$

Set

$$\frac{(z(\text{Đ}^\sigma(\mathfrak{g} * \ell))')'(z)}{(\text{Đ}^\sigma(\mathfrak{g} * \mathfrak{u}))'(z)} = h(z). \tag{3.11}$$

Then $h(z)$ is analytic in $\mathtt{E}$ with $h(0) = 1$. We shall show that

$$\Re h(z) > 0, \quad z \in \mathtt{E}.$$

First we show that

$$\mathtt{C}_{\mathfrak{g}}[\sigma+1,\mathfrak{a},\mathfrak{b}] \subset \mathtt{C}_{\mathfrak{g}}[\sigma,\mathfrak{a},\mathfrak{b}].$$

For this, let $\mathfrak{u} \in \mathtt{C}_{\mathfrak{g}}[\sigma+1,\mathfrak{a},\mathfrak{b}]$ and set

$$\frac{(z(\text{Đ}^{\sigma}(\mathfrak{g}*\mathfrak{u}))')'(z)}{(\text{Đ}^{\sigma}(\mathfrak{g}*\mathfrak{u}))'(z)} = H_o(z). \tag{3.12}$$

Then $H_o(z)$ is analytic and $H_o(0)=1$. Using identity (3.10) for $\mathfrak{u}$ together with (3.12) we have

$$\frac{(\sigma+1)(\text{Đ}^{\sigma+1}(\mathfrak{g}*\mathfrak{u}))'(z)}{(\text{Đ}^{\sigma}(\mathfrak{g}*\mathfrak{u}))'(z)} = H_o(z)+\sigma. \tag{3.13}$$

Differentiating both sides of (3.13) logarithmically and using (3.12), with some computations we obtain

$$\frac{(z(\text{Đ}^{\sigma+1}(\mathfrak{g}*\mathfrak{u}))'(z)}{(\text{Đ}^{\sigma+1}(\mathfrak{g}*\mathfrak{u}))'(z)} = H_o(z)+\frac{zH_o'(z)}{H_o(z)+\sigma}. \tag{3.14}$$

Since $\mathfrak{u} \in \mathtt{C}_{\mathfrak{g}}[\sigma+1,\mathfrak{a},\mathfrak{b}]$, therefore right hand of (3.14) belongs to $\mathtt{P}[\mathfrak{a},\mathfrak{b}]$. From (3.12), (3.14) and a well-known Lemma 2.2 due to Miller [8], also see [9], it follows that $\Re H_o(z) \in \mathtt{P}[\mathfrak{a},\mathfrak{b}]$ in $\mathtt{E}$. This proves that $\mathfrak{u} \in \mathtt{C}[\sigma,\mathfrak{a},\mathfrak{b}]$.

Now with similar procedure and from (3.11), we get

$$\frac{(z(\text{Đ}^{\sigma+1}(\mathfrak{g}*\ell))')'(z)}{(\text{Đ}^{\sigma+1}(\mathfrak{g}*\mathfrak{u}))'(z)} = h(z)+\frac{zh'(z)}{h_o(z)+\sigma}. \tag{3.15}$$

Again applying Lemma 2.2, we obtain from (3.15) that $\Re h(z) \in \mathtt{P}[\mathfrak{a},\mathfrak{b}]$ in $\mathtt{E}$, which proves that $\ell \in \mathtt{C}^{\star}_{\mathfrak{g}}[\sigma,\mathfrak{a},\mathfrak{b}]$ in $\mathtt{E}$. This establishes our required inclusion result. □

Theorem 3.13. *The class* $\mathtt{C}^{\star}_{\mathfrak{g}}[\mathfrak{a},\mathfrak{b}]$ *is preserved under the following integral operators defined in* [2, 7]

(a)

$$\begin{aligned}\ell_1(z) =& \int_0^z \frac{\ell(\tau)}{\tau}d\tau\\ =&(\mathfrak{g}_1*\ell)(z), \quad (\mathfrak{g}_1)(z) = -log(1-z).\end{aligned}$$

(b)

$$\begin{aligned}\ell_2(z) =& \frac{2}{z}\int_0^z \frac{\ell(\tau)}{\tau} d\tau \\ =& (\mathfrak{g}_2 * \ell)(z), \quad (\mathfrak{g}_2)(z) = -2[z + log(1-z)].\end{aligned}$$

(c)

$$\begin{aligned}\ell_3(z) =& \frac{1+\varsigma}{z^{\varsigma}}\int_0^z \tau^{\varsigma-1}\ell(\tau) d\tau \\ =& (\mathfrak{g}_3 * \ell)(z), \quad (\mathfrak{g}_3)(z) = \sum_{\sigma+1}^{\infty} \frac{1+\varsigma}{\sigma+\varsigma} z^{\sigma}, \quad where \quad \sigma \in \mathbb{N}.\end{aligned}$$

It can be easily seen that $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathtt{C}$. *In* [15]*, Ruscheweyh showed that* $\mathfrak{g}_3 \in \mathtt{C}$ *in* $\mathtt{E}$. *Consequently using Theorem* 3.3*, it follows that* $\ell_{\mathfrak{i}} \in \mathtt{C}^{\star}_{\mathfrak{g}}[\mathfrak{a}, \mathfrak{b}]$*, where* $\mathfrak{i} = 1, 2, 3$.

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