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On Artin Cokernel of the Quaternion Group Q_{2m} when $m = 2^h \cdot p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_n^{r_n}$, such that p_i are Primes, $g.c.d(p_i, p_j) = 1$ and $p_i \neq 2$ for all $i = 1, 2, \ldots, n$, h and r_i any Positive Integer Numbers

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Abstract

In this article, we find the cyclic decomposition of the finite abelian factor group $AC(G) = \overline{R}(G)/T(G)$, where $G = Q_{2m}$ and m is an even number and Q_{2m} is the quaternion group of order $4m$.

(The group of all Z-valued generalized characters of G over the group of induced unit characters from all cyclic subgroups of G).

We find that the cyclic decomposition $AC(Q_{2m})$ depends on the elementary divisor of m.

We have found that if $m = p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_n^{r_n} \cdot 2^h$, p_i are distinct primes, then:

$$AC(Q_{2m}) = \bigoplus_{i=1}^{(r_1+1)(r_2+1)\ldots(r_n+1)(h+2)-1} C_2.$$

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Moreover, we have also found the general form of Artin characters table $Ar(Q_{2m})$ when m is an even number.

1. Introduction

Representation theory is a branch of mathematics that studies abstract algebra structures by representing their elements as linear transformations of vector spaces. So that representation theory is a powerful tool because it reduces problems in abstract algebra to problems in a linear algebra which is a very well understood theory. Moreover, representation and characters theory provide applications, not only in other branches of mathematics but also in physics and chemistry.

For a finite group G, the factor group $\overline{R}(G)/T(G)$ is called the Artin cokernel of G denoted $AC(G)$, $\overline{R}(G)$ denotes the abelian group generated by Z-valued characters of G under the operation of pointwise addition, $T(G)$ is a subgroup of $\overline{R}(G)$ which is generated by Artin characters.

A well-known theorem which is due to Artin asserted that $T(G)$ has a finite index in $\overline{R}(G)$, i.e., $[\overline{R}(G):T(G)]$ is finite so $AC(G)$ is a finite abelian group.

The exponent of $AC(G)$ is called Artin exponent of G denoted by $A(G)$.

In 1968, Lam [10] proved a sharp form of Artin theorem and he determined the least positive integer $A(G)$ such that $[\overline{R}(G):T(G)] = A(G)$.

In 1976, David [4] studies $A(G)$ of arbitrary characters of cyclic subgroups.

In 1995, Mahmood [8] studied the cyclic decomposition of the factor group $\text{cf}(Q_{2m}, Z)/\overline{R}(Q_{2m})$ and he found the rational valued characters table of the quaternion group Q_{2m}.

In 1996, Knwabuez [6] studied $A(G)$ of p-groups. In 2000, Yassein [5] found $AC(G)$ for the group $\bigoplus_{i=1}^{n} Z_p$. In 2001 Ibraheem [3] studied $A(G)$ of alternating group.

Proposition 1.1 [9]. *If p is a prime number and s is a positive integer, then*

$$M(C_{p^s}) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

which is of order $(s+1)\times(s+1)$.

Example 1.2. Consider the matrix $M(C_{64})$, we can find it by Proposition 1.1

$$M(C_{64}) = M(C_{2^6}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is 7×7 square matrix.

Lemma 1.3 [7]**.** *Let A and B be two non-singular matrices of the ranks n and m respectively, over a principal domain R and let*:

$$P_1 \cdot A \cdot W_1 = D(A) = diag\{d_1(A), d_2(A), ..., d_n(A)\}$$

and

$$P_2 \cdot B \cdot W_3 = D(B) = diag\{d_1(B), d_2(B), ..., d_m(B)\}$$

be the invariant factor matrices of A and B. Then:

$$(P_1 \otimes P_2)\cdot(A\otimes B)\cdot(W_1\otimes W_2) = D(A)\otimes D(B)$$

and from this we get the invariant factor matrices of $A\otimes B$.

Proposition 1.4 [9]**.** *The general form of the matrices* $P(C_{p^s})$ *and* $W(C_{p^s})$ *are*:

$$P(C_{p^s}) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

which is $(s+1) \times (s+1)$ *square matrix.*

$W(C_{p^s}) = I_{s+1}$, *where* I_{s+1} *is an identity matrix and* $D(C_{p^s}) = diag\underbrace{\{1, 1, 1, ..., 1\}}_{S+1}$.

Remarks 1.5 [2]**.**

(1) If $m = 2^h$, h is any positive integer, then we can write $M(C_m)$ as follows:

$$M(C_m) = \begin{bmatrix} & & & & 1 & 1 \\ & & & & 1 & 1 \\ & & R_1(C_m) & & \vdots & \vdots \\ & & & & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

which is $(h+1) \times (h+1)$ square matrix, $R_1(C_m)$ is the matrix obtained by omitting the last two rows $\{0, 0, ..., 1, 1\}$ and $\{0, 0, ..., 0, 0, 1\}$ and the last two columns $\{1, 1, ..., 1, 0\}$ and $\{1, 1, ..., 1, 1\}$ from the matrix $M(C_{2^h})$ in Proposition 1.1.

(2) In general, if $m = 2^h \cdot p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_n^{r_n}$ such that $p_i, i = 1, 2, ..., n$ are prime numbers $p_i \neq 2$ and $\gcd(p_i, p_j) = 1$, h and r_i are any positive integer numbers for all $i = 1, 2, ..., n$, then we can write C_m in the form:

$$C_m = C_{2^h} \times C_{p_1^{r_1}} \times C_{p_2^{r_2}} \times \cdots \times C_{p_n^{r_n}}.$$

(i) By proposition, we get

$$M(C_m) = M(C_{2^h}) \otimes M(C_{p_1^{r_1}}) \otimes M(C_{p_2^{r_2}}) \otimes \cdots \otimes M(C_{p_n^{r_n}}).$$

We can write $M(C_m)$ in the form:

$$M(C_m)=\begin{bmatrix} & & & h\ times\left\{\begin{matrix}1\\ \vdots\\ 1\end{matrix}\right. & \begin{matrix}1\\ 1\\ \end{matrix}\\ & R_2(C_m) & & 0 & \vdots\\ & & & h\ times\left\{\begin{matrix}1\\ \vdots\\ 1\end{matrix}\right. & \\ & & & 0 & \\ 0 & 0 & \cdots & h\ times\left\{\begin{matrix}1\\ \vdots\\ 1\end{matrix}\right. & \begin{matrix}\\ \\ 1\end{matrix}\\ 0 & 0 & \cdots & 0 & 1\end{bmatrix}$$

which is $(r_1+1)\cdot\ldots\cdot(r_n+1)(h+1)\times(r_1+1)\cdot\ldots\cdot(r_n+1)(h+1)$ square matrix, $R_2(C_m)$ is the matrix obtaining by omitted the last two rows $\{0, 0, ..., 1, 1\}$ and $\{0, 0, ..., 0, 1\}$ and the last two columns $\{1, ..., 1, 0, 1, ..., 1, 0, ..., 1, 0\}$ and $\{1, 1, ..., 1\}$ from the tensor product

$$M(C_{2^h})\otimes M(C_{p_1^{r_1}})\otimes M(C_{p_2^{r_2}})\otimes\cdots\otimes M(C_{p_n^{r_n}}).$$

(ii) By Lemma 1.3 we have:

(1) $P(C_m)=P(C_{2^h})\otimes P(C_{p_1^{r_1}})\otimes P(C_{p_2^{r_2}})\otimes\cdots\otimes P(C_{p_n^{r_n}})$.

(2) $W(C_m)=W(C_{2^h})\otimes W(C_{p_1^{r_1}})\otimes W(C_{p_2^{r_2}})\otimes\cdots\otimes W(C_{p_n^{r_n}})$.

Theorem 1.6 [6]**.** *Let M be an* $n\times n$ *matrix with entries in a principal ideal domain R. Then there exist matrices P and W such that*:

(1) *P and W are invertible*.

(2) $PMW=D$.

(3) *D is a diagonal matrix*.

(4) *If we denote* D_{ii} *by* d_i, *then there exists a natural number m*; $0\le m\le n$ *such that* $j>m$ *implies* $d_j=0$ *and* $j\le m$ *implies* $d_j\neq 0$ *and* $1\le j\le m$ *implies* $d_j\,|\,d_{j+1}$.

2. The Main Results

Theorem 2.1. *The Artin characters table of the quaternion group* Q_{2m} *when m is an even number is given as follows*:

$Ar(Q_{2m}) =$

Γ-CLASSES	Γ-CLASSES OF C_{2m}				$[y]$	$[xy]$
	$[1]$	$[x^m]$				
$\lvert CL_\alpha \rvert$	1	1	2 2	2	m	m
$\lvert C_{Q_{2m}}(CL_\alpha) \rvert$	$4m$	$4m$	$2m$ $2m$	$2m$	4	4
Φ_1	$2Ar(C_{2m})$				0	0
Φ_2					0	0
$\vdots$					$\vdots$	$\vdots$
Φ_l					0	0
Φ_{l+1}	m	m	0 0 …	0	2	0
Φ_{l+2}	m	m	0 0 …	0	0	2

where l *is the number of* Γ*-classes of* C_{2m} *and* $\Phi_j, 1 \le j \le l+2$ *are the Artin characters of the quaternion group* Q_{2m}.

Proof. Let $g \in Q_{2m}$.

Case (I):

If H is a subgroup of $C_{2m} = \langle x \rangle, 1 \le j \le l$ and φ is the principal character of H, then by using theorem

$$\Phi_j(g) = \begin{cases} \dfrac{|C_G(g)|}{|C_H(g)|} \displaystyle\sum_{i=1}^{n} \varphi(h_i) & \text{if } h_i \in H \cap CL(g) \\ 0 & \text{if } H \cap CL(g) = \varnothing \end{cases}$$

(i) If $g = 1$

$$\Phi_j(1) = \frac{|C_{Q_{2m}}(1)|}{|C_H(1)|} \cdot \varphi(1) = \frac{4m}{|C_H(1)|} \cdot 1 = \frac{2 \cdot 2m}{|C_H(1)|} \cdot 1 = \frac{2|C_{C_{2m}}(1)|}{|C_H(1)|} \cdot 1 = 2 \cdot \varphi'_j(1)$$

$$= 2\varphi'_j(1) \text{ since } H \cap CL(1) = \{1\}$$

and φ is the principal character where φ'_j is the Artin characters of C_{2m}.

(ii) If $g = x^m$ and $g \in H$

$$\Phi_j(g) = \frac{|C_{Q_{2m}}(g)|}{|C_H(g)|} \cdot \varphi(g) = \frac{4m}{|C_H(g)|} \cdot 1 \text{ since } H \cap CL(g) = \{g\} \text{ and } \varphi(g) = 1$$

$$= \frac{2 \cdot 2m}{|C_H(g)|} \cdot \varphi(g) = \frac{2|C_{C_{2m}}(g)|}{|C_H(g)|} \cdot \varphi(g) = 2 \cdot \varphi'_j(g)$$

(iii) If $g \neq x^m$ and $g \in H$

$$\Phi_j(g) = \frac{|C_{Q_{2m}}(g)|}{|C_H(g)|}(\varphi(g) + \varphi(g^{-1}))$$

$$= \frac{2m}{|C_H(g)|}(1+1) \text{ since } H \cap CL(g) = \{g, g^{-1}\} \text{ and } \varphi(g) = \varphi(g^{-1}) = 1$$

$$= \frac{2|C_{C_{2m}}(g)|}{|C_H(g)|} = 2 \cdot \varphi'_j(g).$$

(iv) If $g \notin H$

$$\Phi_j(g) = 0 \text{ since } H \cap CL(g) = \varnothing$$

$$= 2 \cdot 0 = 2 \cdot \varphi'_j(g).$$

Case (II):

If $H = \langle y \rangle = \{1, y, y^2, y^3\}$.

(i) If $g = 1$

$$\Phi_{l+1}(1) = \frac{|C_{Q_{2m}}(1)|}{|C_H(1)|} \cdot \varphi(1) = \frac{4m}{4} \cdot 1 = m \text{ since } H \cap CL(1) = \{1\}$$

(ii) If $g = x^m = y^2$ and $g \in H$

$$\Phi_{l+1}(g) = \frac{|C_{Q_{2m}}(g)|}{|C_H(g)|} \cdot \varphi(g) = \frac{4m}{4} \cdot 1 = m \text{ since } H \cap CL(g) = \{g\} \text{ and } \varphi(g) = 1.$$

(iii) If $g \neq x^m$ and $g \in H$, i.e., $\{g = y$ or $g = y^3\}$

$$\Phi_{l+1}(g) = \frac{|C_{Q_{2m}}(g)|}{|C_H(g)|}(\varphi(g) + \varphi(g^{-1}))$$

$$= \frac{4}{4}(1 + 1) = 2 \text{ since } H \cap CL(g) = \{g, g^{-1}\} \text{ and } \varphi(g) = \varphi(g^{-1}) = 1$$

otherwise

$\Phi_{l+1}(g) = 0$ since $H \cap CL(g) = \varnothing$.

Case (III):

If $H = \langle xy \rangle = \{1, xy, (xy)^2 = y^2 = x^m, (xy)^3 = xy^3\}$.

(i) If $g = 1$

$$\Phi_{l+2}(g) = \frac{|C_{Q_{2m}}(1)|}{|C_H(1)|} \cdot \varphi(1) = \frac{4m}{4} \cdot 1 = m \text{ since } H \cap CL(1) = \{1\}.$$

(ii) If $g = (xy)^2 = y^2 = x^m$ and $g \in H$

$$\Phi_{l+2}(g) = \frac{|C_{Q_{2m}}(g)|}{|C_H(g)|} \cdot \varphi(g) = \frac{4m}{4} \cdot 1 = m \text{ since } H \cap CL(g) = \{g\} \text{ and } \varphi(g) = 1.$$

(iii) If $g \neq (xy)^2 = y^2 = x^m$ and $g \in H$, i.e., $\{g = xy$ or $g = (xy)^3\}$

$$\Phi_{l+2}(g) = \frac{|C_{Q_{2m}}(g)|}{|C_H(g)|}(\varphi(g) + \varphi(g^{-1}))$$

$$= \frac{4}{4}(1 + 1) = 2 \text{ since } H \cap CL(g) = \{g, g^{-1}\} \text{ and } \varphi(g) = \varphi(g^{-1}) = 1$$

otherwise

$\Phi_{l+2}(g) = 0$ since $H \cap CL(g) = \varnothing$.

Example 2.2. To construct $Ar(Q_{256})$ by using Theorem 2.1 we get the following table:

$Ar(Q_{256}) = Ar(Q_{2^8}) =$

Γ-CLASSES	[1]	$[x^{128}]$	$[x^{64}]$	$[x^{32}]$	$[x^{16}]$	$[x^{8}]$	$[x^{4}]$	$[x^{2}]$	$[x]$	$[y]$	$[xy]$
$\lvert CL_\alpha \rvert$	1	1	2	2	2	2	2	2	2	128	128
$\lvert C_{Q_{2m}}(CL_\alpha) \rvert$	512	512	256	256	256	256	256	256	256	4	4
Φ_1	512	0	0	0	0	0	0	0	0	0	0
Φ_2	256	256	0	0	0	0	0	0	0	0	0
Φ_3	128	128	128	0	0	0	0	0	0	0	0
Φ_4	64	64	64	64	0	0	0	0	0	0	0
Φ_5	32	32	32	32	32	0	0	0	0	0	0
Φ_6	16	16	16	16	16	16	0	0	0	0	0
Φ_7	8	8	8	8	8	8	8	0	0	0	0
Φ_8	4	4	4	4	4	4	4	4	0	0	0
Φ_9	2	2	2	2	2	2	2	2	2	0	0
Φ_{10}	128	128	0	0	0	0	0	0	0	2	0
Φ_{11}	128	128	0	0	0	0	0	0	0	0	2

Proposition 2.3. *If* $m = 2^h \cdot p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_n^{r_n}$ *such that* p_i *are primes,* $g.c.d(p_i, p_j) = 1$ *and* $p_i \neq 2$ *for all* $i = 1, 2, \ldots, h,$ *h and n any positive integers, then*

$$M(Q_{2m}) = \left[\begin{array}{ccccccccc|l}
 & & & & & & & & & \left\{\begin{matrix} 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{matrix}\right\} h+1\cdot times \\
 & & & & & & & & & \quad 0 \quad 1 \quad 0 \quad 1 \\
 & & & 2\cdot R_2(C_{2m}) & & & & & & \left\{\begin{matrix} 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{matrix}\right\} h+1\cdot times \\
 & & & & & & & & & \quad 0 \quad 1 \quad 0 \quad 1 \\
0 & 0 & \ldots & 0 & 0 & 0 & \cdots & 0 & & \left\{\begin{matrix} 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{matrix}\right\} h+1\cdot times \\
0 & 0 & \ldots & 0 & 0 & 0 & \cdots & 0 & & \quad 0 \quad 1 \quad 0 \quad 1 \\
0 & 0 & \ldots & 0 & 1 & 1 & \cdots & 1 & & \quad 0 \quad 0 \quad 1 \quad 1 \\
0 & 0 & \ldots & 0 & 1 & 1 & \cdots & 1 & & \quad 1 \quad 0 \quad 0 \quad 1
\end{array}\right]$$

which is $[(r_1+1)(r_2+1)\cdots(r_n+1)(h+2)+2]\times[(r_1+1)(r_2+1)\cdots(r_n+1)(h+2)+2]$ *square matrix*.

$R_2(C_{2m})$ *is similar to the matrix in Remark* 1.5.

Proof. By Theorem 2.1, we obtain the Artin character table $A(Q_{2m})$ of the quaternion group, and from previous proposition we get the rational valued character table $\equiv^* (Q_{2m})$ of the quaternion group.

Thus, by the definition of the matrix $M(Q_{2m})$

$$M(Q_{2m}) = Ar(Q_{2m}).\,(\equiv^* (Q_{2m}))^{-1}$$

$$M(Q_{2m}) = \begin{bmatrix} & & & & & & & & \left.\begin{Bmatrix} 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{Bmatrix}\right\} h+1 \cdot times \\ & & & & & & & & 0 \quad 1 \quad 0 \quad 1 \\ & & & 2 \cdot R_2(C_{2m}) & & & & & \left.\begin{Bmatrix} 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{Bmatrix}\right\} h+1 \cdot times \\ & & & & & & & & 0 \quad 1 \quad 0 \quad 1 \\ & & & & & & & & \left.\begin{Bmatrix} 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{Bmatrix}\right\} h+1 \cdot times \\ 0 & 0 & \dots & 0 & 0 & 0 & \cdots & 0 & \\ 0 & 0 & \dots & 0 & 0 & 0 & \cdots & 0 & 0 \quad 1 \quad 0 \quad 1 \\ 0 & 0 & \dots & 0 & 1 & 1 & \cdots & 1 & 0 \quad 0 \quad 1 \quad 1 \\ 0 & 0 & \dots & 0 & 1 & 1 & \cdots & 1 & 1 \quad 0 \quad 0 \quad 1 \end{bmatrix}$$

which is $[(r_1+1)(r_2+1)\cdots(r_n+1)(h+2)+2] \times [(r_1+1)(r_2+1)\cdots(r_n+1)(h+2)+2]$ square matrix.

Example 2.4. Consider the quaternion group Q_{48}, we can find matrix $M(Q_{48})$ by two ways:

First: by the definition of $M(Q_{48})$

$$M(Q_{48}) = M(Q_{3.2^4}).$$

We must find $Ar(Q_{3.2^4})$ and $(\equiv^* (Q_{3.2^4}))^{-1}$.

By using corollary we get

$$Ar(C_{48}) = Ar(C_{3.2^4}) = Ar(C_3) \otimes Ar(C_{2^4})$$

$$= \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 16 & 0 & 0 & 0 & 0 \\ 8 & 8 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 16 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ 8 & 8 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 & 4 & 4 & 4 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then from Theorem 2.1 we find $Ar(Q_{3.2^4})$ as follows:

$$Ar(Q_{3.2^4}) = \begin{bmatrix} 96 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 48 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 24 & 0 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & 16 & 16 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 12 & 0 & 12 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 8 & 8 & 8 & 8 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 0 & 6 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 & 4 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 \\ 24 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 24 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Now, we find $\overset{*}{\equiv}(C_{3.2^4})$ as

$$\overset{*}{\equiv}(C_{48}) = (\overset{*}{\equiv}(C_{3.2^4})) = (\overset{*}{\equiv}(C_3)) \otimes (\overset{*}{\equiv}(C_{2^4}))$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 8 & -8 & 0 & 0 & 0 \\ 4 & 4 & -4 & 0 & 0 \\ 2 & 2 & 2 & -2 & 0 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -16 & 0 & 0 & 0 & -8 & 8 & 0 & 0 & 0 \\ 8 & 8 & -8 & 0 & 0 & -4 & -4 & 4 & 0 & 0 \\ 4 & 4 & 4 & -4 & 0 & -2 & -2 & -2 & 2 & 0 \\ 2 & 2 & 2 & 2 & -2 & -1 & -1 & -1 & -1 & 1 \\ 2 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 \\ 8 & -8 & 0 & 0 & 0 & 8 & -8 & 0 & 0 & 0 \\ 4 & 4 & -4 & 0 & 0 & 4 & 4 & -4 & 0 & 0 \\ 2 & 2 & 2 & -2 & 0 & 2 & 2 & 2 & -2 & 0 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

By pervious proposition we get $(\overset{*}{\equiv}(Q_{3.2^4}))$ as:

$$= \begin{bmatrix} 16 & -16 & -8 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & -8 & 8 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 8 & -4 & -8 & -4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & -4 & 4 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & -2 & 4 & -2 & -4 & -2 & 0 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & -2 & 2 & 0 & -2 & 0 & 0 & 0 \\ 2 & 2 & -1 & 2 & -1 & 2 & -1 & -2 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Now,

$$M(Q_{48}) = Ar(Q_{48}) \cdot (\overset{*}{\equiv}(Q_{48}))^{-1}$$

$$
= \begin{bmatrix}
96 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
48 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
32 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 24 & 0 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 16 & 16 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 12 & 0 & 12 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 8 & 8 & 8 & 8 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
6 & 6 & 0 & 6 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 & 4 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 \\
24 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
24 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
$$

$$
\bullet \begin{bmatrix}
1/48 & 1/48 & 1/48 & 1/48 & 1/48 & 1/48 & 1/48 & 1/48 & 1/96 & 1/96 & 1/96 & 1/96 \\
-1/48 & -1/48 & 1/48 & 1/48 & 1/48 & 1/48 & 1/48 & 1/48 & 1/96 & 1/96 & 1/96 & 1/96 \\
-1/48 & 1/24 & -1/48 & 1/24 & -1/48 & 1/24 & -1/48 & -1/48 & 1/48 & 1/48 & 1/48 & 1/48 \\
0 & 0 & -1/24 & -1/24 & 1/24 & 1/24 & 1/24 & 1/24 & 1/48 & 1/48 & 1/48 & 1/48 \\
1/48 & -1/24 & -1/48 & 1/24 & -1/48 & 1/24 & -1/48 & -1/48 & 1/48 & 1/48 & 1/48 & 1/48 \\
0 & 0 & 0 & 0 & -1/12 & -1/12 & 1/12 & 1/12 & 1/24 & 1/24 & 1/24 & 1/24 \\
0 & 0 & 1/24 & -1/12 & -1/24 & 1/12 & -1/24 & -1/24 & 1/24 & 1/24 & 1/24 & 1/24 \\
0 & 0 & 0 & 0 & 1/12 & -1/6 & -1/12 & -1/12 & 1/12 & 1/12 & 1/12 & 1/12 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/6 & 1/6 & -1/12 & 1/12 & -1/12 & 1/12 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/6 & -1/6 & -1/6 & 1/6 & -1/6 & 1/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/4 & -1/4 & 1/4 & 1/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & -1/4 & -1/4 & 1/4
\end{bmatrix}
$$

$$
= \begin{bmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 2 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
$$

which is 12×12 square matrix.

Second:

By Proposition 2.3, we find $R(C_{3.2^4})$ by using Remark 1.5 as:

$$M(C_{3.2^4}) = M(C_3) \otimes M(C_{2^4})$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, by omitting the last two columns and the last two rows of this matrix, we get:

$$R_2(C_{3.2^4}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which is 8×8 square matrix.

Then, by Proposition 2.3 we have:

$$M\left(Q_{3.2^4}\right)=\left[\begin{array}{cccccccccccccc} & & 2R_2\left(C_{3.2^4}\right) & & & & & & & & \left.\begin{matrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{matrix}\right\} & 4\cdot times \\ & & & & & & & & & & \begin{matrix}0&1&0&1\end{matrix} & \\ & & & & & & & & & & \left.\begin{matrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{matrix}\right\} & 4\cdot times \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & & \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & \begin{matrix}0&1&0&1\end{matrix} & \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1 & \begin{matrix}0&0&1&1\end{matrix} & \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1 & \begin{matrix}1&0&0&1\end{matrix} & \end{array}\right]$$

which is 12×12 square matrix.

Then:

$$M(Q_{3\cdot 2^4}) = \begin{bmatrix} 2&2&2&2&2&2&2&2&1&1&1&1\\ 0&2&2&2&2&0&2&2&1&1&1&1\\ 0&0&2&2&2&0&0&2&1&1&1&1\\ 0&0&0&2&2&0&0&0&1&1&1&1\\ 0&0&0&0&2&0&0&0&0&1&0&1\\ 0&0&0&0&0&2&2&2&1&1&1&1\\ 0&0&0&0&0&0&2&2&1&1&1&1\\ 0&0&0&0&0&0&0&2&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&0&1\\ 0&0&1&1&1&1&1&1&0&0&1&1\\ 0&0&1&1&1&1&1&1&1&0&0&1 \end{bmatrix}.$$

Proposition 2.5. *If* $m = 2^h \cdot p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_n^{r_n}$ *such that* p_i *are primes,* $g.c.d(p_i,\ p_j) = 1$ *and* $p_i \neq 2$ *for all* $i = 1, 2, \ldots, n,$ h *and* r_i *any positive integers, then the matrices* $P(Q_{2m})$ *and* $W(Q_{2m})$ *take the forms*:

$$P(Q_{2m}) = \begin{bmatrix} & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & \vdots & \vdots \\ & & P(C_{2m}) & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ & & & & & -1 & 1 \\ & & & & & 0 & -1 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix}$$

and

$$W(Q_{2m}) = \begin{bmatrix} & & & & & & & & & 0 & 0 & 0 \\ & & & & & & & & & 0 & 0 & 0 \\ & & & & & I_k & & & & \vdots & \vdots & \vdots \\ & & & & & & & & & \vdots & \vdots & \vdots \\ & & & & & & & & & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & 0 & 0 & 1 \end{bmatrix},$$

where $k = [(r_1+1)(r_2+1)\cdots(r_n+1)(h+2)]-1$ *and* I_k *is an identity matrix of the order k. These matrices are*

$$[(r_1+1)(r_2+1)\cdots(r_n+1)(h+2)+2]\times[((r_1+1)(r_2+1)\cdots(r_n+1)(h+2)+2]$$

square matrix.

Proof. By using Theorem 1.6 and taking the form of $M(Q_{2m})$ from Proposition 2.3 and the above forms of $P(Q_{2m})$ and $W(Q_{2m})$, then:

$$P(Q_{2m})\cdot M(Q_{2m})\cdot W(Q_{2m}) = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= D(Q_{2m}) = diag\{2, 2, ..., 2, 1, 1, 1\}$$

which is $[(r_1+1)(r_2+1)\cdots(r_n+1)(h+2)+2]\times[((r_1+1)(r_2+1)\cdots(r_n+1)(h+2)+2]$ square matrix.

Example 2.6. Consider the Q_{96}, then we can find the matrices $P(Q_{96})$ and $W(Q_{96})$ immediately by using Proposition 2.5 and we can find $M(Q_{96})$ by Proposition 2.3, where $Q_{96} = Q_{2^5 \cdot 3}$:

$$P(Q_{96}) \cdot M(Q_{96}) \cdot W(Q_{96})$$

$$= \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$\bullet \begin{bmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$= diag\{2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1\}$

which is 14×14 square matrix.

Theorem 2.7. *If* $m = 2^h \cdot p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_n^{r_n}$ *such that* p_i *are primes* $g.c.d(p_i, p_j) = 1$ *and* $p_i \neq 2$ *for all* $i = 1, 2, \ldots, n$, h *and* r_i *any positive integers, then the cyclic decomposition of* $AC(Q_{2m})$ *is*:

$$AC(Q_{2m}) = \bigoplus_{i=1}^{(r_1+1)(r_2+1)\ldots(r_n+1)(h+2)-1} C_2.$$

Proof. By pervious proposition we find $M(Q_{2m})$ and by Proposition 2.5 we have $P(Q_{2m})$ and $W(Q_{2m})$. Hence

$$P(Q_{2m}) \cdot M(Q_{2m}) \cdot W(Q_{2m}) = diag\{2, 2, ..., 2, 1, 1, 1\}$$

$$= \{d_1, d_2, ..., d_{(r_1+1)(r_2+1)\ldots(r_n+1)(h+2)-1}, d_{(r_1+1)(r_2+1)\ldots(r_n+1)(h+2)},$$

$$d_{(r_1+1)(r_2+1)\ldots(r_n+1)(h+2)-1}, d_{(r_1+1)(r_2+1)\ldots(r_n+1)(h+2)+2}\}.$$

Then by theorem we have

$$AC(Q_{2m}) = \bigoplus_{i=1}^{(r_1+1)(r_2+1)\ldots(r_n+1)(h+2)-1} C_2.$$

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