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On Certain Generalizations of Close-to-convex Functions

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Abstract

The aim of this article is to introduce and study certain subclasses of analytic functions and we investigate various properties of these classes such as inclusion properties and convex convolution preserving properties. Also, some related applications are discussed.

1 Introduction

Let **A** be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disk $E = \{z : |z| < 1\}$. We denote S, S^*, C, K and C^* the classes of univalent, starlike, convex, close-to-convex and quasi-convex functions, respectively. If f and g are analytic in E, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a schwartz function w in E such that $f(z) = g(w(z))$.

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The convolution or Hadamard product of two functions $f, g \in \mathbf{A}$ is denoted by $f * g$ and is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in E.$$

Analytic functions p in the class $P[A, B]$ can be defined by using subordination as follows [5].

Let p be analytic in E with $p(0) = 1$. Then $p \in P[A, B]$, if and only if,

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \qquad -1 \leq B < A \leq 1,\ z \in E.$$

For $k \geqslant 0$, the conic domains Ω_k, defined as;

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

The domains Ω_k $(k = 0)$ represents right half plane, Ω_k $(0 < k < 1)$ represents hyperbola, Ω_k $(k = 1)$ represents a parabola and Ω_k $(k > 1)$ represents an ellipse. The extremal functions for these conic regions are given as

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0 \\ 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & k = 1 \\ 1 + \frac{2}{1-k^2} \left[\left(\frac{2}{\pi} \arccos k \right) \arctan h\sqrt{z} \right], & 0 < k < 1 \\ 1 + \frac{1}{k^2-1} \sin \left(\frac{\pi}{2R(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}} dx \right) + \frac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.2}$$

where $u(z) = \frac{z-\sqrt{t}}{z-\sqrt{tz}}$, $t \in (0,1)$, $z \in E$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. See [6, 7] for more information. These conic regions are being studied by several authors. See [1, 14, 17].

Let $\phi_\lambda(z) = \frac{z}{(1-z)^{\lambda+1}}$, $(\lambda > -1)$ and $\phi_\lambda^{-1}(z)$ be defined such that $\phi_\lambda(z) * \phi_\lambda^{-1}(z) = \frac{z}{1-z}$. Then

$$\begin{aligned} I_\lambda f(z) &= \phi_\lambda^{-1}(z) * f(z) \\ &= \left(\frac{z}{(1-z)^{\lambda+1}} \right)^{-1} * f(z). \end{aligned} \tag{1.3}$$

The operator I_λ is known as Noor integral operator, we refer to [12, 15]. We can easily verify the following recursive relation for this operator by using (1.3).

$$z(I_{\lambda+1}f)' = (\lambda+1)I_\lambda f - \lambda I_{\lambda+1}f. \tag{1.4}$$

Dziok and Noor [3], introduced the concepts of some general classes as follows:

Let A_0 be the class of functions $f \in \mathbf{A}$ with $f(0) = 1$. Assume that μ, δ, m be real parameters, $\mu \geq 0$, $m \geq 2$ and let $\Phi = (\phi, \varphi) \in \mathbf{A} \times \mathbf{A}$, $\xi \in \mathbf{A}$, $G = (g_1, g_2)$ and $H = (h_1, h_2)$, where g_i, h_i $(i = 1, 2)$ are analytic, univalent convex functions with $g_i(0) = 1$ and $h_i(0) = 1$ $(i = 1, 2)$. Then

$$P(h) = \{q \in A_0 : q \prec h\},$$

$$P_\mu(H) = \{\mu q_1 + (1-\mu)q_2 : q_1 \prec h_1,\ q_2 \prec h_2\},$$

$$P_\mu((h,h)) = P_\mu(h) \text{ and } P_\mu(\frac{1+z}{1-z}) = P_m, \quad \left(\mu = \frac{m}{4} + \frac{1}{2}\right).$$

Here P_m is the class introduced and studied by Pinchuk [18].

A function $f \in \mathbf{A}$ is said to be in the class $M_\mu^\delta(\Phi, \xi, H)$, if and only if, $J_\delta(f(z)) \in P_\mu(H)$, where

$$J_\delta(f(z)) = (1-\delta)\frac{\phi * \xi * f}{\varphi * \xi * f} + \delta\frac{\phi * f}{\varphi * f}.$$

We denote by $W_\mu(\Phi, \xi, H) = M_\mu^0(\Phi, \xi, H)$, the class of functions $f \in \mathbf{A}$ such that

$$\frac{\phi * \xi * f}{\varphi * \xi * f} \in P_\mu(H).$$

A function $f \in \mathbf{A}$ is said to be in the class $CM_{\mu,\vartheta}^\delta(\Phi, \xi, G, H)$, if there exists a function $g \in W_\vartheta(\Phi, \xi, H)$ such that

$$(1-\delta)\frac{\phi * \xi * f}{\varphi * \xi * g} + \delta\frac{\phi * f}{\varphi * g} \in P_\mu(H).$$

Moreover, let us define

$$S^*(\varphi, H) = W_1((z\varphi', \varphi), \xi_1, H), \qquad C(\varphi, H) = W_1((\varphi_2, \varphi_1), \xi_1, H)$$

$$T_\mu(\varphi, G, H) = CM_{\mu,1}^0((z\varphi', \varphi), \xi_1, G, H), \quad T_\mu^*(\varphi, G, H) = CM_{\mu,1}^0((\varphi_2, \varphi_1), \xi_1, G, H),$$

where $\varphi_1(z) = z\varphi'(z)$, $\varphi_2(z) = z\varphi_1'$ and $\xi_1 = \frac{z}{1-z}$.

Definition 1. A function $f \in \mathbf{A}$ is said to be in class $T_\mu(\varphi, G, H)$ if there exists a function $g \in S^*(\varphi, G)$ such that

$$\frac{z(\varphi * f)'}{(\varphi * g)} \in P_\mu(H).$$

Analogous to this class in terms of Alexander type relation, we can define the class $T^*_\mu(\varphi, G, H)$ as following.

Definition 2. Let $f \in \mathbf{A}$. Then

$$f \in T^*_\mu(\varphi, G, H) \text{ if and only if } zf' \in T_\mu(\varphi, G, H). \tag{1.5}$$

For different values of μ, φ, G and H, we can obtain the well-known classes, referred as [4, 8, 13, 15, 16, 23].

2 Main Results

To prove our main results we use the following lemmas:

Lemma 1. [11] *Let h be analytic, univalent convex function in E with $h(0) = 1$ and $Re(\gamma h(z) + \sigma) > 0$, $\sigma, \gamma \in \mathbb{C}$ and $\gamma \neq 0$. If $p(z)$ is analytic in E and $p(0) = h(0)$, then*

$$\left\{p(z) + \frac{zp'(z)}{\gamma p(z) + \sigma}\right\} \prec h(z),$$

implies $p(z) \prec q(z) \prec h(z)$, where $q(z)$ is best dominant and is given as,

$$q(z) = \left[\left\{\int_0^1 \left(\exp \int_t^{tz} \frac{h(u) - 1}{u} du\right) dt\right\}^{-1} - \frac{\sigma}{\gamma}\right].$$

Lemma 2. [3] *Let $H = (h_1, h_2)$, where h_i $(i = 1, 2)$ are analytic, univalent convex functions with $h_i(0) = 1$ $(i = 1, 2)$ and let $\varkappa : E \to \mathbb{C}$ (set of complex numbers) with $Re(\varkappa) > 0$. If $p(z)$ is analytic, with $p(0) = 1$ in E, satisfies*

$$p(z) + \varkappa z p'(z) \in P_\mu(H),$$

then $p(z) \in P_\mu(H)$.

Lemma 3. [10] *Let h be convex functions with $h(0) = 1$ and let $\varkappa : E \to \mathbb{C}$ (set of complex numbers) with $Re\,(\varkappa) > 0$. If $p(z)$ is analytic, with $p(0) = 1$ in E, satisfies*

$$p(z) + \varkappa z p'(z) \prec h(z),$$

then $p(z) \prec h(z)$.

Lemma 4. [22] *If $f \in C$, $g \in S^*$, then for each h analytic in E with $h(0) = 1$,*

$$\frac{(f * hg)\,(E)}{(f * g)\,(E)} \subset \overline{Co}h(E),$$

where $\overline{Co}h(E)$ denotes the convex hull of $h(E)$.

Lemma 5. [21] *Let p be an analytic function in E with $p(0) = 1$ and $Re\{p(z)\} > 0$, $z \in E$. Then, for $\eta > 0$ and $\nu \neq -1$ (complex),*

$$Re\left\{p(z) + \frac{\eta z p'(z)}{p(z) + \nu}\right\} > 0, \text{ for } |z| < r_0,$$

where r_0 is given by

$$r_0 = \frac{|\nu + 1|}{\sqrt{s + \sqrt{s^2 - |\nu^2 - 1|^2}}}, \quad s = 2\,(\eta + 1)^2 + |\nu|^2 - 1$$

and this radius is best possible.

Lemma 6. *Let $Re\,\{h(z) + m\} > 0$. Then*

$$S^*(\varphi_\lambda^{-1}, h) \subset S^*(\varphi_{\lambda+1}^{-1}, h).$$

Proof. Let $f \in S^*(\varphi_\lambda^{-1}, h)$. Then, for $I_{\lambda+1} f(z) = \varphi_{\lambda+1}^{-1} * f(z)$, we set

$$\frac{z\,(I_{\lambda+1} f(z))'}{I_{\lambda+1} f(z)} = p(z), \tag{2.1}$$

where $p(z)$ is analytic with $p(0) = 1$.

Using identity (1.4) and (2.1), we have

$$(1 + \lambda)\frac{(I_\lambda f(z))}{(I_{\lambda+1} f(z))} = p(z) + \lambda.$$

Logarithmic differentiation yields

$$\frac{z\left(I_\lambda f(z)\right)'}{\left(I_\lambda f(z)\right)} = p(z) + \frac{zp'(z)}{p(z)+\lambda}. \tag{2.2}$$

Since $f \in S^*(\varphi_\lambda^{-1}, h)$, from (2.2) we have

$$p(z) + \frac{zp'(z)}{p(z)+\lambda} \prec h(z). \tag{2.3}$$

By applying Lemma 1, we conclude that $p(z) \prec h(z)$ and consequently,

$$\frac{z(I_{\lambda+1} f(z))'}{I_{\lambda+1} f(z)} \prec h(z).$$

This implies $f(z) \in S^*(\varphi_{\lambda+1}^{-1}, h)$. □

2.1 Inclusion Properties

We assume μ, $k \geq 0$, $m \geq 2$, $-1 \leq B < A \leq 1$, $\varphi(z) = \varphi_\lambda^{-1}(z) = \left(\frac{z}{(1-z)^{\lambda+1}}\right)^{-1}$, $(\lambda > -1)$, and $H = (h_1, h_2)$ where h_i $(i = 1, 2)$ and g_1 are analytic, univalent convex functions with $g_1(0) = 1$ and $h_i(0) = 1$ $(i = 1, 2)$ throughout our investigations, otherwise stated.

Theorem 1. *For* $\lambda \geq 0$

$$T_\mu(\varphi_\lambda^{-1}, g_1, H) \subset T_\mu(\varphi_{\lambda+1}^{-1}, g_1, H).$$

Proof. Let $f \in T_\mu(\varphi_\lambda^{-1}, g_1, H)$. Then, by Definition 1, there exists $g \in S^*(\varphi_\lambda^{-1}, g_1)$ such that

$$\frac{z(I_\lambda f(z))'}{I_\lambda g(z)} \in P_\mu\left(H\right), \tag{2.4}$$

where $I_\lambda f(z) = \varphi_\lambda^{-1} * f(z)$ and $I_\lambda g(z) = \varphi_\lambda^{-1} * g(z)$.

Consider

$$\frac{z(I_{\lambda+1} f(z))'}{I_{\lambda+1} g(z)} = p(z), \tag{2.5}$$

where $p(z)$ is analytic in E with $p(0) = 1$.

Using identity (1.4), we get

$$\frac{z(I_\lambda f(z))'}{I_\lambda g(z)} = \frac{\frac{z(I_{\lambda+1}(zf(z))')'}{I_{\lambda+1}g(z)} + \frac{\lambda z(I_{\lambda+1}f(z))'}{I_{\lambda+1}g(z)}}{\frac{z(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)} + \lambda}. \tag{2.6}$$

Differentiate logarithmically both sides of (2.5), we obtain

$$\frac{(z\,(I_{\lambda+1}f(z))')'}{z(I_{\lambda+1}f(z))'} = \frac{(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)} + \frac{p'(z)}{p(z)}$$

$$\frac{z(I_{\lambda+1}\,(zf(z))')'}{I_{\lambda+1}g(z)} = \frac{z(I_{\lambda+1}f(z))'}{I_{\lambda+1}g(z)}\left[\frac{z(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)} + \frac{zp'(z)}{p(z)}\right]$$

$$\frac{(z(I_{\lambda+1}f(z))')'}{I_{\lambda+1}g(z)} = p(z).q(z) + zp'(z), \tag{2.7}$$

where $q(z) = \frac{z(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)}$. From (2.6) and (2.7), we get

$$\frac{z(I_\lambda f(z))'}{I_\lambda g(z)} = \frac{p(z).q(z) + zp'(z) + \lambda p(z)}{q(z)+\lambda}$$

$$\frac{z(I_\lambda f(z))'}{I_\lambda g(z)} = p(z) + \frac{zp'(z)}{q(z)+\lambda}. \tag{2.8}$$

From (2.4) and (2.8), we have

$$p(z) + \frac{zp'(z)}{q(z)+\lambda} \in P_\mu\,(H)\,. \tag{2.9}$$

Since $g \in S^*(\varphi_\lambda^{-1}, g_1)$, by Lemma 6, this implies $g \in S^*(\varphi_{\lambda+1}^{-1}, g_1)$. Thus $q \in P(g_1) \subset P$ implies $Re(q(z)) > 0$ and $Re(\frac{1}{q(z)+\lambda}) > 0$ in E. By Lemma 2 and (2.9), we conclude $p(z) \in P_\mu\,(H)$. Hence $f \in T_\mu(\varphi_{\lambda+1}^{-1}, g_1, H)$. □

Corollary 1. *For* $\lambda \geq 0$, $\mu = \frac{m}{2} + \frac{1}{2}$ $(m \geq 2)$ *and* $H = (h_1, h_1)$

$$T_m(\varphi_\lambda^{-1}, g_1, h_1) \subset T_m(\varphi_{\lambda+1}^{-1}, g_1, h_1).$$

Proof. Let $f \in T_m(\varphi_\lambda^{-1}, g_1, h_1)$. Then, by Definition 1, there exists $g \in S^*(\varphi_\lambda^{-1}, g_1)$ such that

$$\frac{z(I_\lambda f(z))'}{I_\lambda g(z)} \in P_m\,(h_1)\,, \tag{2.10}$$

where $I_\lambda f(z) = \varphi_\lambda^{-1}(z) * f(z)$ and $I_\lambda g(z) = \varphi_\lambda^{-1}(z) * g(z)$.

Consider

$$\frac{z(I_{\lambda+1}f(z))'}{I_{\lambda+1}g(z)} = Q(z) \tag{2.11}$$

$$= \left(\frac{m}{2}+\frac{1}{2}\right) q_1(z) - \left(\frac{m}{2}-\frac{1}{2}\right) q_2(z), \tag{2.12}$$

where $Q(z)$ is analytic with $Q(0) = 1$.

Using identity (1.4), to get

$$\frac{z(I_\lambda f(z))'}{I_\lambda g(z)} = \frac{\frac{z(I_{\lambda+1}(zf(z))')'}{I_{\lambda+1}g(z)} + \frac{\lambda z(I_{\lambda+1}f(z))'}{I_{\lambda+1}g(z)}}{\frac{z(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)} + \lambda}. \tag{2.13}$$

Differentiate logarithmically both sides of (2.11), we get

$$\frac{\left(z\left(I_{\lambda+1}f(z)\right)'\right)'}{z(I_{\lambda+1}f(z))'} = \frac{(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)} + \frac{Q'(z)}{Q(z)},$$

this implies

$$\frac{z(I_{\lambda+1}\left(zf(z)\right)')'}{I_{\lambda+1}g(z)} = \frac{z(I_{\lambda+1}f(z))'}{I_{\lambda+1}g(z)} \left[\frac{z(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)} + \frac{zQ'(z)}{Q(z)}\right]$$

$$\frac{\left(z(I_{\lambda+1}f(z))'\right)'}{I_{\lambda+1}g(z)} = Q(z).R(z) + zQ'(z), \tag{2.14}$$

where $R(z) = \frac{z(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)}$. From (2.13) and (2.14), we obtain

$$\frac{z(I_\lambda f(z))'}{I_\lambda g(z)} = \frac{Q(z).R(z) + zQ'(z) + \lambda Q(z)}{R(z)+\lambda}$$

$$= Q(z) + \frac{zQ'(z)}{R(z)+\lambda}. \tag{2.15}$$

From (2.12) and (2.15), we have

$$\frac{z(I_\lambda f(z))'}{I_\lambda g(z)} = \left(\frac{m}{2}+\frac{1}{2}\right)\left(q_1(z) + \frac{zq_1'(z)}{R(z)+\lambda}\right) - \left(\frac{m}{2}-\frac{1}{2}\right)\left(q_2(z) + \frac{zq_2'(z)}{R(z)+\lambda}\right). \tag{2.16}$$

Thus (2.10) and (2.16), we conclude that

$$\left(\frac{m}{2}+\frac{1}{2}\right)\left(q_1(z)+\frac{zq_1'(z)}{R(z)+\lambda}\right)-\left(\frac{m}{2}-\frac{1}{2}\right)\left(q_2(z)+\frac{zq_2'(z)}{R(z)+\lambda}\right)\in P_m\left(h_1\right),$$

this implies

$$q_i(z)+\frac{zq_i'(z)}{R(z)+\lambda}\prec h_1(z),\quad \text{for } i=1,2. \tag{2.17}$$

Since $g\in S^*(\varphi_\lambda^{-1},g_1)$, by Lemma 6, we have $g\in S^*(\varphi_{\lambda+1}^{-1},g_1)$, implies $R(z)\prec g_1(z)\in P$. This means $Re(R(z))>0$ or $Re(\frac{1}{R(z)+\lambda})>0$ in E. Thus, by Lemma 3 and (2.17), we conclude $q_i(z)\prec h_1$ $(i=1,2)$. Hence $f\in T_m(\varphi_{\lambda+1}^{-1},g_1,h_1)$. □

Theorem 2. *For* $\lambda\geq 0$

$$T_\mu^*(\varphi_\lambda^{-1},g_1,H)\subset T_\mu^*(\varphi_{\lambda+1}^{-1},g_1,H).$$

Proof. Let

$$\begin{aligned}
& f\in T_\mu^*(\varphi_\lambda^{-1},g_1,H). \\
\Leftrightarrow\ & zf'\in T_\mu(\varphi_\lambda^{-1},g_1,H), && \text{(by Definition 2)}\\
\Rightarrow\ & zf'\in T_\mu(\varphi_{\lambda+1}^{-1},g_1,H), && \text{(by Theorem 1)}\\
\Leftrightarrow\ & f\in T_\mu^*(\varphi_{\lambda+1}^{-1},g_1,H). && \text{(by Definition 2)}
\end{aligned}$$

□

We can easily prove the following corollary by using similar technique as used in Theorem 2.

Corollary 2. *For* $\lambda\geq 0$, $\mu=\frac{m}{2}+\frac{1}{2}$, $(m\geq 2)$, *and* $H=(h_1,h_1)$

$$T_m^*(\varphi_\lambda^{-1},g_1,h_1)\subset T_m^*(\varphi_{\lambda+1}^{-1},g_1,h_1).$$

Special casses:

On using the same technique as used in Corollary 1 and Corollary 2, we can easily prove same result for the following different choices of g_1 and h_1.

$$\begin{aligned}
\text{(i) } g_1 &= h_1 = \frac{1+Az}{1+Bz}. \\
\text{(ii) } g_1 &= h_1 = p_k(z), \text{ where } p_k(z) \text{ is given by (1.2).} \\
\text{(iii) } g_1 &= \frac{1+Az}{1+Bz} \text{ and } h_1 = p_k(z). \\
\text{(iv) } g_1 &= p_k(z) \text{ and } h_1 = \frac{1+Az}{1+Bz}.
\end{aligned}$$

2.2 Convex Convolution Preserving Properties

Theorem 3. *Let $f \in T_\mu(\varphi, g_1, H)$ and ψ be any convex univalent function in E. Then*

$$\psi * f \in T_\mu(\varphi, g_1, H).$$

Proof. Let $f \in T_\mu(\varphi, g_1, H)$. Then, by Definition 1, there exists $g \in S^*(\varphi, g_1)$ such that

$$\frac{z(\varphi * f)'(z)}{(\varphi * g)(z)} = F(z) \in P_\mu(H).$$

Consider, for $\psi \in C$

$$\begin{aligned}
\frac{z(\varphi * (\psi * f))'(z)}{(\varphi * (\psi * g))(z)} &= \frac{z(\psi * (\varphi * f))'(z)}{(\psi * (\varphi * g))(z)} \\
&= \frac{\psi(z) * z(\varphi * f)'(z)}{(\psi * (\varphi * g))(z)} \\
&= \frac{\psi(z) * \frac{z(\varphi * f)'(z)}{(\varphi * g)(z)} (\varphi * g)(z)}{(\psi * (\varphi * g))(z)} \\
&= \frac{\psi(z) * F(z).(\varphi * g)(z)}{(\psi * (\varphi * g))(z)}.
\end{aligned}$$

Since $g \in S^*(\varphi, g_1)$ implies $(\varphi * g) \in S^*(g_1) \subset S^*$, by Lemma 4, It conclude $\psi * f \in T_\mu(\varphi, g_1, H)$. □

Theorem 4. *Let $f \in T^*_\mu(\varphi, g_1, H)$ and ψ be any convex univalent function in E. Then $\psi * f \in T^*_\mu(\varphi, g_1, H)$.*

Proof. We can easily prove this result by using Theorem 3 along with relation (1.5). $\square$

We can deduce some special casses of Theorem 3 and Theorem 4, for different choices of φ, g_1 and $H = (h_1, h_1)$. We mention some of the cases as follows.

(i) g_1 and h_1 be analytic, univalent convex functions in E and $\varphi(z) = \varphi_\lambda^{-1}$, for $\lambda \geq 0$.

(ii) $\varphi \in \mathbf{A}$ and $g_1(z) = h_1(z) = \dfrac{1+Az}{1+Bz}, \ -1 \leq B < A \leq 1.$

(iii) $\varphi \in \mathbf{A}$ and $g_1(z) = h_1(z) = p_k(z),$ where $p_k(z)$ is given by (1.3).

(iv) $\varphi \in \mathbf{A}$ and $g_1(z) = \dfrac{1+Az}{1+Bz}$ and $h_1(z) = p_k(z)$.

(v) $\varphi \in \mathbf{A}$ and $g_1(z) = p_k(z)$ and $h_1(z) = \dfrac{1+Az}{1+Bz}$.

2.2.1 Application of Convex Convolution Preserving Properties

Corollary 3. *The classes $T_\mu(\varphi, g_1, H)$ and $T_\mu^*(\varphi, g_1, H)$ are closed under the following operators.*

$$\begin{aligned}
\text{(i) } f_1(z) &= \int_0^z \frac{f(t)}{t} dt. \\
\text{(ii) } f_2(z) &= \frac{2}{z}\int_0^z f(t)dt, \qquad (\textit{Libera's operator } [9]). \\
\text{(iii) } f_3(z) &= \int_0^z \frac{f(t)-f(xt)}{t-xt} dt, \quad |x| \leq 1, \ x \neq 1. \\
\text{(iv) } f_4(z) &= \frac{c+1}{z^c}\int_0^z t^{c-1} f(t), \quad Re(c) \geq 0, \ (\textit{Generalized Bernardi operator } [2]).
\end{aligned}$$

Proof. We may write, $f_i(z) = \varsigma_i(z) * f(z)$, where $\varsigma_i(z)$, $i = 1, 2, 3, 4$, are convex and given by

$$\begin{aligned}
\varsigma_1(z) &= -\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n, \\
\varsigma_2(z) &= \frac{-2\left[z - \log(1-z)\right]}{z} = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n,
\end{aligned}$$

$$\begin{aligned}\varsigma_3(z) &= \frac{1}{1-x}\log\left(\frac{1-xz}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1-x^n}{(1-x)^n}z^n, \qquad |x|\leq 1, \quad x\neq 1,\\ \varsigma_4(z) &= \sum_{n=1}^{\infty}\frac{1+c}{n+c}z^n, \quad Re(c)\geq 0.\end{aligned}$$

The proof follows easily by using Theorem 3 and Theorem 4. □

2.3 Radius Problem

Theorem 5. *Let* $f\in T_\mu(\varphi_{\lambda+1}^{-1},\frac{1+Az}{1+Bz},\frac{1+Az}{1+Bz})$. *Then*

$$f\in T_\mu(\varphi_{\lambda}^{-1},\frac{1+Az}{1+Bz},\frac{1+z}{1-z}) \text{ for } |z|<r_\lambda$$

$$r_\lambda=\frac{2(1+\lambda)}{L+\sqrt{L^2-4M}}, \tag{2.18}$$

where $L=3A^2+\lambda(A+B)-B, \quad M=(1+\lambda)\left(A^2+\lambda AB\right)$. *The value of* r_λ *is sharp.*

Proof. Let $f\in T_\mu(\varphi_{\lambda+1}^{-1},\frac{1+Az}{1+Bz},\frac{1+Az}{1+Bz})$. Then, by Definition 1, there exists $g\in S^*(\varphi_{\lambda+1}^{-1},\frac{1+Az}{1+Bz})$ such that

$$\frac{z(I_{\lambda+1}f(z))'}{I_{\lambda+1}g(z)}=p(z)\in P_\mu\left(\frac{1+Az}{1+Bz}\right), \tag{2.19}$$

where $I_{\lambda+1}f(z)=\varphi_{\lambda+1}^{-1}(z)*f(z)$ and $I_{\lambda+1}g(z)=\varphi_{\lambda+1}^{-1}(z)*g(z)$.

Using identity (1.4), we get

$$\frac{z(I_\lambda f(z))'}{I_\lambda g(z)}=\frac{\frac{z(I_{\lambda+1}(zf(z))')'}{I_{\lambda+1}g(z)}+\frac{\lambda z(I_{\lambda+1}f(z))'}{I_{\lambda+1}g(z)}}{\frac{z(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)}+\lambda}. \tag{2.20}$$

Logarithmic Differentiating (2.19), we get

$$\frac{(z(I_{\lambda+1}f(z))')'}{z(I_{\lambda+1}f(z))'}=\frac{(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)}+\frac{p'(z)}{p(z)}$$

or equivalently,

$$\frac{z(I_{\lambda+1}(zf(z))')'}{I_{\lambda+1}g(z)}=\frac{z(I_{\lambda+1}f(z))'}{I_{\lambda+1}g(z)}\left[\frac{z(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)}+\frac{zp'(z)}{p(z)}\right].$$

From (2.19), this implies

$$\frac{(z(I_{\lambda+1}f(z))')'}{I_{\lambda+1}g(z)} = p(z).q(z) + zp'(z), \tag{2.21}$$

where $q(z) = \frac{z(I_{\lambda+1}g(z))'}{I_{\lambda+1}g(z)}$. From (2.20) and (2.21), we get

$$\frac{z(I_{\lambda+1}f(z))'}{I_{\lambda+1}g(z)} = \frac{p(z).q(z) + zp'(z) + \lambda p(z)}{q(z)+\lambda}$$

$$\frac{z(I_{\lambda}f(z))'}{I_{\lambda}g(z)} = p(z) + \frac{zp'(z)}{q(z)+\lambda}. \tag{2.22}$$

If we take $p(z) = \mu q_1 + (1-\mu)q_2$ in (2.22), where $q_1, q_2 \in P_1\left(\frac{1+Az}{1+Bz}\right)$, then we obtain

$$p(z) + \frac{zp'(z)}{q(z)+\lambda} = \mu\left(q_1(z) + \frac{zq_1'(z)}{q(z)+\lambda}\right) + (1-\mu)\left(\left(q_2(z) + \frac{zq_2'(z)}{q(z)+\lambda}\right)\right)$$

Since $g \in S^*(\varphi_{\lambda+1}^{-1}, \frac{1+Az}{1+Bz})$, it implies $q(z) \in P\left(\frac{1+Az}{1+Bz}\right) = P\left[A,B\right]$. To prove $q_i(z) + \frac{zq_i'(z)}{q(z)+\lambda} \prec \frac{1+z}{1-z}$ $(i = 1,2)$, we use distortion result for the functions of class $P[A,B]$ (see [19]),

$$Re\left(q_i(z) + \frac{zq_i'(z)}{q(z)+\lambda}\right) \geq Re\left(q_i(z)\right)\left[\frac{(1-Ar)\left\{(1-Ar)+\lambda\left(1-Br\right)\right\} - (A-B)\,r}{(1-Ar)\left\{(1-Ar)+\lambda\left(1-Br\right)\right\}}\right].$$

The right hand side of the above inequality is positive, for $|z| < r_\lambda$, where r_λ is given by (2.18). Thus

$$p(z) + \frac{zp'(z)}{q(z)+\lambda} = \mu h_1 + (1-\mu)h_2, \text{ where } h_1, h_2 \prec \frac{1+z}{1-z}. \tag{2.23}$$

Consequently, from (2.22) and (2.23), we conclude that

$$f \in T_\mu\left(\varphi_\lambda^{-1}, \frac{1+Az}{1+Bz}, \frac{1+z}{1-z}\right) \text{ for } |z| < r_\lambda,$$

where r_λ is given by (2.18). □

Corollary 4. *For* $\mu = \frac{m}{4} + \frac{1}{2}$ $(m \geq 2)$*,* $A = 1$ *and* $B = -1$*, if* $f \in T_m(\varphi_{\lambda+1}^{-1}, \frac{1+z}{1-z}, \frac{1+z}{1-z})$*, then*

$$f \in T_m\left(\varphi_{\lambda}^{-1}, \frac{1+z}{1-z}, \frac{1+z}{1-z}\right), \textit{ for } |z| < r_\lambda, \textit{ where}$$

$$r_\lambda = \frac{(1+\lambda)}{2+\sqrt{3+\lambda^2}}. \tag{2.24}$$

The value of r_λ *is sharp.*

Corollary 5. *For* $m = 2$*,* $A = 1$ *and* $B = -1$*, if* $f \in T_2(\varphi_{\lambda+1}^{-1}, \frac{1+Az}{1+Bz}, \frac{1+Az}{1+Bz})$*, then*

$$f \in T_2\left(\varphi_{\lambda}^{-1}, \frac{1+z}{1-z}, \frac{1+z}{1-z}\right), \textit{ for } |z| < r_\lambda,$$

where r_λ *is given by* (2.24)*. Furthermore, for* $\lambda = 0$*, we have well-known radius problem*

$$K \subset C^*, \textit{ for } |z| < r_0,$$

where $r_0 = \frac{1}{2+\sqrt{3}}$*.*

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