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A Study on Generalized Fibonacci Numbers: Sum Formulas $\sum_{k=0}^{n} kx^k W_k^3$ and $\sum_{k=1}^{n} kx^k W_{-k}^3$ for the Cubes of Terms

Yüksel Soykan

Abstract

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} kx^k W_k^3$ and $\sum_{k=1}^{n} kx^k W_{-k}^3$ for the cubes of generalized Fibonacci numbers are presented. As special cases, we give sum formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers.

1 Introduction

There are so many studies in the literature that concern about special second order recurrence sequences such as Fibonacci and Lucas. The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \; F_1 = 1$$

and the sequence of Lucas numbers $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \; L_1 = 1.$$

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field. Horadam [8] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \quad W_0 = a, \; W_1 = b, \quad (n \geq 2) \tag{1.1}$$

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where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [7], [9] and [10]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences.

Name of sequence	Notation: $W_n(a, b; r, s)$	OEIS: [17]
Fibonacci	$F_n = W_n(0, 1; 1, 1)$	A000045
Lucas	$L_n = W_n(2, 1; 1, 1)$	A000032
Pell	$P_n = W_n(0, 1; 2, 1)$	A000129
Pell-Lucas	$Q_n = W_n(2, 2; 2, 1)$	A002203
Jacobsthal	$J_n = W_n(0, 1; 1, 2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2, 1; 1, 2)$	A014551

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\begin{aligned}\sum_{k=0}^{n} k(-1)^k F_k^3 &= \frac{1}{4}((-1)^n((2n-3)F_{n+2}^3 + (2n+5)F_{n+1}^3 + 3F_{n+2}^2F_{n+1} \\ &\quad -6nF_{n+1}^2F_{n+2}) - 5)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n} k(-1)^k L_{-k}^3 &= \frac{1}{4}((-1)^n((2n+5)\,L_{-n+1}^3 + (2n-3)\,L_{-n}^3 - 3L_{-n+1}^2L_{-n} \\ &\quad -3\,(2n+2)\,L_{-n}^2L_{-n+1}) + 49).\end{aligned}$$

In this work, we derive expressions for sums of second powers of generalized Fibonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers.

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[1,2,6,11,12,18]	[5,19,21,22,23]	[3,4,13]
Generalized Tribonacci	[15,20]		
Generalized Tetranacci	[14,16]		

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 1.1. *Let x be a complex number. For $n \geq 0$ we have the following formulas: If $(-s^3x^2+rsx+1)(r^3x+s^3x^2+3rsx-1) \neq 0$, then*

(a)

$$\sum_{k=0}^{n} x^k W_k^3 = \frac{\Lambda_1}{(-s^3x^2+rsx+1)(r^3x+s^3x^2+3rsx-1)}$$

where

$$\begin{aligned}
\Lambda_1 = {} & -x^{n+2}(s^3x^2+2rsx-1)W_{n+2}^3 \\
& -x^{n+1}(r^3x+s^3x^2+3r^2s^2x^2-r^3s^3x^3+r^4sx^2+2rsx-1)W_{n+1}^3 \\
& +3rsx^{n+3}(r+s^2x)W_{n+2}^2W_{n+1} \\
& -3rs^2x^{n+3}(rsx-1)W_{n+1}^2W_{n+2}+x(s^3x^2+2rsx-1)W_1^3 \\
& +(r^3x+s^3x^2+3r^2s^2x^2-r^3s^3x^3+r^4sx^2+2rsx-1)W_0^3 \\
& -3rsx^2(r+s^2x)W_1^2W_0+3rs^2x^2\left(rsx-1\right)W_0^2W_1.
\end{aligned}$$

(b)

$$\sum_{k=0}^{n} x^k W_k^2 W_{k+1} = \frac{\Lambda_2}{(-s^3x^2+rsx+1)(r^3x+s^3x^2+3rsx-1)}$$

where

$$\begin{aligned}\Lambda_2 &= -rx^{n+2}(rsx-1)W_{n+2}^3 - rs^3x^{n+3}(rsx-1)W_{n+1}^3 \\ &\quad +sx^{n+2}(2r^3x - s^3x^2+1)W_{n+2}^2W_{n+1} \\ &\quad -x^{n+1}(r^3x+s^3x^2+r^4sx^2-2rs^4x^3+2rsx-1)W_{n+1}^2W_{n+2} \\ &\quad +rx(rsx-1)W_1^3 \\ &\quad +rs^3x^2(rsx-1)W_0^3 - sx(2r^3x-s^3x^2+1)W_1^2W_0 \\ &\quad +(r^3x+s^3x^2+r^4sx^2-2rs^4x^3+2rsx-1)W_0^2W_1.\end{aligned}$$

(c)

$$\sum_{k=0}^{n} x^k W_{k+1}^2 W_k = \frac{\Lambda_3}{(-s^3x^2+rsx+1)(r^3x+s^3x^2+3rsx-1)}$$

where

$$\begin{aligned}\Lambda_3 &= rx^{n+2}(r+s^2x)W_{n+2}^3 + rs^3x^{n+3}(r+s^2x)W_{n+1}^3 \\ &\quad -x^{n+1}(r^3x+s^3x^2+3r^2s^2x^2-1)W_{n+2}^2W_{n+1} \\ &\quad +s^2x^{n+2}(2r^3x-s^3x^2+1)W_{n+1}^2W_{n+2} - rx(r+s^2x)W_1^3 \\ &\quad -rs^3x^2(r+s^2x)W_0^3 \\ &\quad +(r^3x+s^3x^2+3r^2s^2x^2-1)W_1^2W_0 - s^2x(2r^3x-s^3x^2+1)W_0^2W_1.\end{aligned}$$

Proof. It is given in [21].

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 1.2. *Let x be a complex number. For $n \geq 1$ we have the following formulas: If $(-s^3+x^2+rsx)(r^3x+s^3-x^2+3rsx) \neq 0$, then*

(a)

$$\sum_{k=1}^{n} x^k W_{-k}^3 = \frac{\Lambda_4}{(-s^3+x^2+rsx)(r^3x+s^3-x^2+3rsx)}$$

where

$$\begin{aligned}\Lambda_4 &= x^{n+1}(s^3-x^2+2rsx)W_{-n+1}^3 \\ &+x^{n+1}(s^3x-r^3s^3+r^3x^2-x^3+2rsx^2+r^4sx+3r^2s^2x)W_{-n}^3 \\ &-3rsx^{n+1}(rx+s^2)W_{-n+1}^2W_{-n}+3rs^2x^{n+1}(-x+rs)W_{-n}^2W_{-n+1} \\ &-x(s^3-x^2+2rsx)W_1^3 \\ &+x(-s^3x+r^3s^3-r^3x^2+x^3-2rsx^2-r^4sx-3r^2s^2x)W_0^3 \\ &+3rsx(rx+s^2)W_1^2W_0-3rs^2x(-x+rs)W_0^2W_1.\end{aligned}$$

(b)

$$\sum_{k=1}^{n}x^kW_{-k+1}^2W_{-k}=\frac{\Lambda_5}{(-s^3+x^2+rsx)(r^3x+s^3-x^2+3rsx)}$$

where

$$\begin{aligned}\Lambda_5 &= -rx^{n+2}(rx+s^2)W_{-n+1}^3-rs^3x^{n+1}(rx+s^2)W_{-n}^3 \\ &+x^{n+2}(r^3x+3r^2s^2+s^3-x^2)W_{-n+1}^2W_{-n} \\ &+s^2x^{n+1}(-2r^3x+s^3-x^2)W_{-n}^2W_{-n+1} \\ &+rx^2(rx+s^2)W_1^3+rs^3x(rx+s^2)W_0^3 \\ &-x^2(r^3x+3r^2s^2+s^3-x^2)W_1^2W_0 \\ &+s^2x(2r^3x-s^3+x^2)W_0^2W_1.\end{aligned}$$

(c)

$$\sum_{k=1}^{n}x^kW_{-k}^2W_{-k+1}=\frac{\Lambda_6}{(-s^3+x^2+rsx)(r^3x+s^3-x^2+3rsx)}$$

$$\begin{aligned}\Lambda_6 &= rx^{n+2}\left(-x+rs\right)W_{-n+1}^3+rs^3x^{n+1}\left(-x+rs\right)W_{-n}^3 \\ &+sx^{n+1}(-2r^3x+s^3-x^2)W_{-n+1}^2W_{-n} \\ &+x^{n+1}(-2rs^4+s^3x+r^3x^2-x^3+2rsx^2+r^4sx)W_{-n}^2W_{-n+1} \\ &+rx^2(x-rs)W_1^3+rs^3x(x-rs)W_0^3+sx(2r^3x-s^3+x^2)W_1^2W_0 \\ &+x(2rs^4-s^3x-r^3x^2+x^3-2rsx^2-r^4sx)W_0^2W_1.\end{aligned}$$

Proof. It is given in [21].

2 Sum Formulas of Generalized Fibonacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.1. *Let x be a complex number. For $n \geq 0$ we have the following formulas: If* $(-s^3x^2 + rsx + 1)(r^3x + s^3x^2 + 3rsx - 1) \neq 0$, *then*

(a)

$$\sum_{k=0}^{n} kx^k W_k^3 = \frac{\Gamma_1}{(-s^3x^2 + rsx + 1)^2(r^3x + s^3x^2 + 3rsx - 1)^2}$$

where

$$\begin{aligned}
\Gamma_1 &= -x^{n+2}(n(-s^3x^2 + rsx + 1)(s^3x^2 + 2rsx - 1)(r^3x + s^3x^2 + 3rsx - 1) \\
&\quad -r^3x - 4s^3x^2 + 2s^6x^4 + 8r^2s^2x^2 + 8r^3s^3x^3 + 2r^5s^2x^3 + 6r^2s^5x^4 \\
&\quad +2r^4s^4x^4 - r^3s^6x^5 + 4r^4sx^2 + 8rs^4x^3 - 8rsx + 2)W_{n+2}^3 \\
&\quad -x^{n+1}(n(-s^3x^2 + rsx + 1)(r^3x + s^3x^2 + 3rsx - 1) \\
&\quad (r^3x + s^3x^2 + 3r^2s^2x^2 - r^3s^3x^3 + r^4sx^2 + 2rsx - 1) \\
&\quad -2r^3x - s^3x^2 + r^6x^2 - s^6x^4 + s^9x^6 - 2r^2s^2x^2 + 16r^3s^3x^3 \\
&\quad +10r^5s^2x^3 + 13r^2s^5x^4 + 10r^4s^4x^4 + 4r^6s^3x^4 - 2r^3s^6x^5 + r^8s^2x^4 \\
&\quad -6r^5s^5x^5 + 3r^2s^8x^6 - 2r^7s^4x^5 \\
&\quad +3r^4s^7x^6 + r^6s^6x^6 + 1 + 2r^4sx^2 + 2r^7sx^3 + 4rs^7x^5 - 4rsx)W_{n+1}^3 \\
&\quad +3rsx^{n+3}(n(-s^3x^2 + rsx + 1)(r + s^2x)(r^3x + s^3x^2 + 3rsx - 1) \\
&\quad -3r + 2r^4x - 4s^2x + 4s^5x^3 + 6r^3s^2x^2 + 6r^2s^4x^3 + 2r^4s^3x^3 - r^3s^5x^4 \\
&\quad +8rs^3x^2 + r^5sx^2 - rs^6x^4 + 4r^2sx)W_{n+2}^2W_{n+1} \\
&\quad -3rs^2x^{n+3}(n(-s^3x^2 + rsx + 1)(rsx - 1)(r^3x + s^3x^2 + 3rsx - 1) \\
&\quad +(2r^3x - s^3x^2 - r^2s^2x^2 + 5rsx - 3)(s^3x^2 + r^2s^2x^2 + rsx - 1))W_{n+1}^2W_{n+2} \\
&\quad +x(s^3x^2 + 1)(-2s^3x^2 + s^6x^4 + 7r^2s^2x^2 + 3r^4sx^2 + 4rs^4x^3 - 4rsx + 1)W_1^3
\end{aligned}$$

$$
\begin{aligned}
&+s^3x^2(-r^3x-4s^3x^2+2s^6x^4+8r^2s^2x^2+8r^3s^3x^3+2r^5s^2x^3+6r^2s^5x^4\\
&+2r^4s^4x^4-r^3s^6x^5+4r^4sx^2+8rs^4x^3-8rsx+2)W_0^3\\
&-3rsx^2(-2r+r^4x-3s^2x+2s^5x^3+s^8x^5+2r^3s^2x^2+5r^2s^4x^3\\
&+2r^4s^3x^3+2r^2sx+4rs^3x^2+2rs^6x^4)W_1^2W_0\\
&+3rs^2x^2(-r^3x-2s^6x^4+4r^2s^2x^2+2r^3s^3x^3+r^5s^2x^3+2r^4sx^2\\
&+rs^7x^5-5rsx+2)W_0^2W_1.
\end{aligned}
$$

(b)

$$
\sum_{k=0}^{n} kx^kW_k^2W_{k+1}=\frac{\Gamma_2}{(-s^3x^2+rsx+1)^2(r^3x+s^3x^2+3rsx-1)^2}
$$

where

$$
\begin{aligned}
\Gamma_2 \;=\;& -rx^{n+2}(n(-s^3x^2+rsx+1)(rsx-1)(r^3x+s^3x^2+3rsx-1)-r^3x\\
&-2s^6x^4+4r^2s^2x^2+2r^3s^3x^3+r^5s^2x^3+2r^4sx^2+rs^7x^5-5rsx+2)W_{n+2}^3\\
&-rs^3x^{n+3}(n(rsx-1)(-s^3x^2+rsx+1)(r^3x+s^3x^2+3rsx-1)\\
&+(s^3x^2+r^2s^2x^2+rsx-1)(2r^3x-s^3x^2-r^2s^2x^2+5rsx-3))W_{n+1}^3\\
&+sx^{n+2}(n(-s^3x^2+rsx+1)(2r^3x-s^3x^2+1)(r^3x+s^3x^2+3rsx-1)\\
&-5r^3x+4s^3x^2+4r^6x^2-2s^6x^4+2r^3s^3x^3+6r^5s^2x^3-6r^2s^5x^4-2r^4s^4x^4\\
&+3r^3s^6x^5+8r^4sx^2-4rs^4x^3+2r^7sx^3+2rs^7x^5+2rsx-2)W_{n+2}^2W_{n+1}\\
&-x^{n+1}(n(-s^3x^2+rsx+1)(r^3x+s^3x^2+3rsx-1)\\
&(r^3x+s^3x^2+r^4sx^2-2rs^4x^3+2rsx-1)\\
&-2r^3x-s^3x^2+r^6x^2-s^6x^4+s^9x^6+7r^2s^2x^2+4r^5s^2x^3-5r^2s^5x^4\\
&+r^4s^4x^4+4r^6s^3x^4-10r^3s^6x^5+r^8s^2x^4-4r^5s^5x^5+4r^2s^8x^6+3r^4s^7x^6\\
&+2r^4sx^2+8rs^4x^3+2r^7sx^3-4rs^7x^5-4rsx+1)W_{n+1}^2W_{n+2}\\
&+rx(2s^3x^2-3s^6x^4+5r^2s^2x^2-2r^3s^3x^3+2r^2s^5x^4+r^4s^4x^4\\
&+2r^4sx^2-4rs^4x^3+2rs^7x^5-2rsx+1)W_1^3\\
&+rs^3x^2(-r^3x-2s^6x^4+4r^2s^2x^2+2r^3s^3x^3+r^5s^2x^3+2r^4sx^2\\
&+rs^7x^5-5rsx+2)W_0^3
\end{aligned}
$$

$$+sx(s^3x^2+1)(4r^3x-2s^3x^2-2r^6x^2+s^6x^4+3r^2s^2x^2$$
$$-4r^3s^3x^3-3r^4sx^2+1)W_1^2W_0$$
$$+s^2x^2(rsx-1)(-2s+3r^5x+4s^4x^2-2s^7x^4-6r^2$$
$$+2r^3s^4x^3+4r^3sx+r^6sx^2)W_0^2W_1.$$

(c)

$$\sum_{k=0}^{n} kx^kW_{k+1}^2W_k=\frac{\Gamma_3}{(-s^3x^2+rsx+1)^2(r^3x+s^3x^2+3rsx-1)^2}$$

where

$$\begin{aligned}
\Gamma_3 \;=\; & rx^{n+2}(n(-s^3x^2+rsx+1)(r+s^2x)(r^3x+s^3x^2+3rsx-1)\\
& -2r+r^4x-3s^2x+2s^5x^3+s^8x^5+2r^3s^2x^2+5r^2s^4x^3+2r^4s^3x^3\\
& +2r^2sx+4rs^3x^2+2rs^6x^4)W_{n+2}^3\\
& +rs^3x^{n+3}(n(r+s^2x)(-s^3x^2+rsx+1)(r^3x+s^3x^2\\
& +3rsx-1)-3r+2r^4x-4s^2x\\
& +4s^5x^3+6r^3s^2x^2+6r^2s^4x^3+2r^4s^3x^3-r^3s^5x^4+4r^2sx+8rs^3x^2\\
& +r^5sx^2-rs^6x^4)W_{n+1}^3\\
& -x^{n+1}(n(-s^3x^2+rsx+1)(r^3x+s^3x^2+3rsx-1)\\
& (r^3x+s^3x^2+3r^2s^2x^2-1)\\
& -2r^3x-s^3x^2+r^6x^2-s^6x^4+s^9x^6-6r^2s^2x^2+12r^3s^3x^3\\
& +6r^5s^2x^3+9r^2s^5x^4\\
& +12r^4s^4x^4+4r^6s^3x^4+2r^3s^6x^5+3r^2s^8x^6+3r^4sx^2+1)W_{n+2}^2W_{n+1}\\
& +s^2x^{n+2}(n(r^3x+s^3x^2+3rsx-1)(-s^3x^2+rsx+1)(2r^3x-s^3x^2+1)\\
& -5r^3x+4s^3x^2+4r^6x^2-2s^6x^4+2r^3s^3x^3+6r^5s^2x^3-6r^2s^5x^4-2r^4s^4x^4\\
& +3r^3s^6x^5+8r^4sx^2-4rs^4x^3+2r^7sx^3+2rs^7x^5+2rsx-2)W_{n+1}^2W_{n+2}\\
& +rx(r+2s^2x-2s^8x^5+2r^3s^2x^2-4r^2s^4x^3-2r^4s^3x^3-r^3s^5x^4\\
& +r^5sx^2-5rs^6x^4)W_1^3\\
& +rs^3x^2(2r-r^4x+3s^2x-2s^5x^3-s^8x^5-2r^3s^2x^2-5r^2s^4x^3\\
& -2r^4s^3x^3-2r^2sx-4rs^3x^2-2rs^6x^4)W_0^3
\end{aligned}$$

$$+sx(r+s^2x)(2r^3x-4s^3x^2-r^6x^2+2s^6x^4+4r^3s^3x^3+3r^5s^2x^3$$
$$+6r^2s^5x^4+2)W_1^2W_0$$
$$+s^2x(s^3x^2+1)(4r^3x-2s^3x^2-2r^6x^2+s^6x^4+3r^2s^2x^2$$
$$-4r^3s^3x^3-3r^4sx^2+1)W_0^2W_1.$$

Proof. Using the recurrence relation

$$W_{n+2}=rW_{n+1}+sW_n$$

i.e.

$$sW_n=W_{n+2}-rW_{n+1}$$

we obtain

$$s^3W_n^3=W_{n+2}^3-3rW_{n+2}^2W_{n+1}+3r^2W_{n+1}^2W_{n+2}-r^3W_{n+1}^3$$

and so

$$\begin{aligned}
s^3\times n\times x^nW_n^3 &= n\times x^nW_{n+2}^3-3r\times n\times x^nW_{n+2}^2W_{n+1}\\
&\quad +3r^2\times n\times x^nW_{n+1}^2W_{n+2}-r^3\times n\times x^nW_{n+1}^3\\
s^3(n-1)x^{n-1}W_{n-1}^3 &= (n-1)x^{n-1}W_{n+1}^3-3r(n-1)x^{n-1}W_{n+1}^2W_n\\
&\quad +3r^2(n-1)x^{n-1}W_n^2W_{n+1}-r^3(n-1)x^{n-1}W_n^3\\
s^3(n-2)x^{n-2}W_{n-2}^3 &= (n-2)x^{n-2}W_n^3-3r(n-2)x^{n-2}W_n^2W_{n-1}\\
&\quad +3r^2(n-2)x^{n-2}W_{n-1}^2W_n-r^3(n-2)x^{n-2}W_{n-1}^3\\
&\vdots\\
s^3\times 2\times x^2W_2^3 &= 2\times x^2W_4^3-3r\times 2\times x^2W_4^2W_3+3r^2\times 2\times x^2W_3^2W_4\\
&\quad -r^3\times 2\times x^2W_3^3\\
s^3\times 1\times x^1W_1^3 &= 1\times x^1W_3^3-3r\times 1\times x^1W_3^2W_2+3r^2\times 1\times x^1W_2^2W_3\\
&\quad -r^3\times 1\times x^1W_2^3\\
s^3\times 0\times x^0W_0^3 &= 0\times x^0W_2^3-3r\times 0\times x^0W_2^2W_1+3r^2\times 0\times x^0W_1^2W_2\\
&\quad -r^3\times 0\times x^0W_1^3.
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
&s^3\sum_{k=0}^{n}kx^kW_k^3\\
=\;&(nx^nW_{n+2}^3+(n-1)x^{n-1}W_{n+1}^3-(-1)x^{-1}W_1^3-(-2)x^{-2}W_0^3\\
&+x^{-2}\sum_{k=0}^{n}kx^kW_k^3-2x^{-2}\sum_{k=0}^{n}x^kW_k^3)-3r(nx^nW_{n+2}^2W_{n+1}-(-1)x^{-1}W_1^2W_0\\
&+x^{-1}\sum_{k=0}^{n}kx^kW_{k+1}^2W_k-x^{-1}\sum_{k=0}^{n}x^kW_{k+1}^2W_k)\\
&+3r^2(nx^nW_{n+1}^2W_{n+2}-(-1)x^{-1}W_0^2W_1\\
&+x^{-1}\sum_{k=0}^{n}kx^kW_k^2W_{k+1}-x^{-1}\sum_{k=0}^{n}x^kW_k^2W_{k+1})-r^3(nx^nW_{n+1}^3-(-1)x^{-1}W_0^3\\
&+x^{-1}\sum_{k=0}^{n}kx^kW_k^3-x^{-1}\sum_{k=0}^{n}x^kW_k^3). \qquad (2.1)
\end{aligned}$$

Next we calculate $\sum_{k=0}^{n}kx^kW_{k+1}^2W_k$. Again, using the recurrence relation

$$W_{n+2}=rW_{n+1}+sW_n$$

i.e.

$$sW_n=W_{n+2}-rW_{n+1}$$

we obtain

$$sW_{n+1}^2W_n=W_{n+1}^2W_{n+2}-rW_{n+1}^3$$

and so

$$\begin{aligned}
s\times n\times x^nW_{n+1}^2W_n &= nx^nW_{n+1}^2W_{n+2}-r\times n\times x^nW_{n+1}^3\\
s(n-1)x^{n-1}W_n^2W_{n-1} &= (n-1)x^{n-1}W_n^2W_{n+1}-r(n-1)x^{n-1}W_n^3\\
s(n-2)x^{n-2}W_{n-1}^2W_{n-2} &= (n-2)x^{n-2}W_{n-1}^2W_n-r(n-2)x^{n-2}W_{n-1}^3\\
&\vdots\\
s\times 2\times x^2W_3^2W_2 &= 2\times x^2W_3^2W_4-r\times 2\times x^2W_3^3\\
s\times 1\times x^1W_2^2W_1 &= 1\times x^1W_2^2W_3-r\times 1\times x^1W_2^3\\
s\times 0\times x^0W_1^2W_0 &= 0\times x^0W_1^2W_2-r\times 0\times x^0W_1^3.
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
& s\sum_{k=0}^{n} kx^k W_{k+1}^2 W_k \\
= \; & (nx^n W_{n+1}^2 W_{n+2} - (-1)x^{-1} W_0^2 W_1 + x^{-1}\sum_{k=0}^{n} kx^k W_k^2 W_{k+1} \\
& -x^{-1}\sum_{k=0}^{n} x^k W_k^2 W_{k+1}) \\
& -r(nx^n W_{n+1}^3 - (-1)x^{-1} W_0^3 + x^{-1}\sum_{k=0}^{n} kx^k W_k^3 - x^{-1}\sum_{k=0}^{n} x^k W_k^3). \quad (2.2)
\end{aligned}$$

Next we calculate $\sum_{k=0}^{n} kx^k W_k^2 W_{k+1}$. Again, using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1} \Rightarrow s^2 W_n^2 = W_{n+2}^2 + r^2 W_{n+1}^2 - 2rW_{n+2}W_{n+1}$$

we obtain

$$s^2 W_n^2 W_{n+1} = W_{n+2}^2 W_{n+1} + r^2 W_{n+1}^3 - 2rW_{n+1}^2 W_{n+2}$$

and so

$$\begin{aligned}
s^2 \times n \times x^n W_n^2 W_{n+1} &= n \times x^n W_{n+2}^2 W_{n+1} + r^2 \times n \times x^n W_{n+1}^3 \\
&\quad -2r \times n \times x^n W_{n+1}^2 W_{n+2} \\
s^2 (n-1) x^{n-1} W_{n-1}^2 W_n &= (n-1)x^{n-1} W_{n+1}^2 W_n + r^2 (n-1) x^{n-1} W_n^3 \\
&\quad -2r(n-1)x^{n-1} W_n^2 W_{n+1} \\
s^2 (n-2) x^{n-2} W_{n-2}^2 W_{n-1} &= (n-2)x^{n-2} W_n^2 W_{n-1} + r^2 (n-2) x^{n-2} W_{n-1}^3 \\
&\quad -2r(n-2)x^{n-2} W_{n-1}^2 W_n \\
&\;\;\vdots
\end{aligned}$$

$$
\begin{aligned}
s^2 \times 3 \times x^3 W_3^2 W_4 &= 3 \times x^3 W_5^2 W_4 + r^2 \times 3 \times x^3 W_4^3 - 2r \times 3 \times x^3 W_4^2 W_5 \\
s^2 \times 2 \times x^2 W_2^2 W_3 &= 2 \times x^2 W_4^2 W_3 + r^2 \times 2 \times x^2 W_3^3 - 2r \times 2 \times x^2 W_3^2 W_4 \\
s^2 \times 1 \times x^1 W_1^2 W_2 &= 1 \times x^1 W_3^2 W_2 + r^2 \times 1 \times x^1 W_2^3 - 2r \times 1 \times x^1 W_2^2 W_3 \\
s^2 \times 0 \times x^0 W_0^2 W_1 &= 0 \times x^0 W_2^2 W_1 + r^2 \times 0 \times x^0 W_1^3 - 2r \times 0 \times x^0 W_1^2 W_2.
\end{aligned}
$$

If we add the above equations by side by, we get

$$
\begin{aligned}
& s^2 \sum_{k=0}^{n} k x^k W_k^2 W_{k+1} \\
= \; & (n x^n W_{n+2}^2 W_{n+1} - (-1) x^{-1} W_1^2 W_0 + x^{-1} \sum_{k=0}^{n} k x^k W_{k+1}^2 W_k - x^{-1} \sum_{k=0}^{n} x^k W_{k+1}^2 W_k) \\
& + r^2 (n x^n W_{n+1}^3 - (-1) x^{-1} W_0^3 + x^{-1} \sum_{k=0}^{n} k x^k W_k^3 - x^{-1} \sum_{k=0}^{n} x^k W_k^3) \\
& - 2r (n x^n W_{n+1}^2 W_{n+2} - (-1) x^{-1} W_0^2 W_1 + x^{-1} \sum_{k=0}^{n} k x^k W_k^2 W_{k+1} \\
& - x^{-1} \sum_{k=0}^{n} x^k W_k^2 W_{k+1}). \qquad (2.3)
\end{aligned}
$$

Using Theorem 1.1 and solving the system (2.1)-(2.2)-(2.3), the required results of (a),(b) and (c) follow.

2.1 The case $x = 1$

See [22] for the case $x = 1$.

2.2 The case $x = -1$

In this subsection we consider the special case $x = -1$.

Taking $x = -1, r = s = 1$ in Theorem 2.1 (a), (b) and (c)), we obtain the following proposition.

Proposition 2.2. *If $x = -1, r = s = 1$, then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k^3 = \frac{1}{4}((-1)^n ((2n-3)W_{n+2}^3 + (2n+5)W_{n+1}^3 + 3W_{n+2}^2 W_{n+1} - 6n W_{n+1}^2 W_{n+2}) - 5W_1^3 + 3W_0^3 + 3W_1^2 W_0 + 6W_0^2 W_1)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_k^2 W_{k+1} = \frac{1}{4}((-1)^n ((2n-2) W_{n+2}^3 - 2nW_{n+1}^3 - (2n+1) W_{n+2}^2 W_{n+1} + 7W_{n+1}^2 W_{n+2}) - 4W_1^3 + 2W_0^3 + W_1^2 W_0 + 7W_0^2 W_1)$.

(c) $\sum_{k=0}^{n} k(-1)^k W_{k+1}^2 W_k = \frac{1}{4}((-1)^n (-W_{n+2}^3 + W_{n+1}^3 + (2n+2) W_{n+2}^2 W_{n+1} - (2n+1) W_{n+1}^2 W_{n+2}) - W_1^3 + W_0^3 + W_0^2 W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.3. *For $n \geq 0$, Fibonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k F_k^3 = \frac{1}{4}((-1)^n ((2n-3)F_{n+2}^3 + (2n+5)F_{n+1}^3 + 3F_{n+2}^2 F_{n+1} - 6nF_{n+1}^2 F_{n+2}) - 5)$.

(b) $\sum_{k=0}^{n} k(-1)^k F_k^2 F_{k+1} = \frac{1}{4}((-1)^n ((2n-2) F_{n+2}^3 - 2nF_{n+1}^3 - (2n+1) F_{n+2}^2 F_{n+1} + 7F_{n+1}^2 F_{n+2}) - 4)$.

(c) $\sum_{k=0}^{n} k(-1)^k F_{k+1}^2 F_k = \frac{1}{4}((-1)^n (-F_{n+2}^3 + F_{n+1}^3 + (2n+2) F_{n+2}^2 F_{n+1} - (2n+1) F_{n+1}^2 F_{n+2}) - 1)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.4. *For $n \geq 0$, Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k L_k^3 = \frac{1}{4}((-1)^n ((2n-3)L_{n+2}^3 + (2n+5)L_{n+1}^3 + 3L_{n+2}^2 L_{n+1} - 6nL_{n+1}^2 L_{n+2}) + 49)$.

(b) $\sum_{k=0}^{n} k(-1)^k L_k^2 L_{k+1} = \frac{1}{4}((-1)^n ((2n-2) L_{n+2}^3 - 2nL_{n+1}^3 - (2n+1) L_{n+2}^2 L_{n+1} + 7L_{n+1}^2 L_{n+2}) + 42)$.

(c) $\sum_{k=0}^{n} k(-1)^k L_{k+1}^2 L_k = \frac{1}{4}((-1)^n (-L_{n+2}^3 + L_{n+1}^3 + (2n+2) L_{n+2}^2 L_{n+1} - (2n+1) L_{n+1}^2 L_{n+2}) + 11)$.

Taking $x = -1, r = 2, s = 1$ in Theorem 2.1 (a),(b) and (c), we obtain the following proposition.

Proposition 2.5. *If $x = -1, r = 2, s = 1$, then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k^3 = \frac{1}{98}((-1)^n ((14n-5)W_{n+2}^3 + (84n+89)W_{n+1}^3 - (63n+51) W_{n+1}^2 W_{n+2} - (21n-18)W_{n+2}^2 W_{n+1}) - 19W_1^3 + 5W_0^3 + 39W_1^2 W_0 + 12W_0^2 W_1)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_k^2 W_{k+1} = \frac{1}{98}((-1)^n ((21n-4) W_{n+2}^3 - (21n+17) W_{n+1}^3 - (56n+15) W_{n+2}^2 W_{n+1} + (28n+67) W_{n+1}^2 W_{n+2}) - 25W_1^3 + 4W_0^3 + 39W_0^2 W_1 + 41W_1^2 W_0)$.

(c) $\sum_{k=0}^{n} k(-1)^k W_{k+1}^2 W_k = \frac{1}{98}((-1)^n ((7n-13) W_{n+2}^3 - (7n-6) W_{n+1}^3 + (14n+37) W_{n+2}^2 W_{n+1} - (56n+15) W_{n+1}^2 W_{n+2}) - 20W_1^3 + 13W_0^3 + 23W_1^2 W_0 + 41W_0^2 W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.6. *For $n \geq 0$, Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k P_k^3 = \frac{1}{98}((-1)^n ((14n-5)P_{n+2}^3 + (84n+89)P_{n+1}^3 - (63n+51)P_{n+1}^2 P_{n+2} - (21n-18)P_{n+2}^2 P_{n+1}) - 19)$.

(b) $\sum_{k=0}^{n} k(-1)^k P_k^2 P_{k+1} = \frac{1}{98}((-1)^n ((21n-4) P_{n+2}^3 - (21n+17) P_{n+1}^3 - (56n+15) P_{n+2}^2 P_{n+1} + (28n+67) P_{n+1}^2 P_{n+2}) - 25)$.

(c) $\sum_{k=0}^{n} k(-1)^k P_{k+1}^2 P_k = \frac{1}{98}((-1)^n ((7n-13) P_{n+2}^3 - (7n-6) P_{n+1}^3 + (14n+37) P_{n+2}^2 P_{n+1} - (56n+15) P_{n+1}^2 P_{n+2}) - 20)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.7. *For $n \geq 0$, Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k Q_k^3 = \frac{1}{98}((-1)^n ((14n-5)Q_{n+2}^3 + (84n+89)Q_{n+1}^3 - (63n+51) Q_{n+1}^2 Q_{n+2} - (21n-18)Q_{n+2}^2 Q_{n+1}) + 296)$.

(b) $\sum_{k=0}^{n} k(-1)^k Q_k^2 Q_{k+1} = \frac{1}{98}((-1)^n ((21n-4) Q_{n+2}^3 - (21n+17) Q_{n+1}^3 - (56n+15) Q_{n+2}^2 Q_{n+1} + (28n+67) Q_{n+1}^2 Q_{n+2}) + 472)$.

(c) $\sum_{k=0}^{n} k(-1)^k Q_{k+1}^2 Q_k = \frac{1}{98}((-1)^n ((7n-13) Q_{n+2}^3 - (7n-6) Q_{n+1}^3 + (14n+37) Q_{n+2}^2 Q_{n+1} - (56n+15) Q_{n+1}^2 Q_{n+2}) + 456)$.

If $x = -1, r = 1, s = 2$, then $(-s^3x^2 + rsx + 1)(r^3x + s^3x^2 + 3rsx - 1) = 0$ so we can't use Theorem 2.1 directly. But we can give another method to find $\sum_{k=0}^{n} k(-1)^k W_k^3, \sum_{k=0}^{n} k(-1)^k W_k^2 W_{k+1}$ and $\sum_{k=0}^{n} k(-1)^k W_{k+1}^2 W_k$ by using Theorem 2.1.

Theorem 2.8. *If $x = -1, r = 1, s = 2$, then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k^3 = \frac{1}{486}((-1)^n ((9n^2+47n-350)W_{n+2}^3 - (72n^2+34n-2838)W_{n+1}^3 - 2(27n^2+51n-1042)W_{n+2}^2 W_{n+1} + 4(27n^2+15n-1046)W_{n+1}^2 W_{n+2}) - 388W_1^3 + 2800W_0^3 - 4136W_0^2 W_1 + 2132W_1^2 W_0)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_k^2 W_{k+1} = \frac{1}{1458}((-1)^n ((-27n^2+39n+1034)W_{n+2}^3 + 8(27n^2+15n-1046)W_{n+1}^3 + 2(81n^2+99n-3150)W_{n+2}^2 W_{n+1} - (324n^2+450n-12666)W_{n+1}^2 W_{n+2}) + 968W_1^3 - 8272W_0^3 + 12792W_0^2 W_1 - 6336W_1^2 W_0)$.

(c) $\sum_{k=0}^{n} k(-1)^k W_{k+1}^2 W_k = \frac{1}{1458}((-1)^n (-(-27n^2+3n+1066)W_{n+2}^3 - 8(27n^2+51n-1042)W_{n+1}^3 + (-162n^2+72n+6366)W_{n+2}^2 W_{n+1} + 4(81n^2+99n-3150)W_{n+1}^2 W_{n+2}) - 1036W_1^3 + 8528W_0^3 + 6132W_1^2 W_0 - 12672W_0^2 W_1)$.

Proof.

(a) We use Theorem 2.1 (a). If we set $r=1, s=2$ in Theorem 2.1 (a), then we have

$$\sum_{k=0}^{n} kx^k W_k^3 = \frac{f_1(x)}{(-8x^2+2x+1)^2(8x^2+7x-1)^2}$$

where

$$\begin{aligned} f_1(x) &= -(8x^2-17x+200x^3+352x^4-64x^5+2 \\ &\quad +n(-8x^2+2x+1)(8x^2+4x-1)(8x^2+7x-1))x^{n+2}W_{n+2}^3 \\ &\quad -(172x^3-11x^2-10x+548x^4+160x^5+1728x^6 \\ &\quad +1-n(-8x^2+2x+1)(8x^2+7x-1)(8x^3-22x^2-5x+1))x^{n+1}W_{n+1}^3 \\ &\quad +6(90x^2-6x+240x^3-96x^4-3+n(4x+1) \\ &\quad (-8x^2+2x+1)(8x^2+7x-1))x^{n+3}W_{n+2}^2 W_{n+1} \\ &\quad +12((12x^2+2x-1)(12x^2-12x+3)-n(2x-1) \\ &\quad (-8x^2+2x+1)(8x^2+7x-1))x^{n+3}W_{n+1}^2 W_{n+2} \\ &\quad +x(8x^2+1)(64x^4+64x^3+18x^2-8x+1)W_1^3 \\ &\quad +8x^2(-64x^5+352x^4+200x^3+8x^2-17x+2)W_0^3 \\ &\quad +12x^2(128x^5-128x^4+20x^3+20x^2-11x+2)W_1 W_0^2 \\ &\quad -6x^2(256x^5+128x^4+160x^3+40x^2-7x-2)W_1^2 W_0. \end{aligned}$$

For $x=-1$, the right hand side of the above sum formulas is an indeterminate

form. Now, we can use L'Hospital rule (by applying twice). Then we get

$$\begin{aligned}
&\sum_{k=0}^{n} k(-1)^k W_k^3 \\
= \;& \left.\frac{\frac{d}{dx}(f_1(x))}{\frac{d}{dx}((-8x^2+2x+1)^2(8x^2+7x-1)^2)}\right|_{x=-1} \\
= \;& \frac{1}{486}((-1)^n ((9n^2+47n-350)W_{n+2}^3 - (72n^2+34n-2838)W_{n+1}^3 \\
& -2(27n^2+51n-1042)W_{n+2}^2 W_{n+1} + 4(27n^2+15n-1046)W_{n+1}^2 W_{n+2}) \\
& -388W_1^3 + 2800W_0^3 - 4136W_0^2 W_1 + 2132W_1^2 W_0).
\end{aligned}$$

(b) We use Theorem 2.1 (b). If we set $r = 1, s = 2$ in Theorem 2.1 (b), then we have

$$\sum_{k=0}^{n} kx^k W_k^2 W_{k+1} = \frac{f_2(x)}{(-8x^2+2x+1)^2(8x^2+7x-1)^2}$$

where

$$\begin{aligned}
f_2(x) \;=\; & -(20x^2 - 11x + 20x^3 - 128x^4 + 128x^5 + 2 + n(2x-1) \\
& (-8x^2+2x+1)(8x^2+7x-1))x^{n+2}W_{n+2}^3 \\
& +8((12x^2+2x-1)(12x^2-12x+3) - n(2x-1) \\
& (-8x^2+2x+1)(8x^2+7x-1))x^{n+3}W_{n+1}^3 \\
& -2(x - 52x^2 + 20x^3 + 352x^4 - 448x^5 + 2 \\
& -n(-8x^2+2x+1)^2(8x^2+7x-1))x^{n+2}W_{n+2}^2 W_{n+1} \\
& -x^{n+1}(25x^2 - 10x + 148x^3 - 172x^4 - 1280x^5 + 1920x^6 \\
& +1 + n(-8x^2+2x+1)(8x^2+7x-1) \\
& (-32x^3 + 10x^2 + 5x - 1))W_{n+1}^2 W_{n+2} \\
& +x(256x^5 - 112x^4 - 80x^3 + 40x^2 - 4x + 1)W_1^3 \\
& +8x^2(128x^5 - 128x^4 + 20x^3 + 20x^2 - 11x + 2)W_0^3 \\
& +2x(8x^2+1)(64x^4 - 32x^3 - 12x^2 + 4x + 1)W_1^2 W_0 \\
& +4(2x-1)(-256x^4 + 32x^3 + 66x^2 + 11x - 10)x^2 W_0^2 W_1.
\end{aligned}$$

For $x = -1$, the right hand side of the above sum formulas is an indeterminate

form. Now, we can use L'Hospital rule (by applying twice). Then we get

$$\begin{aligned}
&\sum_{k=0}^{n} k(-1)^k W_k^2 W_{k+1} \\
= &\left. \frac{\frac{d}{dx}(f_2(x))}{\frac{d}{dx}((-8x^2+2x+1)^2(8x^2+7x-1)^2)} \right|_{x=-1} \\
= &\frac{1}{1458}((-1)^n((-27n^2+39n+1034)W_{n+2}^3+8(27n^2+15n-1046)W_{n+1}^3 \\
&+2(81n^2+99n-3150)W_{n+2}^2W_{n+1}-(324n^2+450n-12666)W_{n+1}^2W_{n+2}) \\
&+968W_1^3-8272W_0^3+12792W_0^2W_1-6336W_1^2W_0).
\end{aligned}$$

(c) We use Theorem 2.1 (c). If we set $r=1, s=2$ in Theorem 2.1 (c), then we have

$$\sum_{k=0}^{n} kx^k W_{k+1}^2 W_k = \frac{f_3(x)}{(-8x^2+2x+1)^2(8x^2+7x-1)^2}$$

where

$$\begin{aligned}
f_3(x) = \; &(40x^2-7x+160x^3+128x^4+256x^5-2+n(4x+1) \\
&(-8x^2+2x+1)(8x^2+7x-1))x^{n+2}W_{n+2}^3 \\
&+8(90x^2-6x+240x^3-96x^4-3+n(4x+1) \\
&(-8x^2+2x+1)(8x^2+7x-1))x^{n+3}W_{n+1}^3 \\
&-x^{n+1}(120x^3-25x^2-2x+448x^4+128x^5+1280x^6 \\
&+1+n(-8x^2+2x+1)(8x^2+7x-1)(20x^2+x-1))W_{n+2}^2W_{n+1} \\
&-4(x-52x^2+20x^3+352x^4-448x^5+2 \\
&-n(-8x^2+2x+1)^2(8x^2+7x-1))x^{n+2}W_{n+1}^2W_{n+2} \\
&+x(-512x^5-352x^4-80x^3+10x^2+8x+1)W_1^3 \\
&-8x^2(256x^5+128x^4+160x^3+40x^2-7x-2)W_0^3 \\
&+2x(4x+1)(320x^4+44x^3-33x^2+2x+2)W_1^2W_0 \\
&+4x(8x^2+1)(64x^4-32x^3-12x^2+4x+1)W_0^2W_1.
\end{aligned}$$

For $x=-1$, the right hand side of the above sum formulas is an indeterminate

form. Now, we can use L'Hospital rule (by applying twice). Then we get

$$\begin{aligned}
&\sum_{k=0}^{n} k(-1)^k W_{k+1}^2 W_k \\
= &\left.\frac{\frac{d}{dx}(f_3(x))}{\frac{d}{dx}((-8x^2+2x+1)^2(8x^2+7x-1)^2)}\right|_{x=-1} \\
= &\frac{1}{1458}((-1)^n(-(-27n^2+3n+1066)W_{n+2}^3 - 8(27n^2+51n-1042)W_{n+1}^3 \\
&+(-162n^2+72n+6366)W_{n+2}^2 W_{n+1} + 4(81n^2+99n-3150)W_{n+1}^2 W_{n+2}) \\
&-1036W_1^3 + 8528W_0^3 + 6132W_1^2 W_0 - 12672W_0^2 W_1).
\end{aligned}$$

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.9. *For $n \geq 0$, Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k J_k^3 = \frac{1}{486}((-1)^n((9n^2+47n-350)J_{n+2}^3 - (72n^2+34n-2838)J_{n+1}^3 - 2(27n^2+51n-1042)J_{n+2}^2 J_{n+1} + 4(27n^2+15n-1046)J_{n+1}^2 J_{n+2}) - 388)$.

(b) $\sum_{k=0}^{n} k(-1)^k J_k^2 J_{k+1} = \frac{1}{1458}((-1)^n((-27n^2+39n+1034)J_{n+2}^3 + 8(27n^2+15n-1046) J_{n+1}^3 + 2(81n^2+99n-3150)J_{n+2}^2 J_{n+1} - (324n^2+450n-12666)J_{n+1}^2 J_{n+2}) + 968)$.

(c) $\sum_{k=0}^{n} k(-1)^k J_{k+1}^2 J_k = \frac{1}{1458}((-1)^n(-(-27n^2+3n+1066)J_{n+2}^3 - 8(27n^2+51n-1042) J_{n+1}^3 + (-162n^2+72n+6366)J_{n+2}^2 J_{n+1} + 4(81n^2+99n-3150)J_{n+1}^2 J_{n+2}) - 1036)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.10. *For $n \geq 0$, Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k j_k^3 = \frac{1}{486}((-1)^n((9n^2+47n-350)j_{n+2}^3 - (72n^2+34n-2838)j_{n+1}^3 - 2(27n^2+51n-1042)j_{n+2}^2 j_{n+1} + 4(27n^2+15n-1046)j_{n+1}^2 j_{n+2}) + 9732)$.

(b) $\sum_{k=0}^{n} k(-1)^k j_k^2 j_{k+1} = \frac{1}{1458}((-1)^n((-27n^2+39n+1034)j_{n+2}^3 + 8(27n^2+15n-1046) j_{n+1}^3 + 2(81n^2+99n-3150)j_{n+2}^2 j_{n+1} - (324n^2+450n-12666)j_{n+1}^2 j_{n+2}) - 26712)$.

(c) $\sum_{k=0}^{n} k(-1)^k j_{k+1}^2 j_k = \frac{1}{1458}((-1)^n(-(-27n^2+3n+1066)j_{n+2}^3 - 8(27n^2+51n-1042) j_{n+1}^3 + (-162n^2+72n+6366)j_{n+2}^2 j_{n+1} + 4(81n^2+99n-3150)j_{n+1}^2 j_{n+2}) + 28764)$.

2.3 The case $x = 1 + i$

In this subsection we consider the special case $x = 1 + i$.

Taking $x = 1 + i, r = s = 1$ in Theorem 2.1 (a),(b) and (c)), we obtain the following proposition.

Proposition 2.11. *If $x = 1+i, r = s = 1$, then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(1+i)^k W_k^3 = \frac{1}{63+216i}((1+i)^n (2i((24-57i)n+79-47i)W_{n+2}^3 + (1+i)((51-168i)n+145-10i)W_{n+1}^3 + 3(-2+2i)((15+30i)n-17+56i) W_{n+2}^2 W_{n+1} + 3(-2+2i)((9-12i)n+15-20i)W_{n+1}^2 W_{n+2}) - (45+65i)W_1^3 - (94+158i)W_0^3 + (156+192i)W_1^2 W_0 - (48+36i)W_0^2 W_1)$.

(b) $\sum_{k=0}^{n} k(1+i)^k W_k^2 W_{k+1} = \frac{1}{63+216i}((1+i)^n (2i((9-12i)n+6-8i)W_{n+2}^3 + (-2+2i)((9-12i)n+15-20i)W_{n+1}^3 + 2i((36+27i)n+3+21i)W_{n+2}^2 W_{n+1} - (1+i)((45+90i)n+39+48i)W_{n+1}^2 W_{n+2}) + (7-i)W_1^3 - (16+12i)W_0^3 + (27+39i)W_1^2 W_0 - (6+42i)W_0^2 W_1)$.

(c) $\sum_{k=0}^{n} k(1+i)^k W_{k+1}^2 W_k = \frac{1}{63+216i}(2i(1+i)^n((15+30i)n-32+26i)W_{n+2}^3 + (-2+2i)(1+i)^n((15+30i)n-17+56i)W_{n+1}^3 + (1+i)(1+i)^n ((81-108i)n+141+12i)W_{n+2}^2 W_{n+1} + 2i(1+i)^n((36+27i)n+3+21i)W_{n+1}^2 W_{n+2} + (43+51i)W_1^3 + (52+64i)W_0^3 - (60+120i)W_1^2 W_0 + (27+39i)W_0^2 W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.12. *For $n \geq 0$, Fibonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(1+i)^k F_k^3 = \frac{1}{63+216i}((1+i)^n (2i((24-57i)n+79-47i)F_{n+2}^3 + (1+i)((51-168i)n+145-10i)F_{n+1}^3 + 3(-2+2i)((15+30i)n-17+56i) F_{n+2}^2 F_{n+1} + 3(-2+2i)((9-12i)n+15-20i)F_{n+1}^2 F_{n+2}) - 45 - 65i)$.

(b) $\sum_{k=0}^{n} k(1+i)^k F_k^2 F_{k+1} = \frac{1}{63+216i}((1+i)^n (2i((9-12i)n+6-8i)F_{n+2}^3 + (-2+2i)((9-12i)n+15-20i)F_{n+1}^3 + 2i((36+27i)n+3+21i)F_{n+2}^2 F_{n+1} - (1+i)((45+90i)n+39+48i)F_{n+1}^2 F_{n+2}) + 7 - i)$.

(c) $\sum_{k=0}^{n} k(1+i)^k F_{k+1}^2 F_k = \frac{1}{63+216i}(2i(1+i)^n((15+30i)n-32+26i)F_{n+2}^3 + (-2+2i)(1+i)^n((15+30i)n-17+56i)F_{n+1}^3 + (1+i)(1+i)^n ((81-108i)n+141+12i)F_{n+2}^2 F_{n+1} + 2i(1+i)^n((36+27i)n+3+21i)F_{n+1}^2 F_{n+2} + 43 + 51i)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.13. *For $n \geq 0$, Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(1+i)^k L_k^3 = \frac{1}{63+216i}((1+i)^n(2i((24-57i)n+79-47i)L_{n+2}^3 + (1+i)((51-168i)n+145-10i)L_{n+1}^3 + 3(-2+2i)((15+30i)n-17+56i) L_{n+2}^2 L_{n+1} + 3(-2+2i)((9-12i)n+15-20i)L_{n+1}^2 L_{n+2}) - 677 - 1089i)$.

(b) $\sum_{k=0}^{n} k(1+i)^k L_k^2 L_{k+1} = \frac{1}{63+216i}((1+i)^n(2i((9-12i)n+6-8i)L_{n+2}^3 + (-2+2i)((9-12i)n+15-20i)L_{n+1}^3 + 2i((36+27i)n+3+21i)L_{n+2}^2 L_{n+1} - (1+i)((45+90i)n+39+48i)L_{n+1}^2 L_{n+2}) - 91 - 187i)$.

(c) $\sum_{k=0}^{n} k(1+i)^k L_{k+1}^2 L_k = \frac{1}{63+216i}(2i(1+i)^n((15+30i)n-32+26i)L_{n+2}^3 + (-2+2i)(1+i)^n((15+30i)n-17+56i)L_{n+1}^3 + (1+i)(1+i)^n ((81-108i)n+141+12i)L_{n+2}^2 L_{n+1} + 2i(1+i)^n((36+27i)n+3+21i)L_{n+1}^2 L_{n+2} + 447 + 479i)$.

Corresponding sums of the other second order linear sequences can be calculated similarly when $x = 1+i$.

3 Sum Formulas of Generalized Fibonacci Numbers with Negative Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. *Let x be a complex number. For $n \geq 1$ we have the following formulas: If $(-s^3+x^2+rsx)(r^3x+s^3-x^2+3rsx) \neq 0$, then*

(a)

$$\sum_{k=1}^{n} kx^k W_{-k}^3 = \frac{\Gamma_4}{(-s^3+x^2+rsx)^2(r^3x+s^3-x^2+3rsx)^2}$$

where

$$\begin{aligned}
\Gamma_4 &= x^{n+1}(n(-s^3+x^2+rsx)(s^3-x^2+2rsx)(r^3x+s^3-x^2+3rsx) \\
&\quad -(s^3+x^2)(-2s^3x^2+s^6+x^4+7r^2s^2x^2-4rsx^3+4rs^4x+3r^4sx^2))W_{-n+1}^3 \\
&\quad +x^{n+1}(n(r^3x+s^3-x^2+3rsx)(-s^3+x^2+rsx) \\
&\quad (s^3x-r^3s^3+r^3x^2-x^3+2rsx^2+r^4sx+3r^2s^2x) \\
&\quad -s^3(2s^6x-r^3s^6-r^3x^4-4s^3x^3+2x^5+8r^2s^2x^3+8r^3s^3x^2 \\
&\quad +2r^5s^2x^2-8rsx^4+8rs^4x^2+6r^2s^5x+4r^4sx^3+2r^4s^4x))W_{-n}^3 \\
&\quad -3rsx^{n+1}(n(rx+s^2)(r^3x+s^3-x^2+3rsx)(-s^3+x^2+rsx) \\
&\quad +2rx^5-r^4x^4+3s^2x^4-2s^5x^2-s^8-5r^2s^4x^2-2r^3s^2x^3 \\
&\quad -2r^4s^3x^2-2rs^6x-2r^2sx^4-4rs^3x^3)W_{-n+1}^2W_{-n} \\
&\quad +3rs^2x^{n+1}(n(-x+rs)(r^3x+s^3-x^2+3rsx)(-s^3+x^2+rsx)+2s^6x+r^3x^4 \\
&\quad -2x^5-4r^2s^2x^3-2r^3s^3x^2-r^5s^2x^2-2r^4sx^3-rs^7+5rsx^4)W_{-n}^2W_{-n+1} \\
&\quad +x(s^3+x^2)(-2s^3x^2+s^6+x^4+7r^2s^2x^2-4rsx^3+4rs^4x+3r^4sx^2)W_1^3 \\
&\quad +s^3x(2s^6x-r^3s^6-r^3x^4-4s^3x^3+2x^5+8r^2s^2x^3+8r^3s^3x^2+2r^5s^2x^2 \\
&\quad -8rsx^4+8rs^4x^2+6r^2s^5x+4r^4sx^3+2r^4s^4x)W_0^3 \\
&\quad -3rsx(-2rx^5+r^4x^4-3s^2x^4+2s^5x^2+s^8+5r^2s^4x^2+2r^3s^2x^3+2r^4s^3x^2 \\
&\quad +2rs^6x+4rs^3x^3+2r^2sx^4)W_1^2W_0 \\
&\quad +3rs^2x(rs^7-2s^6x-r^3x^4+2x^5+4r^2s^2x^3+2r^3s^3x^2+r^5s^2x^2 \\
&\quad -5rsx^4+2r^4sx^3)W_0^2W_1.
\end{aligned}$$

(b)

$$\sum_{k=1}^{n} kx^kW_{-k+1}^2W_{-k} = \frac{\Gamma_5}{(-s^3+x^2+rsx)^2(r^3x+s^3-x^2+3rsx)^2}$$

where

$$\begin{aligned}
\Gamma_5 &= rx^{n+2}(n(rx+s^2)(s^3-x^2-rsx)(r^3x+s^3-x^2+3rsx)-2s^2x^4+2s^8 \\
&\quad +4r^2s^4x^2-2r^3s^2x^3+2r^4s^3x^2+r^3s^5x-r^5sx^3+5rs^6x-rx^5)W_{-n+1}^3 \\
&\quad +rs^3x^{n+1}(n(rx+s^2)(s^3-x^2-rsx)(r^3x+s^3-x^2+3rsx)-2rx^5+r^4x^4 \\
&\quad -3s^2x^4+2s^5x^2+s^8+5r^2s^4x^2+2r^3s^2x^3+2r^4s^3x^2+4rs^3x^3 \\
&\quad +2r^2sx^4+2rs^6x)W_{-n}^3 \\
&\quad +x^{n+2}(n(r^3x+s^3-x^2+3rsx)(-s^3+x^2+rsx)(r^3x+3r^2s^2+s^3-x^2)
\end{aligned}$$

$$+s(rx+s^2)(-6r^2s^5-2r^3x^3+4s^3x^2+r^6x^2-2s^6-2x^4-4r^3s^3x$$
$$-3r^5s^2x))W_{-n+1}^2W_{-n}$$
$$-s^2x^{n+1}(n(-s^3+x^2+rsx)(2r^3x-s^3+x^2)(r^3x+s^3-x^2+3rsx)$$
$$+(s^3+x^2)(4r^3x^3-2s^3x^2-2r^6x^2+s^6+x^4+3r^2s^2x^2-4r^3s^3x$$
$$-3r^4sx^2))W_{-n}^2W_{-n+1}$$
$$+rx^2(rx^5+2s^2x^4-2s^8-4r^2s^4x^2+2r^3s^2x^3-2r^4s^3x^2-5rs^6x$$
$$-r^3s^5x+r^5sx^3)W_1^3$$
$$-rs^3x(-2rx^5+r^4x^4-3s^2x^4+2s^5x^2+s^8+5r^2s^4x^2$$
$$+2r^3s^2x^3+2r^4s^3x^2+2rs^6x+4rs^3x^3+2r^2sx^4)W_0^3$$
$$+sx^2(rx+s^2)(6r^2s^5+2r^3x^3-4s^3x^2-r^6x^2+2s^6+2x^4$$
$$+4r^3s^3x+3r^5s^2x)W_1^2W_0$$
$$+s^2x(s^3+x^2)(4r^3x^3-2s^3x^2-2r^6x^2+s^6+x^4+3r^2s^2x^2$$
$$-4r^3s^3x-3r^4sx^2)W_0^2W_1.$$

(c)

$$\sum_{k=1}^{n}kx^kW_{-k}^2W_{-k+1}=\frac{\Gamma_6}{(-s^3+x^2+rsx)^2(r^3x+s^3-x^2+3rsx)^2}$$

where

$$\begin{aligned}\Gamma_6 = {} & rx^{n+2}(n(-x+rs)(-s^3+x^2+rsx)(r^3x+s^3-x^2+3rsx)-2rs^7+3s^6x\\ & -2s^3x^3-x^5-5r^2s^2x^3+2r^3s^3x^2+2rsx^4-2r^2s^5x-2r^4sx^3\\ & -r^4s^4x+4rs^4x^2)W_{-n+1}^3\\ & +rs^3x^{n+1}(n(-x+rs)(-s^3+x^2+rsx)(r^3x+s^3-x^2+3rsx)-rs^7+2s^6x\\ & +r^3x^4-2x^5-4r^2s^2x^3-2r^3s^3x^2-r^5s^2x^2-2r^4sx^3+5rsx^4)W_{-n}^3\\ & -sx^{n+1}(n(-s^3+x^2+rsx)(2r^3x-s^3+x^2)(r^3x+s^3-x^2+3rsx)\\ & +(s^3+x^2)(4r^3x^3-2s^3x^2-2r^6x^2+s^6+x^4+3r^2s^2x^2-4r^3s^3x-3r^4sx^2))\\ & W_{-n+1}^2W_{-n}+x^{n+1}(n(r^3x+s^3-x^2+3rsx)\\ & (-s^3+x^2+rsx)(-2rs^4+s^3x+r^3x^2-x^3+2rsx^2+r^4sx)\\ & +s^2(x-rs)(-2sx^4-6r^2x^4+3r^5x^3+4s^4x^2-2s^7+4r^3sx^3+2r^3s^4x+r^6sx^2))\\ & W_{-n}^2W_{-n+1}\end{aligned}$$

$$+rx^2(2rs^7-3s^6x+2s^3x^3+x^5+5r^2s^2x^3-2r^3s^3x^2-2rsx^4-4rs^4x^2$$
$$+2r^2s^5x+2r^4sx^3+r^4s^4x)W_1^3$$
$$+rs^3x(rs^7-2s^6x-r^3x^4+2x^5+4r^2s^2x^3+2r^3s^3x^2+r^5s^2x^2-5rsx^4$$
$$+2r^4sx^3)W_0^3$$
$$+sx(s^3+x^2)(4r^3x^3-2s^3x^2-2r^6x^2+s^6+x^4+3r^2s^2x^2-4r^3s^3x-3r^4sx^2)$$
$$W_1^2W_0+s^2x(-x+rs)(-2sx^4-6r^2x^4+3r^5x^3+4s^4x^2-2s^7+4r^3sx^3$$
$$+2r^3s^4x+r^6sx^2)W_0^2W_1.$$

Proof. Using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}\Rightarrow W_{-n}=-\frac{r}{s}W_{-n+1}+\frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n}=W_{-n+2}-rW_{-n+1}$$

we obtain

$$s^3W_{-n}^3=W_{-n+2}^3-3rW_{-n+2}^2W_{-n+1}+3r^2W_{-n+1}^2W_{-n+2}-r^3W_{-n+1}^3$$

and so

$$\begin{aligned}
s^3\times n\times x^nW_{-n}^3 &= nx^nW_{-n+2}^3-3r\times n\times x^nW_{-n+2}^2W_{-n+1}\\
&\quad +3r^2\times n\times x^nW_{-n+1}^2W_{-n+2}-r^3\times n\times x^nW_{-n+1}^3\\
s^3(n-1)x^{n-1}W_{-n+1}^3 &= (n-1)x^{n-1}W_{-n+3}^3-3r(n-1)x^{n-1}W_{-n+3}^2W_{-n+2}\\
&\quad +3r^2(n-1)x^{n-1}W_{-n+2}^2W_{-n+3}-r^3(n-1)x^{n-1}W_{-n+2}^3\\
s^3(n-2)x^{n-2}W_{-n+2}^3 &= (n-2)x^{n-2}W_{-n+4}^3-3r(n-2)x^{n-2}W_{-n+4}^2W_{-n+3}\\
&\quad +3r^2(n-2)x^{n-2}W_{-n+3}^2W_{-n+4}-r^3(n-2)x^{n-2}W_{-n+3}^3\\
&\vdots\\
s^3\times 3\times x^3W_{-3}^3 &= 3\times x^3W_{-1}^3-3r\times 3\times x^3W_{-1}^2W_{-2}\\
&\quad +3r^2\times 3\times x^3W_{-2}^2W_{-1}-r^3\times 3\times x^3W_{-2}^3\\
s^3\times 2\times x^2W_{-2}^3 &= 2\times x^2W_0^3-3r\times 2\times x^2W_0^2W_{-1}+3r^2\times 2\times x^2W_{-1}^2W_0\\
&\quad -r^3\times 2\times x^2W_{-1}^3\\
s^3\times 1\times x^1W_{-1}^3 &= 1\times x^1W_1^3-3r\times 1\times x^1W_1^2W_0+3r^2\times 1\times x^1W_0^2W_1\\
&\quad -r^3\times 1\times x^1W_0^3.
\end{aligned}$$

If we add the above equations by side by, we get

$$
\begin{aligned}
& s^3(\sum_{k=1}^{n} kx^k W_{-k}^3) \\
= \; & ((-n-1)x^{n+1}W_{-n+1}^3 + (-n-2)x^{n+2}W_{-n}^3 + x^1W_1^3 + 2\times x^2W_0^3 \\
& +x^2\sum_{k=1}^{n} kx^kW_{-k}^3 + 2x^2\sum_{k=1}^{n} x^kW_{-k}^3) - 3r((-n-1)x^{n+1}W_{-n+1}^2W_{-n} \\
& +x^1W_1^2W_0 + x^1\sum_{k=1}^{n} kx^kW_{-k+1}^2W_{-k} + x^1\sum_{k=1}^{n} x^kW_{-k+1}^2W_{-k}) \\
& +3r^2((-n-1)x^{n+1}W_{-n}^2W_{-n+1} + x^1W_0^2W_1 + x^1\sum_{k=1}^{n} kx^kW_{-k}^2W_{-k+1} \\
& +x^1\sum_{k=1}^{n} x^kW_{-k}^2W_{-k+1}) - r^3((-n-1)x^{n+1}W_{-n}^3 + x^1W_0^3 \\
& +x^1\sum_{k=1}^{n} kx^kW_{-k}^3 + x^1\sum_{k=1}^{n} x^kW_{-k}^3). \qquad (3.1)
\end{aligned}
$$

Next we calculate $\sum_{k=1}^{n} kx^kW_{-k+1}^2W_{-k}$. Again using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$sW_{-n+1}^2W_{-n} = W_{-n+1}^2W_{-n+2} - rW_{-n+1}^3$$

and so

$$
\begin{aligned}
s\times n\times x^nW_{-n+1}^2W_{-n} &= nx^nW_{-n+1}^2W_{-n+2} - r\times n\times x^nW_{-n+1}^3 \\
s(n-1)x^{n-1}W_{-n+2}^2W_{-n+1} &= (n-1)x^{n-1}W_{-n+2}^2W_{-n+3} - r(n-1)x^{n-1}W_{-n+2}^3 \\
s(n-2)x^{n-2}W_{-n+3}^2W_{-n+2} &= (n-2)x^{n-2}W_{-n+3}^2W_{-n+4} - r(n-2)x^{n-2}W_{-n+3}^3 \\
&\vdots \\
s\times 3\times x^3W_{-2}^2W_{-3} &= 3x^3W_{-2}^2W_{-1} - r\times 3\times x^3W_{-2}^3 \\
s\times 2\times x^2W_{-1}^2W_{-2} &= 2\times x^2W_{-1}^2W_0 - r\times 2\times x^2W_{-1}^3 \\
s\times 1\times x^1W_0^2W_{-1} &= 1\times x^1W_0^2W_1 - r\times 1\times x^1W_0^3.
\end{aligned}
$$

If we add the above equations by side by, we get

$$\begin{aligned} & s\sum_{k=1}^{n}kx^{k}W_{-k+1}^{2}W_{-k} \\ = \ & (-(n+1)x^{n+1}W_{-n}^{2}W_{-n+1}+x^{1}W_{0}^{2}W_{1}+x^{1}\sum_{k=1}^{n}kx^{k}W_{-k}^{2}W_{-k+1} \\ & +x^{1}\sum_{k=1}^{n}x^{k}W_{-k}^{2}W_{-k+1})-r(-(n+1)x^{n+1}W_{-n}^{3}+x^{1}W_{0}^{3} \\ & +x^{1}\sum_{k=1}^{n}kx^{k}W_{-k}^{3}+x^{1}\sum_{k=1}^{n}x^{k}W_{-k}^{3}). \end{aligned} \tag{3.2}$$

Next we calculate $\sum_{k=1}^{n}x^{k}W_{-k+1}^{2}W_{-k}$. Again using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$sW_{-n}=W_{-n+2}-rW_{-n+1}$$

we obtain

$$\begin{aligned} s^{2}W_{-n}^{2} &= W_{-n+2}^{2}-2rW_{-n+2}W_{-n+1}+r^{2}W_{-n+1}^{2} \\ &\Rightarrow s^{2}W_{-n}^{2}W_{-n+1}=W_{-n+2}^{2}W_{-n+1}-2rW_{-n+1}^{2}W_{-n+2}+r^{2}W_{-n+1}^{3} \end{aligned}$$

and so

$$\begin{aligned} s^{2}\times n\times x^{n}W_{-n}^{2}W_{-n+1} &= nx^{n}W_{-n+2}^{2}W_{-n+1}-2r\times n\times x^{n}W_{-n+1}^{2}W_{-n+2} \\ &\quad +r^{2}\times n\times x^{n}W_{-n+1}^{3} \\ s^{2}(n-1)x^{n-1}W_{-n+1}^{2}W_{-n+2} &= (n-1)x^{n-1}W_{-n+3}^{2}W_{-n+2}-2r(n-1)x^{n-1}W_{-n+2}^{2}W_{-n+3} \\ &\quad +r^{2}(n-1)x^{n-1}W_{-n+2}^{3} \\ s^{2}(n-2)x^{n-2}W_{-n+2}^{2}W_{-n+3} &= (n-2)x^{n-2}W_{-n+4}^{2}W_{-n+3}-2r(n-2)x^{n-2}W_{-n+3}^{2}W_{-n+4} \\ &\quad +r^{2}(n-2)x^{n-2}W_{-n+3}^{3} \\ &\ \ \vdots \\ s^{2}\times 3\times x^{3}W_{-3}^{2}W_{-2} &= 3\times x^{3}W_{-1}^{2}W_{-2}-2r\times 3\times x^{3}W_{-2}^{2}W_{-1}+r^{2}\times 3\times x^{3}W_{-2}^{3} \\ s^{2}\times 2\times x^{2}W_{-2}^{2}W_{-1} &= 2\times x^{2}W_{0}^{2}W_{-1}-2r\times 2\times x^{2}W_{-1}^{2}W_{0}+r^{2}\times 2\times x^{2}W_{-1}^{3} \\ s^{2}\times 1\times x^{1}W_{-1}^{2}W_{0} &= 1\times x^{1}W_{1}^{2}W_{0}-2r\times 1\times x^{1}W_{0}^{2}W_{1}+r^{2}\times 1\times x^{1}W_{0}^{3}. \end{aligned}$$

If we add the above equations by side by, we get

$$
\begin{aligned}
& s^2 \sum_{k=1}^{n} kx^k W_{-k}^2 W_{-k+1} \\
= \quad & (-(n+1)x^{n+1} W_{-n+1}^2 W_{-n} + x^1 W_1^2 W_0 + x^1 \sum_{k=1}^{n} kx^k W_{-k+1}^2 W_{-k} \\
& + x^1 \sum_{k=1}^{n} x^k W_{-k+1}^2 W_{-k}) - 2r(-(n+1)x^{n+1} W_{-n}^2 W_{-n+1} + x^1 W_0^2 W_1 \\
& + x^1 \sum_{k=1}^{n} kx^k W_{-k}^2 W_{-k+1} + x^1 \sum_{k=1}^{n} x^k W_{-k}^2 W_{-k+1}) \\
& + r^2(-(n+1)x^{n+1} W_{-n}^3 \\
& + x^1 W_0^3 + x^1 \sum_{k=1}^{n} kx^k W_{-k}^3 + x^1 \sum_{k=1}^{n} x^k W_{-k}^3). \qquad (3.3)
\end{aligned}
$$

Then, using Theorem 1.2 and solving the system (3.1)-(3.2)-(3.3), the required results of (a), (b) and (c) follow.

3.1 The case $x = 1$

See [22] for the case $x = 1$.

3.2 The case $x = -1$

In this subsection we consider the special case $x = -1$.

Taking $x = -1, r = s = 1$ in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.2. *If $x = -1, r = s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} k(-1)^k W_{-k}^3 = \frac{1}{4}((-1)^n ((2n+5) W_{-n+1}^3 + (2n-3) W_{-n}^3 - 3W_{-n+1}^2 W_{-n} - 3(2n+2) W_{-n}^2 W_{-n+1}) - 5W_1^3 + 3W_0^3 + 3W_1^2 W_0 + 6W_0^2 W_1)$.

(b) $\sum_{k=1}^{n} k(-1)^k W_{-k+1}^2 W_{-k} = \frac{1}{4}((-1)^n (W_{-n+1}^3 - W_{-n}^3 + 2nW_{-n+1}^2 W_{-n} - (2n+1) W_{-n}^2 W_{-n+1}) - W_1^3 + W_0^3 + W_0^2 W_1)$.

(c) $\sum_{k=1}^{n} k(-1)^k W_{-k}^2 W_{-k+1} = \frac{1}{4}((-1)^n ((2n+4) W_{-n+1}^3 - (2n+2) W_{-n}^3 - (2n+1) W_{-n+1}^2 W_{-n} - 7W_{-n}^2 W_{-n+1}) - 4W_1^3 + 2W_0^3 + W_1^2 W_0 + 7W_0^2 W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.3. *For $n \geq 1$, Fibonacci numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(-1)^k F_{-k}^3 = \frac{1}{4}((-1)^n((2n+5)F_{-n+1}^3 + (2n-3)F_{-n}^3 - 3F_{-n+1}^2 F_{-n} - 3(2n+2)F_{-n}^2 F_{-n+1}) - 5)$.

(b) $\sum_{k=1}^{n} k(-1)^k F_{-k+1}^2 F_{-k} = \frac{1}{4}((-1)^n(F_{-n+1}^3 - F_{-n}^3 + 2nF_{-n+1}^2 F_{-n} - (2n+1) F_{-n}^2 F_{-n+1}) - 1)$.

(c) $\sum_{k=1}^{n} k(-1)^k F_{-k}^2 F_{-k+1} = \frac{1}{4}((-1)^n((2n+4)F_{-n+1}^3 - (2n+2)F_{-n}^3 - (2n+1) F_{-n+1}^2 F_{-n} - 7F_{-n}^2 F_{-n+1}) - 4)$.

Taking $F_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.4. *For $n \geq 1$, Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(-1)^k L_{-k}^3 = \frac{1}{4}((-1)^n((2n+5)L_{-n+1}^3 + (2n-3)L_{-n}^3 - 3L_{-n+1}^2 L_{-n} - 3(2n+2)L_{-n}^2 L_{-n+1}) + 49)$.

(b) $\sum_{k=1}^{n} k(-1)^k L_{-k+1}^2 L_{-k} = \frac{1}{4}((-1)^n(L_{-n+1}^3 - L_{-n}^3 + 2nL_{-n+1}^2 L_{-n} - (2n+1) L_{-n}^2 L_{-n+1}) + 11)$.

(c) $\sum_{k=1}^{n} k(-1)^k L_{-k}^2 L_{-k+1} = \frac{1}{4}((-1)^n((2n+4)L_{-n+1}^3 - (2n+2)L_{-n}^3 - (2n+1) L_{-n+1}^2 L_{-n} - 7L_{-n}^2 L_{-n+1}) + 42)$.

Taking $x = -1, r = 2, s = 1$ in Theorem 3.1 (a),(b) and (c), we obtain the following proposition.

Proposition 3.5. *If $x = -1, r = 2, s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} k(-1)^k W_{-k}^3 = \frac{1}{98}((-1)^n((14n+19)W_{-n+1}^3 + (84n-5)W_{-n}^3 - 3(7n+13) W_{-n+1}^2 W_{-n} - 3(21n+4)W_{-n}^2 W_{-n+1}) - 19W_1^3 + 5W_0^3 + 39W_1^2 W_0 + 12W_0^2 W_1)$.

(b) $\sum_{k=1}^{n} k(-1)^k W_{-k+1}^2 W_{-k} = \frac{1}{98}((-1)^n((7n+20)W_{-n+1}^3 - (7n+13)W_{-n}^3 + (14n-23)W_{-n+1}^2 W_{-n} - (56n+41)W_{-n}^2 W_{-n+1}) - 20W_1^3 + 13W_0^3 + 23W_1^2 W_0 + 41W_0^2 W_1)$.

(c) $\sum_{k=1}^{n} k(-1)^k W_{-k}^2 W_{-k+1} = \frac{1}{98}((-1)^n((21n+25)W_{-n+1}^3 - (21n+4)W_{-n}^3 - (56n+41)W_{-n+1}^2 W_{-n} + (28n-39)W_{-n}^2 W_{-n+1}) - 25W_1^3 + 4W_0^3 + 41W_1^2 W_0 + 39W_0^2 W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 3.6. *For $n \geq 1$, Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(-1)^k P_{-k}^3 = \frac{1}{98}((-1)^n ((14n+19) P_{-n+1}^3 + (84n-5) P_{-n}^3 - 3(7n+13) P_{-n+1}^2 P_{-n} - 3(21n+4) P_{-n}^2 P_{-n+1}) - 19)$.

(b) $\sum_{k=1}^{n} k(-1)^k P_{-k+1}^2 P_{-k} = \frac{1}{98}((-1)^n ((7n+20) P_{-n+1}^3 - (7n+13) P_{-n}^3 + (14n-23) P_{-n+1}^2 P_{-n} - (56n+41) P_{-n}^2 P_{-n+1}) - 20)$.

(c) $\sum_{k=1}^{n} k(-1)^k P_{-k}^2 P_{-k+1} = \frac{1}{98}((-1)^n ((21n+25) P_{-n+1}^3 - (21n+4) P_{-n}^3 - (56n+41) P_{-n+1}^2 P_{-n} + (28n-39) P_{-n}^2 P_{-n+1}) - 25)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.7. *For $n \geq 1$, Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(-1)^k Q_{-k}^3 = \frac{1}{98}((-1)^n ((14n+19) Q_{-n+1}^3 + (84n-5) Q_{-n}^3 - 3(7n+13) Q_{-n+1}^2 Q_{-n} - 3(21n+4) Q_{-n}^2 Q_{-n+1}) + 296)$.

(b) $\sum_{k=1}^{n} k(-1)^k Q_{-k+1}^2 Q_{-k} = \frac{1}{98}((-1)^n ((7n+20) Q_{-n+1}^3 - (7n+13) Q_{-n}^3 + (14n-23) Q_{-n+1}^2 Q_{-n} - (56n+41) Q_{-n}^2 Q_{-n+1}) + 456)$.

(c) $\sum_{k=1}^{n} k(-1)^k Q_{-k}^2 Q_{-k+1} = \frac{1}{98}((-1)^n ((21n+25) Q_{-n+1}^3 - (21n+4) Q_{-n}^3 - (56n+41) Q_{-n+1}^2 Q_{-n} + (28n-39) Q_{-n}^2 Q_{-n+1}) + 472)$.

If $x = -1, r = 1, s = 2$, then $(-s^3 + x^2 + rsx)^2 (r^3 x + s^3 - x^2 + 3rsx)^2 = 0$ so we cannot use Theorem 3.1 directly. But we can give another method to find $\sum_{k=1}^{n} k(-1)^k W_{-k}^3, \sum_{k=1}^{n} k(-1)^k W_{-k+1}^2 W_{-k}$ and $\sum_{k=1}^{n} k(-1)^k W_{-k}^2 W_{-k+1}$ by using Theorem 3.1.

Theorem 3.8. *If $x = -1, r = 1, s = 2$, then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} k(-1)^k W_{-k}^3 = \frac{1}{486}((-1)^n ((-9n^2 + 29n + 38) W_{-n+1}^3 + (72n^2 + 110n) W_{-n}^3 + 2(27n^2 + 3n - 16) W_{-n} W_{-n+1}^2 - 4(27n^2 + 39n + 16) W_{-n}^2 W_{-n+1}) - 38 W_1^3 + 32 W_1^2 W_0 + 64 W_0^2 W_1)$.

(b) $\sum_{k=1}^{n} k(-1)^k W_{-k+1}^2 W_{-k} = \frac{1}{1458}((-1)^n (-(27n^2 + 57n + 14) W_{-n+1}^3 + 8(27n^2 + 3n - 16) W_{-n}^3 + (162n^2 + 396n + 168) W_{-n+1}^2 W_{-n} - 4(81n^2 + 63n - 18) W_{-n}^2 W_{-n+1}) + 14 W_1^3 + 128 W_0^3 - 168 W_1^2 W_0 - 72 W_0^2 W_1)$.

(c) $\sum_{k=1}^{n} k(-1)^k W_{-k}^2 W_{-k+1} = \frac{1}{1458}((-1)^n ((27n^2+93n+82)W_{-n+1}^3 - 8(27n^2+39n+16)W_{-n}^3 - 2(81n^2+63n-18)W_{-n+1}^2 W_{-n} + (324n^2+198n-192)W_{-n}^2 W_{-n+1}) - 82 W_1^3 + 128W_0^3 - 36W_1^2 W_0 + 192W_0^2 W_1)$.

Proof.

(a) We use Theorem 3.1 (a). If we set $r=1, s=2$ in Theorem 3.1 (a), then we have

$$\sum_{k=1}^{n} kx^k W_{-k}^3 = \frac{f_4(x)}{(-x^2+7x+8)^2(x^2+2x-8)^2}$$

where

$$\begin{aligned} f_4(x) &= -((x^2+8)(x^4-8x^3+18x^2+64x+64) \\ &\quad -n(-x^2+4x+8)(-x^2+7x+8)(x^2+2x-8))x^{n+1}W_{-n+1}^3 \\ &\quad -(2816x+1600x^2+64x^3-136x^4+16x^5-512 \\ &\quad -n(-x^2+7x+8)(x^2+2x-8)(-x^3+5x^2+22x-8))x^{n+1}W_{-n}^3 \\ &\quad +6(128x+160x^2+40x^3-7x^4-2x^5+256-n(x+4)(-x^2 \\ &\quad +7x+8)(x^2+2x-8))x^{n+1}W_{-n+1}^2 W_{-n} \\ &\quad -12(20x^2-128x+20x^3-11x^4+2x^5+128+n(x-2)(-x^2 \\ &\quad +7x+8)(x^2+2x-8))x^{n+1}W_{-n}^2 W_{-n+1} \\ &\quad +x(x^2+8)(x^4-8x^3+18x^2+64x+64)W_1^3 \\ &\quad +8x(2x^5-17x^4+8x^3+200x^2+352x-64)W_0^3 \\ &\quad -6x(-2x^5-7x^4+40x^3+160x^2+128x+256)W_1^2 W_0 \\ &\quad +12x(2x^5-11x^4+20x^3+20x^2-128x+128)W_0^2 W_1. \end{aligned}$$

For $x=-1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule (by applying twice). Then we get

$$\begin{aligned} \sum_{k=1}^{n} k(-1)^k W_{-k}^3 &= \left. \frac{\frac{d}{dx}(f_4(x))}{\frac{d}{dx}((-x^2+7x+8)^2(x^2+2x-8)^2)} \right|_{x=-1} \\ &= \frac{1}{486}((-1)^n ((-9n^2+29n+38)W_{-n+1}^3 + (72n^2+110n)W_{-n}^3 \\ &\quad +2(27n^2+3n-16)W_{-n}W_{-n+1}^2 \\ &\quad -4(27n^2+39n+16)W_{-n}^2 W_{-n+1}) \\ &\quad -38W_1^3 + 32W_1^2 W_0 + 64W_0^2 W_1). \end{aligned}$$

(b) We use Theorem 3.1 (b). If we set $r = 1, s = 2$ in Theorem 3.1 (b) then we have

$$\sum_{k=1}^{n} k(-1)^k W_{-k+1}^2 W_{-k} = \frac{f_5(x)}{(-x^2+7x+8)^2(x^2+2x-8)^2}$$

where

$$\begin{aligned} f_5(x) &= -(10x^3 - 80x^2 - 352x + 8x^4 + x^5 - 512 \\ &\quad +n(x+4)(-x^2+7x+8)(x^2+2x-8))x^{n+2}W_{-n+1}^3 \\ &\quad +8(128x + 160x^2 + 40x^3 - 7x^4 - 2x^5 + 256 \\ &\quad -n(x+4)(-x^2+7x+8)(x^2+2x-8))x^{n+1}W_{-n}^3 \\ &\quad -(2(x+4)(2x^4+2x^3-33x^2+44x+320) \\ &\quad -n(-x^2+7x+8)(x^2+2x-8)(-x^2+x+20))x^{n+2}W_{-n+1}^2 W_{-n} \\ &\quad -4((x^2+8)(x^4+4x^3-12x^2-32x+64) \\ &\quad +n(-x^2+7x+8)(x^2+2x-8)^2)x^{n+1}W_{-n}^2 W_{-n+1} \\ &\quad +(x^5+8x^4+10x^3-80x^2-352x-512)x^2 W_1^3 \\ &\quad -8x(-2x^5-7x^4+40x^3+160x^2+128x+256)W_0^3 \\ &\quad +2x^2(x+4)(2x^4+2x^3-33x^2+44x+320)W_1^2 W_0 \\ &\quad +4x(x^2+8)(x^4+4x^3-12x^2-32x+64)W_0^2 W_1. \end{aligned}$$

For $x = -1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule (by applying twice). Then we get

$$\begin{aligned} &\sum_{k=1}^{n} k(-1)^k W_{-k+1}^2 W_{-k} \\ &= \left. \frac{\frac{d}{dx}(f_5(x))}{\frac{d}{dx}((-x^2+7x+8)^2(x^2+2x-8)^2)} \right|_{x=-1} \\ &= \frac{1}{1458}((-1)^n(-(27n^2+57n+14)W_{-n+1}^3 \\ &\quad +8(27n^2+3n-16)W_{-n}^3 + (162n^2+396n+168)W_{-n+1}^2 W_{-n} \\ &\quad -4(81n^2+63n-18)W_{-n}^2 W_{-n+1}) \\ &\quad +14W_1^3 + 128W_0^3 - 168W_1^2 W_0 - 72W_0^2 W_1). \end{aligned}$$

(c) We use Theorem 3.1 (c). If we set $r = 1, s = 2$ in Theorem 3.1 (c) then we have

$$\sum_{k=1}^{n} kx^k W_{-k}^2 W_{-k+1} = \frac{f_6(x)}{(-x^2+7x+8)^2(x^2+2x-8)^2}$$

where

$$\begin{aligned} f_6(x) &= -(40x^3-80x^2-112x-4x^4+x^5+256 \\ &+n(x-2)(-x^2+7x+8)(x^2+2x-8))x^{n+2}W_{-n+1}^3 \\ &-8(20x^2-128x+20x^3-11x^4+2x^5+128 \\ &+n(x-2)(-x^2+7x+8)(x^2+2x-8))x^{n+1}W_{-n}^3 \\ &-2((x^2+8)(x^4+4x^3-12x^2-32x+64) \\ &+n(-x^2+7x+8)(x^2+2x-8)^2)x^{n+1}W_{-n+1}^2W_{-n} \\ &+(4(x-2)(-10x^4+11x^3+66x^2+32x-256) \\ &+n(-x^2+7x+8)(x^2+2x-8)(-x^3 \\ &+5x^2+10x-32))x^{n+1}W_{-n}^2W_{-n+1} \\ &-x^2(-x^5+4x^4-40x^3+80x^2+112x-256)W_1^3 \\ &+8x(2x^5-11x^4+20x^3+20x^2-128x+128)W_0^3 \\ &+(x^2+8)(x^4+4x^3-12x^2-32x+64)2xW_1^2W_0 \\ &-4x(x-2)(-10x^4+11x^3+66x^2+32x-256)W_0^2W_1. \end{aligned}$$

For $x=-1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule (by applying twice). Then we get

$$\begin{aligned} \sum_{k=1}^{n} k(-1)^k W_{-k}^2 W_{-k+1} &= \left.\frac{\frac{d}{dx}(f_6(x))}{\frac{d}{dx}((-x^2+7x+8)^2(x^2+2x-8)^2)}\right|_{x=-1} \\ &= \frac{1}{1458}((-1)^n\,((27n^2+93n+82)W_{-n+1}^3 \\ &-8(27n^2+39n+16)W_{-n}^3 \\ &-2(81n^2+63n-18)W_{-n+1}^2W_{-n} \\ &+(324n^2+198n-192)W_{-n}^2W_{-n+1}) \\ &-82W_1^3+128W_0^3-36W_1^2W_0+192W_0^2W_1). \end{aligned}$$

From the last theorem, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n=J_n$ with $J_0=0, J_1=1$).

Corollary 3.9. *For $n\geq 1$, Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^k J_{-k}^3 = \frac{1}{486}((-1)^n\,((-9n^2+29n+38)J_{-n+1}^3+(72n^2+110n)J_{-n}^3+2(27n^2+3n-16)J_{-n}J_{-n+1}^2-4(27n^2+39n+16)J_{-n}^2J_{-n+1})-38)$.

(b) $\sum_{k=1}^{n}k(-1)^{k}J_{-k+1}^{2}J_{-k} = \frac{1}{1458}((-1)^{n}(-(27n^{2}+57n+14)J_{-n+1}^{3}+8(27n^{2}+3n-16)J_{-n}^{3}+(162n^{2}+396n+168)J_{-n+1}^{2}J_{-n}-4(81n^{2}+63n-18)J_{-n}^{2}J_{-n+1})+14)$.

(c) $\sum_{k=1}^{n}k(-1)^{k}J_{-k}^{2}J_{-k+1} = \frac{1}{1458}((-1)^{n}((27n^{2}+93n+82)J_{-n+1}^{3}-8(27n^{2}+39n+16)J_{-n}^{3}-2(81n^{2}+63n-18)J_{-n+1}^{2}J_{-n}+(324n^{2}+198n-192)J_{-n}^{2}J_{-n+1})-82)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.10. *For $n \geq 1$, Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n}k(-1)^{k}j_{-k}^{3} = \frac{1}{486}((-1)^{n}((-9n^{2}+29n+38)j_{-n+1}^{3}+(72n^{2}+110n)j_{-n}^{3}+2(27n^{2}+3n-16)j_{-n}j_{-n+1}^{2}-4(27n^{2}+39n+16)j_{-n}^{2}j_{-n+1})+282)$.

(b) $\sum_{k=1}^{n}k(-1)^{k}j_{-k+1}^{2}j_{-k} = \frac{1}{1458}((-1)^{n}(-(27n^{2}+57n+14)j_{-n+1}^{3}+8(27n^{2}+3n-16)j_{-n}^{3}+(162n^{2}+396n+168)j_{-n+1}^{2}j_{-n}-4(81n^{2}+63n-18)j_{-n}^{2}j_{-n+1})+414)$.

(c) $\sum_{k=1}^{n}k(-1)^{k}j_{-k}^{2}j_{-k+1} = \frac{1}{1458}((-1)^{n}((27n^{2}+93n+82)j_{-n+1}^{3}-8(27n^{2}+39n+16)j_{-n}^{3}-2(81n^{2}+63n-18)j_{-n+1}^{2}j_{-n}+(324n^{2}+198n-192)j_{-n}^{2}j_{-n+1})+1638)$.

3.3 The case $x = 1 + i$

In this subsection we consider the special case $x = 1 + i$.

Taking $x = 1 + i, r = s = 1$ in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.11. *If $x = 1+i, r = s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}k(1+i)^{k}W_{-k}^{3} = \frac{1}{-63-60i}(-(1+i)^{n+1}(((6-15i)n-5+10i)W_{-n+1}^{3}+((57-12i)n+7+18i)W_{-n}^{3}-((27-24i)n-1+40i)W_{-n+1}^{2}W_{-n}-((15+6i)n-3-8i)W_{-n}^{2}W_{-n+1})-(15-5i)W_{1}^{3}-(11-25i)W_{0}^{3}+(41-39i)W_{1}^{2}W_{0}-(5-11i)W_{0}^{2}W_{1})$.

(b) $\sum_{k=1}^{n}k(1+i)^{k}W_{-k+1}^{2}W_{-k} = \frac{1}{-189-180i}((1+i)^{n+1}((1+i)((27-24i)n+26+16i)W_{-n+1}^{3}+((27-24i)n-1+40i)W_{-n}^{3}-(1+i)((15-81i)n+5+5i)W_{-n+1}^{2}W_{-n}+((66+9i)n+7+34i)W_{-n}^{2}W_{-n+1})+(32-52i)W_{1}^{3}+(41-39i)W_{0}^{3}-(10-10i)W_{1}^{2}W_{0}+(27-41i)W_{0}^{2}W_{1})$.

(c) $\sum_{k=1}^{n}k(1+i)^{k}W_{-k}^{2}W_{-k+1} = \frac{1}{-189-180i}((1+i)^{n+1}((1+i)((15+6i)n+12-2i)W_{-n+1}^{3}+((15+6i)n-3-8i)W_{-n}^{3}+((66+9i)n+7+34i)W_{-n+1}^{2}W_{-n}-((102+6i)n+26-18i)W_{-n}^{2}W_{-n+1})-(4+24i)W_{1}^{3}-(5-11i)W_{0}^{3}+(27-41i)W_{1}^{2}W_{0}+(44+8i)W_{0}^{2}W_{1})$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.12. *For $n \geq 1$, Fibonacci numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(1+i)^k F_{-k}^3 = \frac{1}{-63-60i}(-(1+i)^{n+1}(((6-15i)n-5+10i)F_{-n+1}^3 + ((57-12i)n+7+18i)F_{-n}^3 - ((27-24i)n-1+40i)F_{-n+1}^2 F_{-n} - ((15+6i)n-3-8i)F_{-n}^2 F_{-n+1}) - 15+5i)$.

(b) $\sum_{k=1}^{n} k(1+i)^k F_{-k+1}^2 F_{-k} = \frac{1}{-189-180i}((1+i)^{n+1}((1+i)((27-24i)n+26+16i) F_{-n+1}^3 + ((27-24i)n-1+40i)F_{-n}^3 - (1+i)((15-81i)n+5+5i)F_{-n+1}^2 F_{-n} + ((66+9i)n+7+34i)F_{-n}^2 F_{-n+1}) + 32-52i)$.

(c) $\sum_{k=1}^{n} k(1+i)^k F_{-k}^2 F_{-k+1} = \frac{1}{-189-180i}((1+i)^{n+1}((1+i)((15+6i)n+12-2i) F_{-n+1}^3 + ((15+6i)n-3-8i)F_{-n}^3 + ((66+9i)n+7+34i)F_{-n+1}^2 F_{-n} - ((102+6i)n+26-18i)F_{-n}^2 F_{-n+1}) - 4-24i)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.13. *For $n \geq 1$, Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(1+i)^k L_{-k}^3 = \frac{1}{-63-60i}(-(1+i)^{n+1}(((6-15i)n-5+10i)L_{-n+1}^3 + ((57-12i)n+7+18i)L_{-n}^3 - ((27-24i)n-1+40i)L_{-n+1}^2 L_{-n} - ((15+6i)n-3-8i)L_{-n}^2 L_{-n+1}) - 41+171i)$.

(b) $\sum_{k=1}^{n} k(1+i)^k L_{-k+1}^2 L_{-k} = \frac{1}{-189-180i}((1+i)^{n+1}((1+i)((27-24i)n+26+16i) L_{-n+1}^3 + ((27-24i)n-1+40i)L_{-n}^3 - (1+i)((15-81i)n+5+5i)L_{-n+1}^2 L_{-n} + ((66+9i)n+7+34i)L_{-n}^2 L_{-n+1}) + 448-508i)$.

(c) $\sum_{k=1}^{n} k(1+i)^k L_{-k}^2 L_{-k+1} = \frac{1}{-189-180i}((1+i)^{n+1}((1+i)((15+6i)n+12-2i) L_{-n+1}^3 + ((15+6i)n-3-8i)L_{-n}^3 + ((66+9i)n+7+34i)L_{-n+1}^2 L_{-n} - ((102+6i)n+26-18i)L_{-n}^2 L_{-n+1}) + 186+14i)$.

Corresponding sums of the other second order linear sequences can be calculated similarly when $x = 1+i$.

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Yüksel Soykan
Department of Mathematics, Art and Science Faculty,
Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey
e-mail: yuksel_soykan@hotmail.com