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Topological Properties for Harmonic τ-Uniformly Convex Functions of Order ρ Associated with Wanas Differential Operator

Abbas Kareem Wanas

Abstract

The purpose of the present paper is to establish some topological properties for a certain family of harmonic τ-uniformly convex functions of order ρ associated with Wanas differential operator defined in the open unit disk U.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$, if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see Clunie and Sheil-Small [6]).

Denote by $\mathcal{H}$ the family of harmonic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $S_{\mathcal{H}}$ indicate the family of functions $f = h + \overline{g} \in \mathcal{H}$ which are univalent and sense-preserving in the open unit disk U and normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in S_{\mathcal{H}}$ can be expressed as

$$f(z) = h(z) + \overline{g(z)}, \tag{1.1}$$

where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n.$$

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Also note that $\mathcal{H}$ reduces to the family $\mathcal{A}$ of analytic functions in U if co-analytic part of f is identically zero.

A function $f \in S_{\mathcal{H}}$ is said to be *harmonic starlike* in $U(r) = \{z \in \mathbb{C} : |z| < r\}$, if (see [11])

$$\frac{\partial}{\partial t}\left(\arg\left(f\left(re^{it}\right)\right)\right) = Re\left\{\frac{zh'(z) - \overline{z\,g'(z)}}{h(z) + \overline{g(z)}}\right\} > 0, \quad (0 \le t \le 2\pi)$$

i.e., f maps the circle $\partial U(r)$ onto a closed curve that is starlike with respect to the origin.

We consider the usual topology on $\mathcal{H}$ defined by a metric in which a sequence $\{f_n\}$ in $\mathcal{H}$ converges to f if and only if it converges to f uniformly on each compact subset of U. It follows from the theorems of Weierstrass and Montel that this topological space is complete.

Let $\mathcal{M}$ be a sub-family of the set $\mathcal{H}$. A function $f \in \mathcal{M}$ is called an *extreme point* of $\mathcal{M}$ if the condition $f = \lambda f_1 + (1-\lambda) f_2$ $(f_1, f_2 \in \mathcal{M};\ 0 < \lambda < 1)$ implies $f_1 = f_2 = f$.

We denote by $E\mathcal{M}$ the set of all extreme points of $\mathcal{M}$. It is clear that $E\mathcal{M} \subset \mathcal{M}$.

A family $\mathcal{M}$ is *locally uniformly bounded* if for each r $(0 < r < 1)$, there is a real constant $V = V(r)$ so that $|f(z)| \le V$ $(f \in \mathcal{M};\ |z| \le r)$.

A family $\mathcal{M}$ is *convex* if $\gamma f_1 + (1-\gamma) f_2 \in \mathcal{M}$ $(f_1, f_2 \in \mathcal{M};\ 0 \le \gamma \le 1)$.

Moreover, we define the closed convex hull of $\mathcal{M}$ as the intersection of all closed convex subsets of $\mathcal{H}$ (with respect to the topology of locally uniform convergence) that contain $\mathcal{M}$. We denote the closed convex hull of $\mathcal{M}$ by $\overline{co}\,\mathcal{M}$.

A real-valued functional $F : \mathcal{H} \longrightarrow \mathbb{R}$ is called *convex* on a convex family $\mathcal{M} \subset \mathcal{H}$ if

$$F(\gamma f_1 + (1-\gamma) f_2) \le \gamma F(f_1) + (1-\gamma) F(f_2) \quad (f_1, f_2 \in \mathcal{M};\ 0 \le \gamma \le 1).$$

For $\alpha \in \mathbb{R}$, $\beta \ge 0$ with $\alpha + \beta > 0$, $m, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and for analytic part $h \in \mathcal{A}$, Wanas [15] introduced an operator (so-called Wanas operator) $W_{\alpha,\beta}^{k,q} : \mathcal{A} \longrightarrow \mathcal{A}$, defined by

$$W_{\alpha,\beta}^{k,q}\, h(z) = z + \sum_{n=2}^{\infty} [\Phi_n(k,\alpha,\beta)]^q a_n z^n\,, \tag{1.2}$$

where

$$\Phi_n(k,\alpha,\beta)=\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha^m+n\beta^m}{\alpha^m+\beta^m}\right).$$

Special cases of this operator can be found in [1, 2, 3, 4, 5, 8, 10, 12, 13, 14]. For more details see [16].

Now, we extended this operator on the family of harmonic functions. For $f=h+\overline{g}\in\mathcal{H}$, we define the Wanas differential operator $W_{\alpha,\beta}^{k,q}:\mathcal{H}\longrightarrow\mathcal{H}$ as follows:

$$W_{\alpha,\beta}^{k,q}\,f(z)=W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)},$$

where $W_{\alpha,\beta}^{k,q}\,h(z)$ is defined by (1.2) and

$$W_{\alpha,\beta}^{k,q}\,g(z)=\sum_{n=2}^{\infty}[\Phi_n(k,\alpha,\beta)]^q b_n z^n.$$

We denote by $WS_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$ the family of all functions of the form (1.1) that satisfy the condition:

$$Re\left\{\frac{z\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)''}{\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)'}+1\right\}>\tau\left|\frac{z\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)''}{\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)'}\right|+\rho,\quad (1.3)$$

where $0\le\rho<1;\ \tau\ge 0$ and $z\in U$.

Also denote by $T_{\mathcal{H}}$ the sub-family of $S_{\mathcal{H}}$ containing of all functions $f=h+\overline{g}$, where h and g are given by

$$h(z)=z-\sum_{n=2}^{\infty}|a_n|z^n,\quad g(z)=(-1)^q\sum_{n=2}^{\infty}|b_n|z^n\,. \quad (1.4)$$

It is easily verified that if $f\in T_{\mathcal{H}}$, we also have

$$W_{\alpha,\beta}^{k,q}\,f(z)=z-\sum_{n=2}^{\infty}|\Phi_n(k,\alpha,\beta)|^q|a_n|z^n+(-1)^q\sum_{n=2}^{\infty}|\Phi_n(k,\alpha,\beta)|^q|b_n|(\overline{z})^n.$$

Moreover, let $WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$ be the sub-family of $WS_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$, where

$$WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)=T_{\mathcal{H}}\cap WS_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q).$$

We now recall the following lemmas that will be used to prove our main results.

Lemma 1.1 [7]. *Let $\mathcal{M}$ be a nonempty compact convex subset of the family $\mathcal{H}$ and $F : \mathcal{H} \longrightarrow \mathbb{R}$ be a real-valued, continuous and convex functional on $\mathcal{M}$. Then*

$$max\{F(f): f \in \mathcal{M}\} = max\{F(f): f \in E\mathcal{M}\}.$$

Lemma 1.2 [9]. *A family $\mathcal{M} \subset \mathcal{H}$ is compact if and only if $\mathcal{M}$ is closed and locally uniformly bounded.*

2. A Set of Main Results

In the first theorem, we determine the sufficient condition for $f = h + \overline{g}$ to be in the family $WS_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q)$.

Theorem 2.1. *Let $f = h + \overline{g}$ with h and g are given by* (1.1). *If*

$$\sum_{n=2}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q(|a_n|+|b_n|) \leq 1-\rho, \tag{2.1}$$

where $0 \leq \rho < 1, \tau \geq 0$, then f is harmonic univalent in U and $f \in WS_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q)$.

Proof. For proving $f \in WS_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q)$, we must show that (1.3) holds true. Using the fact that $Re\{w\} \geq \tau$ if and only if $|1-\tau+w| \geq |1+\tau-w|$, it is suffices to show that

$$Re\left\{\left(\frac{z\left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)''}{\left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)'} + 1\right)(1+\tau e^{i\theta}) - \tau e^{i\theta}\right\} > \rho \quad (-\pi \leq \theta \leq \pi),$$

or equivalently

$$Re\left\{\frac{(1+\tau e^{i\theta})\left(z\left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)'' + \left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)'\right)}{\left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)'}\right.$$

$$\left. - \frac{\tau e^{i\theta}\left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)'}{\left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)'}\right\} > \rho. \tag{2.2}$$

If we put

$$A(z) = \left(1+\tau e^{i\theta}\right)\left(z\left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)'' + \left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)'\right)$$

$$- \tau e^{i\theta}\left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)'$$

and

$$B(z) = \left(W_{\alpha,\beta}^{k,q}\, h(z) + (-1)^q \overline{W_{\alpha,\beta}^{k,q}\, g(z)}\right)'.$$

We only need to prove that

$$|A(z) + (1-\rho)B(z)| - |A(z) - (1+\rho)B(z)| \geq 0\,.$$

But

$$|A(z) + (1-\rho)B(z)|$$

$$= \Bigg|\left(1+\tau e^{i\theta}\right)\Bigg(\sum_{n=2}^{\infty} n(n-1)[\Phi_n(k,\alpha,\beta)]^q a_n z^{n-1} + (-1)^q \sum_{n=2}^{\infty} n(n-1)[\Phi_n(k,\alpha,\beta)]^q \overline{b_n}(\overline{z})^{n-1}$$

$$+1 + \sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q a_n z^{n-1} + (-1)^q \sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q \overline{b_n}(\overline{z})^{n-1}\Bigg)$$

$$-\tau e^{i\theta}\left(1 + \sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q a_n z^{n-1} + (-1)^q \sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q \overline{b_n}(\overline{z})^{n-1}\right)$$

$$+(1-\rho)\left(1 + \sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q a_n z^{n-1} + (-1)^q \sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q \overline{b_n}(\overline{z})^{n-1}\right)\Bigg|$$

$$= \Bigg|(2-\rho) + \sum_{n=2}^{\infty} n\left(n+1-\rho+(n-1)\tau e^{i\theta}\right)[\Phi_n(k,\alpha,\beta)]^q a_n z^{n-1}$$

$$+(-1)^q \sum_{n=2}^{\infty} n\left(n+1-\rho+(n-1)\tau e^{i\theta}\right)[\Phi_n(k,\alpha,\beta)]^q \overline{b_n}(\overline{z})^{n-1}\Bigg|.$$

Also

$$|A(z)-(1+\rho)B(z)|$$

$$=\Bigg|\left(1+\rho e^{i\theta}\right)\left(\sum_{n=2}^{\infty} n(n-1)[\Phi_n(k,\alpha,\beta)]^q a_n z^{n-1}+(-1)^q \sum_{n=2}^{\infty} n(n-1)[\Phi_n(k,\alpha,\beta)]^q \overline{b_n}(\overline{z})^{n-1}\right.$$

$$\left.+1+\sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q a_n z^{n-1}+(-1)^q \sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q \overline{b_n}(\overline{z})^{n-1}\right)$$

$$-\tau e^{i\theta}\left(1+\sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q a_n z^{n-1}+(-1)^q \sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q \overline{b_n}(\overline{z})^{n-1}\right)$$

$$-(1+\rho)\left(1+\sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q a_n z^{n-1}+(-1)^q \sum_{n=2}^{\infty} n[\Phi_n(k,\alpha,\beta)]^q \overline{b_n}(\overline{z})^{n-1}\right)\Bigg|$$

$$=\Bigg|-\rho+\sum_{n=2}^{\infty} n\left(n-1-\rho+(n-1)\tau e^{i\theta}\right)[\Phi_n(k,\alpha,\beta)]^q a_n z^{n-1}$$

$$+(-1)^q \sum_{n=2}^{\infty} n\left(n-1-\rho+(n-1)\tau e^{i\theta}\right)[\Phi_n(k,\alpha,\beta)]^q \overline{b_n}(\overline{z})^{n-1}\Bigg|.$$

Then

$$|A(z)+(1-\rho)B(z)|-|A(z)-(1+\rho)B(z)|$$

$$\geq 2(1-\rho)-\sum_{n=2}^{\infty} 2n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q |a_n||z|^{n-1}$$

$$-\sum_{n=2}^{\infty} 2n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q |b_n|\,|z|^{n-1}$$

$$>2\left\{(1-\rho)-\sum_{n=2}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q (|a_n|+|b_n|)\right\}>0.$$

The harmonic univalent function

$$f(z)= z+\sum_{n=2}^{\infty} \frac{x_n}{n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q} z^n$$

$$+\sum_{n=2}^{\infty}\frac{\overline{y}_n}{n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q}(\overline{z})^n, \qquad (2.3)$$

where

$$\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1-\rho$$

shows that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.3) are in the family $WS_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$, because

$$\sum_{n=2}^{\infty}n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q\frac{|x_n|}{n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q}$$

$$+\sum_{n=2}^{\infty}n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q\frac{|y_n|}{n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q}$$

$$=\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1-\rho.$$

The restriction placed in Theorem 2.1 on the moduli of the coefficients of $f=h+\overline{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic univalent and $f\in WS_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$.

The next theorem shows that condition (2.1) is also the sufficient condition for functions $f\in T_{\mathcal{H}}$ to be in the family $WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$.

Theorem 2.2. *Let* $f\in T_{\mathcal{H}}$ *be a function of the form* (1.4). *Then* $f\in WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$ *if and only if condition* (2.1) *holds true*.

Proof. In the light of Theorem 2.1, we need only to prove that each function $f\in WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$ satisfies coefficient inequality (2.1). If $f\in WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$, then by (1.3), we have

$$Re\left\{\left(\frac{z\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)''}{\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)'}+1\right)\left(1+\tau e^{i\theta}\right)-\tau e^{i\theta}\right\}>\rho\ (-\pi\leq\theta\leq\pi).$$

This is equivalent to

$$Re\left\{\frac{\left(1+\tau e^{i\theta}\right)\left(z\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)''+\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)'\right)}{\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)'}\right.$$

$$\left.-\frac{\tau e^{i\theta}\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)'}{\left(W_{\alpha,\beta}^{k,q}\,h(z)+(-1)^q\overline{W_{\alpha,\beta}^{k,q}\,g(z)}\right)'}\right\}$$

$$=Re\left\{\frac{(1-\rho)-\sum_{n=2}^{\infty} n\left(n-\rho+(n-1)\tau e^{i\theta}\right)|\Phi_n(k,\alpha,\beta)|^q|a_n|z^{n-1}-}{1-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|a_n|z^{n-1}-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|b_n|(\overline{z})^{n-1}}\right.$$

$$\left.-\frac{\sum_{n=1}^{\infty} n\left(n-\rho+(n-1)\tau e^{i\theta}\right)|\Phi_n(k,\alpha,\beta)|^q|b_n|(\overline{z})^{n-1}}{1-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|a_n|z^{n-1}-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|b_n|(\overline{z})^{n-1}}\right\}\geq 0. \quad (2.4)$$

The above required condition (2.4) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$Re\left\{\frac{(1-\rho)-\left[\sum_{n=2}^{\infty} n(n-\rho)|\Phi_n(k,\alpha,\beta)|^q|a_n|r^{n-1}+\sum_{n=1}^{\infty} n(n-\rho)|\Phi_n(k,\alpha,\beta)|^q|b_n|r^{n-1}\right]}{1-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|a_n|r^{n-1}-\sum_{n=1}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|b_n|r^{n-1}}\right.$$

$$\left.-\frac{\tau e^{i\theta}\left[\sum_{n=2}^{\infty} n(n-1)|\Phi_n(k,\alpha,\beta)|^q|a_n|r^{n-1}+\sum_{n=2}^{\infty} n(n-1)|\Phi_n(k,\alpha,\beta)|^q|b_n|r^{n-1}\right]}{1-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|a_n|r^{n-1}-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|b_n|r^{n-1}}\right\}\geq 0.$$

Since $Re\left(-e^{i\theta}\right)\geq-\left|e^{i\theta}\right|=-1$, and let $r\longrightarrow 1^-$, the above inequality reduces to

$$\frac{1-\rho-\sum_{n=2}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q|a_n|}{1-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|a_n|-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|b_n|}$$

$$-\frac{\sum_{n=1}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q|b_n|}{1-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|a_n|-\sum_{n=2}^{\infty} n|\Phi_n(k,\alpha,\beta)|^q|b_n|}\geq 0.$$

This gives coefficient inequality (2.1), so the proof is complete.

Theorem 2.3. *The family $WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$ is a convex and compact subset of $\mathcal{H}$.*

Proof. Let $0\leq\gamma\leq 1$ and $f_1,f_2\in WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$ be functions of the form:

$$f_j(z)=\sum_{\ell=0}^{\infty}\left(a_{j,\ell}z^{\ell}+\overline{b_{j,\ell}z^{\ell}}\right)\quad (z\in U, j\in\{1,2\}). \tag{2.5}$$

Since

$$\gamma f_1(z)+(1-\gamma)f_2(z)$$
$$=z-\sum_{n=2}^{\infty}(\gamma|a_{1,n}|+(1-\gamma)|a_{2,n}|)z^n-(-1)^q(\gamma|b_{1,n}|+(1-\gamma)|b_{2,n}|)(\overline{z})^n.$$

Thus by Theorem 2.2, we have

$$\sum_{n=2}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q\left((\gamma|a_{1,n}|+(1-\gamma)|a_{2,n}|)+(\gamma|b_{1,n}|+(1-\gamma)|b_{2,n}|)\right)$$

$$=\gamma\sum_{n=2}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q(|a_{1,n}|+|b_{1,n}|)$$

$$+(1-\gamma)\sum_{n=2}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q(|a_{2,n}|+|b_{2,n}|)$$

$$\leq\gamma(1-\rho)+(1-\gamma)(1-\rho)=1-\rho.$$

Therefore

$$\gamma f_1(z) + (1-\gamma) f_2(z) \in WT_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q).$$

Hence the family $WT_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q)$ is convex.

Furthermore, for $f \in WT_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q), |z| \leq r, 0 < r < 1$, we have

$$\begin{aligned} |f(z)| &\leq r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) \\ &< r + \sum_{n=2}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q (|a_n| + |b_n|) \\ &\leq r + 1 - \rho. \end{aligned}$$

Then, we conclude that the family $WT_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q)$ is locally uniformly bounded. By Lemma 1.2, we only need to show that it is closed, i.e., if $f_j \in WT_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q)$ $(j \in \mathbb{N})$ and $f_j \longrightarrow f$, then $f \in WT_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q)$. Let f_j and f be given by (1.2) and (1.1), respectively. In view of Theorem 2.2, we find that

$$\sum_{n=2}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q \left(|a_{j,n}| + |b_{j,n}|\right) \leq 1 - \rho \quad (j \in \mathbb{N}). \tag{2.6}$$

Since $f_j \longrightarrow f$, we conclude that $|a_{j,n}| \longrightarrow |a_n|$ and $|b_{j,n}| \longrightarrow |b_n|$ as $j \longrightarrow \infty$ $(j \in \mathbb{N})$. The sequence of partial sums $\{S_n\}$ associated with the series

$$\sum_{n=2}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q (|a_n| + |b_n|)$$

is a non-decreasing sequence.

Furthermore, by (2.6) it is bounded by $1-\rho$. Thus, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty} n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q (|a_n| + |b_n|) = \lim_{n\to\infty} S_n \leq 1 - \rho.$$

Hence $f \in WT_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q)$ and this completes the proof.

Theorem 2.4. $EWT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)=\{h_n : n\in\mathbb{N}\}\cup\{g_n : n\in\mathbb{N}_2\}$, *where*

$$h_1(z)=z,\qquad h_n(z)=z-\frac{1-\rho}{n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q}z^n,$$

$$g_n(z)=z+\frac{1-\rho}{n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q}(\overline{z})^n. \tag{2.7}$$

Proof. Assume that $0<\lambda<1$ and $g_n=\lambda f_1+(1-\lambda)f_2$, where $f_1,f_2\in WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$ are functions of the form (2.5). Then by (2.1), we obtain

$$|b_{1,n}|=|b_{2,n}|=\frac{1-\rho}{n(n-\rho+(n-1)\tau)|\Phi_n(k,\alpha,\beta)|^q}$$

and $a_{1,j}=a_{2,j}=0$ for $j\in\mathbb{N}_2$ and $b_{1,j}=b_{2,j}=0$ for $j\in\mathbb{N}_2\setminus\{n\}$. Then we have $g_n=f_1=f_2$ and hence $g_n\in WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$. Similarly, we prove that the functions h_n of the form (2.7) are the extreme points of the family $WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$. Assume that $f\in EWT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$ and f is not of the form (2.7), then there are $j\in\mathbb{N}_2$ such that

$$0<|a_j|<\frac{1-\rho}{j(j-\rho+(j-1)\tau)\left|\Phi_j(k,\alpha,\beta)\right|^q}$$

or

$$0<|b_j|<\frac{1-\rho}{j(j-\rho+(j-1)\tau)\left|\Phi_j(k,\alpha,\beta)\right|^q}.$$

If

$$0<|a_j|<\frac{1-\rho}{j(j-\rho+(j-1)\tau)\left|\Phi_j(k,\alpha,\beta)\right|^q},$$

then, we taking

$$\lambda=\frac{j(j-\rho+(j-1)\tau)\left|\Phi_j(k,\alpha,\beta)\right|^q}{1-\rho}|a_j|\quad\text{and}\quad \psi=\frac{1}{1-\lambda}\left(f-\lambda h_j\right).$$

We note that $0<\lambda<1$, $h_j\neq\psi$ and $f=\lambda h_j+(1-\lambda)\psi$. Therefore, $f\notin EWT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$.

If

$$0<|b_j|<\frac{1-\rho}{j(j-\rho+(j-1)\tau)\left|\Phi_j(k,\alpha,\beta)\right|^q},$$

then, we taking

$$\lambda = \frac{j(j-\rho+(j-1)\tau)\left|\Phi_j(k,\alpha,\beta)\right|^q}{1-\rho}\left|b_j\right| \quad \text{and} \quad \phi = \frac{1}{1-\lambda}\left(f-\lambda g_j\right).$$

We note that $0<\lambda<1$, $g_j \neq \phi$ and $f=\lambda g_j+(1-\lambda)\phi$. Therefore, $f \notin EWT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$.

Remark 2.1. If the family $\mathcal{M}=\{f_n \in \mathcal{H} : n \in \mathbb{N}\}$ is locally uniformly bounded, then

$$\overline{co}\,\mathcal{M} = \left\{\sum_{n=1}^{\infty}\gamma_n f_n : \sum_{n=1}^{\infty}\gamma_n = 1,\ \gamma_n \geq 0\ (n \in \mathbb{N})\right\}.$$

Corollary 2.1. *Let* h_n *and* g_n *be defined by* (2.7). *Then*

$$WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q) = \left\{\sum_{n=1}^{\infty}(\gamma_n h_n+\mu_n g_n) : \sum_{n=1}^{\infty}(\gamma_n+\mu_n)=1, \mu_1=0,\ \gamma_n,\mu_n \geq 0\right\}.$$

Remark 2.2. For each fixed value of $n \in \mathbb{N}_2$, $z \in U$, the following real-valued functions

$$F(f)=|a_n|,\quad F(f)=|b_n|,\quad F(f)=|f(z)|,\quad F(f)=\left|W_{\alpha,\beta}^{k,q}f(z)\right| \quad (f \in \mathcal{H})$$

are continuous and convex on $\mathcal{H}$.

Also, for $\gamma \geq 0, 0<r<1$, the real-valued functional

$$F(f)=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\gamma}d\theta\right)^{\frac{1}{\gamma}} \quad (f \in \mathcal{H})$$

is continuous and convex on $\mathcal{H}$.

By making use of Theorem 2.4 and Lemma 1.1, we obtain the following corollaries:

Corollary 2.2. *Let* $f \in WT_{\mathcal{H}}(\rho,\tau,\alpha,\beta,k,q)$, $|z|=r<1$. *Then*

$$r+\frac{1-\rho}{2(2-\rho+\tau)|\Phi_2(k,\alpha,\beta)|^q}r^2 \leq |f(z)| \leq r+\frac{1-\rho}{2(2-\rho+\tau)|\Phi_2(k,\alpha,\beta)|^q}r^2$$

and

$$r+\frac{1-\rho}{2(2-\rho+\tau)}r^2 \leq \left|W_{\alpha,\beta}^{k,q}f(z)\right| \leq r+\frac{1-\rho}{2(2-\rho+\tau)}r^2.$$

The result is sharp. The function h_n *of the form* (2.7) *is the extremal function.*

Corollary 2.3. *Let* $\gamma \geq 0, 0 < r < 1$. *If* $f \in WT_{\mathcal{H}}(\rho, \tau, \alpha, \beta, k, q)$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\gamma} d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left|h_2\left(re^{i\theta}\right)\right|^{\gamma} d\theta$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \left|zf'\left(re^{i\theta}\right)\right|^{\gamma} d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left|zh_2'\left(re^{i\theta}\right)\right|^{\gamma} d\theta.$$

The function h_2 *is the function defined by* (2.7).

3. Conclusion

The results we obtained in this paper which may be considered as a useful tool for those who are interested in the above-mentioned topics for further research. It may also be used to find prospective applications in some areas of mathematics and physics.

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Abbas Kareem Wanas
Department of Mathematics
College of Science
University of Al-Qadisiyah, Iraq
e-mail: abbas.kareem.w@qu.edu.iq