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Some Fixed Point Theory Results for the Interpolative Berinde Weak Operator

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Abstract

Partially inspired by [1] and [2], we introduce a concept of interpolative Berinde weak contraction, and obtain some existence theorems for mappings satisfying such a contractive definition, and some of its extensions.

1 A Result in Metric Space

Definition 1.1. Let (X, d) be a metric space. We say $T : X \mapsto X$ is an *interpolative Berinde weak operator* if it satisfies

$$d(Tx, Ty) \leq \lambda \ d(x, y)^{\alpha} \ d(x, Tx)^{1-\alpha},$$

where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, for all $x, y \in X, \ x, y \notin Fix(T)$.

Theorem 1.2. *Let (X, d) be a metric space. Suppose $T : X \mapsto X$ is an interpolative Berinde weak operator. If (X, d) is complete, then the fixed point of T exists.*

Proof. Define the sequence $\{x_n\} \in X$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, and observe we have the following

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\leq \lambda \ d(x_n, x_{n+1})^{\alpha} \ d(x_n, Tx_n)^{1-\alpha} \\ &= \lambda \ d(x_n, x_{n+1})^{\alpha} \ d(x_n, x_{n+1})^{1-\alpha} \\ &= \lambda d(x_n, x_{n+1}). \end{aligned}$$

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By induction, the following is clear for all $n \in \mathbb{N} \cup \{0\}$,

$$d(x_n, x_{n+1}) \leq \lambda^n \, d(x_0, x_1).$$

Now we show the sequence is Cauchy. For this, let $n, m \in \mathbb{N}$ with $m > n$, and observe we have the following

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq (\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n) d(x_0, x_1) \\ &\leq (\lambda^n + \lambda^{n+1} + \cdots) d(x_0, x_1) \\ &= \lambda^n (1 + \lambda + \cdots) d(x_0, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1). \end{aligned}$$

Now letting $m, n \to \infty$ in the above, it follows that $\{x_n\}$ is Cauchy, and since X is complete, there is $a \in X$ such that

$$\lim_{n\to\infty} x_n = a.$$

Now we show the fixed point exists. First observe that since $\lim_{n\to\infty} x_n = a$, the following is clear,

$$\begin{aligned} \lim_{n\to\infty} d(a, x_{n+1}) &= \lim_{n\to\infty} d(x_n, a) \\ &= \lim_{n\to\infty} d(x_n, x_{n+1}) \\ &= 0. \end{aligned}$$

Now if $a \neq Ta$, then we know $d(a, Ta) > 0$, and we have the following

$$\begin{aligned} 0 &< d(a, Ta) \\ &\leq d(a, x_{n+1}) + d(x_{n+1}, Ta) \\ &= d(a, x_{n+1}) + d(Tx_n, Ta) \\ &\leq d(a, x_{n+1}) + \lambda d(x_n, a)^{\alpha} d(x_n, Tx_n)^{1-\alpha} \\ &= d(a, x_{n+1}) + \lambda d(x_n, a)^{\alpha} d(x_n, x_{n+1})^{1-\alpha}. \end{aligned}$$

Now letting $n \to \infty$ in the above, we deduce that

$$0 < d(a, Ta) \leq 0.$$

Thus, $d(a, Ta)$ is bounded above and below by zero, hence $d(a, Ta) = 0$, and the fixed point exists, that is, $a = Ta$. □

2 A Result in Cone Metric Space over Banach Algebras

This section takes inspiration from [3] with preliminaries as is. Now we begin with the following

Definition 2.1. Let Ω denote a Banach algebra, and let (X, d) denote a cone metric space over Ω, and let $T : X \mapsto X$ be a mapping. If there exists $\lambda, \alpha \in P$, where P is a cone, with $0 \leq \rho(\lambda) < 1$, and $0 < \rho(\alpha) < 1$, where ρ is the spectral radius, such that

$$d(Tx, Ty) \preceq \lambda \; d(x, y)^{\alpha} \; d(x, Tx)^{1-\alpha}$$

for all $x, y \in X,\ x, y \notin Fix(T) = \{x \in X | Tx = x\}$, then we say T is an *interpolative Berinde weak contraction* in the setting of cone metric spaces with Banach algebras.

Our main result is as follows

Theorem 2.2. *Let (X, d) be a complete cone metric space over a Banach algebra Ω. If $T : X \mapsto X$ is an interpolative Berinde weak contraction, then the fixed point of T exists.*

Proof. Define the sequence $\{x_n\} \in X$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, and observe we have the following

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\preceq \lambda \; d(x_n, x_{n+1})^{\alpha} \; d(x_n, Tx_n)^{1-\alpha} \\ &= \lambda \; d(x_n, x_{n+1})^{\alpha} \; d(x_n, x_{n+1})^{1-\alpha} \\ &= \lambda d(x_n, x_{n+1}). \end{aligned}$$

By induction, the following is clear for all $n \in \mathbb{N} \cup \{0\}$,

$$d(x_n, x_{n+1}) \preceq \lambda^n \; d(x_0, x_1).$$

Now for all $n, m \in \mathbb{N}$ with $n < m$, we have the following

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\preceq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) \; d(x_0, x_1) \\ &\preceq \lambda^n (e + \lambda + \lambda^2 + \cdots + \lambda^{m-n-1}) \; d(x_0, x_1) \\ &\preceq \lambda^n (e - \lambda)^{-1} \; d(x_0, x_1). \end{aligned}$$

Now by Lemma 2.7[3] and Lemma 2.10[3], it follows that $\{x_n\}$ is Cauchy. Since (X, d) is complete, there is a point $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now we show the fixed point exists. Since $1 - \alpha \in [0, 1)$ and $2^{1-\alpha} \leq 2$, observe we have the following

$$\begin{aligned} d(x, Tx) &\preceq d(x, x_{n+1}) + d(x_{n+1}, Tx) \\ &= d(x, x_{n+1}) + d(Tx_n, Tx) \\ &\preceq d(x, x_{n+1}) + \lambda \; d(x_n, x)^{\alpha} \; d(x_n, Tx_n)^{1-\alpha} \\ &= d(x, x_{n+1}) + \lambda \; d(x_n, x)^{\alpha} \; d(x_n, x_{n+1})^{1-\alpha} \\ &\preceq d(x, x_{n+1}) + \lambda \; d(x_n, x)^{\alpha} \; \big[d(x_n, x) + d(x, x_{n+1})\big]^{1-\alpha} \\ &\preceq d(x, x_{n+1}) + 2\lambda \; d(x_n, x) \\ &\preceq d(x, x_{n+1}) + 2\lambda \; [d(x_n, x) + 2d(x_{n+1}, x)] \\ &= d(x, x_{n+1}) + 2\lambda \; d(x_n, x) + 4\lambda \; d(x_{n+1}, x) \\ &= (e + 4\lambda) \; d(x, x_{n+1}) + 2\lambda \; d(x_n, x). \end{aligned}$$

Now put $h_n = (e + 4\lambda) \; d(x, x_{n+1}) + 2\lambda \; d(x_n, x)$. It follows from Lemma 2.7, Lemma 2.9, and Lemma 2.6 all contained in [3], that $\{h_n\}$ is a c-sequence, and thus for each $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x, Tx) \preceq h_n \ll c$ for $n \geq n_0$. Thus, by Lemma 2.4[3], the fixed point exists, that is, $Tx = x$. □

3 A Result in Partial Cone Metric Space

This section is inspired by [4] with preliminaries as is. Our main result is as follows.

Theorem 3.1. *Let (X, ρ) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose $T : X \mapsto X$ satisfies*

$$\rho(Tx, Ty) \leq \lambda\ \rho(x, y)^{\alpha}\ \rho(x, Tx)^{1-\alpha},$$

where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, for all $x, y \in X,\ x, y \notin Fix(T)$. Then the fixed point of T exists, and for each $x_0 \in X$, the iterative sequence $\{T^n x_0\}$ converges to the fixed point.

Proof. Define the sequence $\{x_n\} \in X$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, and observe we have the following

$$\begin{aligned}\rho(x_{n+1}, x_{n+2}) &= \rho(Tx_n, Tx_{n+1})\\ &\preceq \lambda\ \rho(x_n, x_{n+1})^{\alpha}\ d(x_n, Tx_n)^{1-\alpha}\\ &= \lambda\ \rho(x_n, x_{n+1})^{\alpha}\ \rho(x_n, x_{n+1})^{1-\alpha}\\ &= \lambda \rho(x_n, x_{n+1}).\end{aligned}$$

By induction, the following is clear for all $n \in \mathbb{N} \cup \{0\}$,

$$\rho(x_n, x_{n+1}) \preceq \lambda^n\ \rho(x_0, x_1).$$

Now we show the sequence is Cauchy. For this, let $m < n$, and observe we have the following

$$\begin{aligned}\rho(x_m, x_n) &\leq \rho(x_m, x_{m-1}) + \rho(x_{m-1}, x_{m-2}) + \cdots + \rho(x_{n+1}, x_n) - \sum_{k=1}^{n-m-1} \rho(x_{n-k}, x_{n-k})\\ &\leq [\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n]\rho(x_0, x_1)\\ &\leq \frac{\lambda^n(1 - \lambda^{n-m})}{1 - \lambda}\rho(x_0, x_1)\\ &\leq \frac{\lambda^n}{1 - \lambda}\rho(x_0, x_1).\end{aligned}$$

Now taking norm to inequality in above, we have

$$\|\rho(x_m, x_n)\| \leq \frac{K\lambda^n}{1-\lambda}\|\rho(x_0, x_1)\|.$$

It now follows that $\{T^n x_0\}$ is a Cauchy sequence in (X, ρ) such that

$$\lim_{n,m\to\infty} \rho(T^n x_9, T^m x_0) = 0.$$

As (X, ρ) is complete, there exists $x_0 \in X$ such that $\{T^n x_0\}$ converges to x and

$$\rho(x, x) = \lim_{n\to\infty} \rho(x_n, x) = \lim_{n\to\infty} \rho(x_n, x_n) = 0.$$

Now we show existence of the fixed point. Observe, for any $n \in \mathbb{N}$, that we have the following

$$\begin{aligned}\rho(Tx, x) &\leq \rho(Tx, T^{n+1}x_0) + \rho(T^{n+1}x_0, x) - \rho(T^{n+1}x_0, T^{n+1}x_9)\\ &\leq \lambda\rho(x, T^n x_0)\rho(x, Tx)^{1-\alpha} + \rho(T^{n+1}x_0, x).\end{aligned}$$

Taking norm to inequality in above, we deduce

$$\|\rho(Tx, x)\| \leq \lambda\|\rho(x, T^n x_0)\|\|\rho(x, Tx)\|^{1-\alpha} + \|\rho(T^{n+1}x_0, x)\| \ \to 0.$$

Hence, $\rho(Tx, x) = 0$. However,

$$\rho(Tx, Tx) \leq \lambda\rho(x, x)^{\alpha}\rho(x, Tx)^{1-\alpha} = 0.$$

Thus,

$$\rho(Tx, Tx) = \rho(x, Tx) = \rho(x, x) = 0.$$

which implies the fixed point exists, that is, $Tx = x$. □

4 A Best Proximity Result

This section takes inspiration from [5] with preliminaries as is. However, some additional notions and notations are necessary.

Notation 4.1. The set of all best proximity points of $T : A \mapsto B$ will be denoted $B_{\text{Prox}}(T) = \{x \in A : d(x, Tx) = d(A, B)\}$.

Definition 4.2. A mapping $T : A \mapsto B$ will be called a *proximal interpolative Berinde weak contraction of the first kind* if there exists $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that for all $x_1, x_2, u_1, u_2 \in A$, $x_1, x_2 \notin B_{\text{Prox}}(T)$, the following implication holds

$$d(u_1, Tx_1) = d(A, B),\ d(u_2, Tx_2) = d(A, B)$$

$\Longrightarrow$

$$d(u_1, u_2) \leq \lambda\ d(x_1, x_2)^{\alpha} d(x_1, u_1)^{1-\alpha}.$$

Definition 4.3. A mapping $T : A \mapsto B$ will be called a *proximal interpolative Berinde weak contraction of the second kind* if there exists $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that for all $x_1, x_2, u_1, u_2 \in A$, $x_1, x_2 \notin B_{\text{Prox}}(T)$, the following implication holds

$$d(u_1, Tx_1) = d(A, B),\ d(u_2, Tx_2) = d(A, B)$$

$\Longrightarrow$

$$d(Tu_1, Tu_2) \leq \lambda\ d(Tx_1, Tx_2)^{\alpha} d(Tx_1, Tu_1)^{1-\alpha}.$$

Our main result is as follows, which extends the Interpolative Berinde Weak Contraction Principle (of the first section) to the case of non-self mappings

Theorem 4.4. *Let X be a complete metric space. Let A and B be nonempty, closed subsets of X such that A is approximately compact with respect to B. Further, suppose that A_0 and B_0 are nonempty. Let $T : A \mapsto B$ and $g : A \mapsto A$ satisfy the following conditions*

(*a*) *T is a continuous proximal interpolative Berinde weak contraction of the second kind.*

(*b*) *g is an isometry.*

(*c*) *$T(A_0)$ is contained in B_0.*

(*d*) *A_0 is contained in $g(A_0)$.*

(*e*) *T preserves isometric distance with respect to g. Then there exists an element $x \in A$ such that*

$$d(gx, Tx) = d(A, B).$$

Proof. Let x_0 be a fixed element in A_0. Since $T(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$, there exists an element x_1 in A_0 such that

$$g(gx_1, Tx_0) = d(A, B).$$

Again since Tx_1 is an element of $T(A_0)$ which is contained in B_0 and A_0 is contained in $g(A_0)$ its follows there exists an element x_2 in A_0 such that

$$d(gx_2, Tx_1) = d(A, B).$$

Since $T(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$, for every positive integer n, haven chosen x_n in A_0, we can also find x_{n+1} in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A, B).$$

As T is a continuous proximal interpolative Berinde weak contraction of the second kind we deduce the following

$$d(Tgx_{n+1}, Tgx_n) \leq \lambda d(Tx_n, Tx_{n-1})^{\alpha} d(Tx_n, Tgx_{n+1})^{1-\alpha}.$$

Since T preserves isometric distance with respect to g, we have,

$$d(Tx_{n+1}, Tx_n) \leq \lambda d(Tx_n, Tx_{n-1})^{\alpha} d(Tx_n, Tx_{n+1})^{1-\alpha}.$$

From the above, we deduce,

$$d(Tx_{n+1}, Tx_n) \leq \lambda^{\frac{1}{\alpha}} d(Tx_n, Tx_{n-1}).$$

Since $\lambda^{\frac{1}{\alpha}} < 1$, it is easy to see that $\{Tx_n\}$ is Cauchy, hence it converges to some element y in B. Further,

$$\begin{aligned} d(y, A) &\leq d(y, gx_n) \\ &\leq d(y, Tx_{n-1}) + d(Tx_{n-1}, gx_n) \\ &= d(y, Tx_{n-1}) + d(A, B) \\ &\leq d(y, Tx_{n-1}) + d(y, A). \end{aligned}$$

Therefore $d(y, gx_n) \to A$. In view of the fact that A is approximately compact with respect to B, $\{gx_n\}$ has a subsequence $\{gx_{n_k}\}$ converging to some $z \in A$. It follows that

$$d(z, y) = \lim_{k\to\infty} d(gx_{n_k}, Tx_{n_k-1}) = d(A, B).$$

Hence z is a member of A_0. Since A_0 is contained in $g(A_0)$, $z = gx$ for some $z \in A_0$. As $g(x_{n_k}) \to g(x)$ and g is an isometry, $x_{n_k} \to x$. Since the mapping T is continuous, it follows that $Tx_{n_k} \to Tx$. Consequently, y and Tx are identical. Thus, it follows that

$$d(gx, Tx) = \lim_{k\to\infty} d(gx_{n_k}, Tx_{n_k-1}) = d(A, B).$$

Hence the best proximity point exists, and the proof is finished. □

If g is the identity in the above theorem, then we get the following

Theorem 4.5. *Let X be a complete metric space. Let A and B be nonempty, closed subsets of X such that A is approximately compact with respect to B. Further, suppose that A_0 and B_0 are nonempty. Let $T : A \mapsto B$ satisfy the following conditions*

(a) *T is a continuous proximal interpolative Berinde weak contraction of the second kind.*

(b) *$T(A_0)$ is contained in B_0.*

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

5 The Alternate Interpolative Berinde Weak Operator in Metric Spaces

Keep $\alpha, \lambda \in (0,1)$. Now

$$d(Tx,Ty) \leq \lambda\ d(x,y)^{\alpha} d(x,Tx)^{1-\alpha}$$

$\implies$

$$\frac{d(Tx,Ty)}{\lambda} \leq d(x,y)^{\alpha} d(x,Tx)^{1-\alpha}$$

$\implies$

$$\begin{aligned}\log\left(\frac{d(Tx,Ty)}{\lambda}\right) &\leq \alpha \log(d(x,y)) + (1-\alpha)\log(d(x,Tx)) \\ &\leq (\alpha + 1 - \alpha)\max\{\log(d(x,y)), \log(d(x,Tx))\} \\ &= \max\{\log(d(x,y)), \log(d(x,Tx))\}\end{aligned}$$

$\implies$

$$2\log\left(\frac{d(Tx,Ty)}{\lambda}\right) \leq \log(d(x,y)\ d(x,Tx))$$

$\implies$

$$\log\left(\frac{d(Tx,Ty)}{\lambda}\right) \leq \log(d(x,y)\ d(x,Tx))^{\frac{1}{2}}$$

$\implies$

$$d(Tx,Ty) \leq \lambda\ d(x,y)^{\frac{1}{2}} d(x,Tx)^{\frac{1}{2}}.$$

Thus we have the following

Definition 5.1. A map $T : X \mapsto X$ will be called an *alternate Interpolative Berinde Weak operator* if it satisfies

$$d(Tx,Ty) \leq \lambda\ d(x,y)^{\frac{1}{2}} d(x,Tx)^{\frac{1}{2}},$$

where $\lambda \in (0,1)$, for all $x, y \in X,\ x, y \notin Fix(T)$.

6 A Contraction Mapping Theorem in the Sense of Istratescu

This section takes inspiration from [6], where the idea of convex contractions first appeared. In particular, we introduce the convex interpolative Berinde weak operator in metric spaces, and obtain a so-called "Convex Interpolative Berinde Weak Contraction Mapping Theorem'."

Definition 6.1. A continuous mapping $T : X \mapsto X$ will be called a *convex interpolative Berinde weak operator* if the following holds for all $x, y \in X,\ x, y \notin \{Fix(T), Fix(T^2)\}$

$$d(T^2x, T^2y) \leq \lambda_1 d(x,y)^{\frac{1}{2}} d(x,Tx)^{\frac{1}{2}} + \lambda_2 d(Tx,Ty)^{\frac{1}{2}} d(Tx,T^2x)^{\frac{1}{2}},$$

where $\lambda_1, \lambda_2 \in (0,1)$ with $\lambda_1 + \lambda_2 < 1$.

Theorem 6.2. *Let (X,d) be a metric space, and $T : X \mapsto X$ be a convex interpolative Berinde weak operator. If (X,d) is complete, then the fixed point exists.*

Proof. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n = T^2x_{n-1}$, and observe we have the following

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(T^2x_{n-1}, T^2x_n) \\ &\leq \lambda_1 d(x_n, x_{n-1})^{\frac{1}{2}} d(x_{n-1}, Tx_{n-1})^{\frac{1}{2}} + \lambda_2 d(Tx_n, Tx_{n-1})^{\frac{1}{2}} d(Tx_{n-1}, T^2x_{n-1})^{\frac{1}{2}} \\ &= \lambda_1 d(x_n, x_{n-1})^{\frac{1}{2}} d(x_{n-1}, x_n)^{\frac{1}{2}} + \lambda_2 d(x_n, x_{n+1})^{\frac{1}{2}} d(x_n, x_{n+1})^{\frac{1}{2}} \\ &= \lambda_1 d(x_n, x_{n-1}) + \lambda_2 d(x_n, x_{n+1}) \\ &\leq (\lambda_1 + \lambda_2) \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} \\ &= (\lambda_1 + \lambda_2) d(x_n, x_{n+1}). \end{aligned}$$

From the above, we deduce that

$$d(x_{n+1}, x_{n+2}) \leq hd(x_n, x_{n+1}),$$

where $h := \lambda_1 + \lambda_2 < 1$. By induction, the following is clear for all $n \in \mathbb{N} \cup \{0\}$

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1).$$

Now we show the sequence is Cauchy. For this, let $n, m \in \mathbb{N}$ with $m > n$, and observe we have the following

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq (h^{m-1} + h^{m-2} + \cdots + h^n) d(x_0, x_1) \\ &\leq (h^n + h^{n+1} + \cdots) d(x_0, x_1) \\ &= h^n (1 + h + \cdots) d(x_0, x_1) \\ &\leq \frac{h^n}{1-h} d(x_0, x_1). \end{aligned}$$

Now letting $m, n \to \infty$ in the above, it follows that $\{x_n\}$ is Cauchy, and since X is complete, there is $a \in X$ such that

$$\lim_{n\to\infty} x_n = a.$$

Now we show the fixed point exists. Suppose $d(a, Ta) = 0$, but $d(a, T^2 a) > 0$. Now observe we have the following

$$\begin{aligned} 0 &< d(a, T^2 a) \\ &\leq d(a, x_{n+1}) + d(x_{n+1}, T^2 a) \\ &= d(a, x_{n+1}) + d(T^2 x_{n-1}, T^2 a) \\ &\leq d(a, x_{n+1}) + \lambda_1 d(x_{n-1}, a)^{\frac{1}{2}} d(x_{n-1}, Tx_{n-1})^{\frac{1}{2}} + \lambda_2 d(Tx_{n-1}, Ta)^{\frac{1}{2}} d(Tx_{n-1}, T^2 x_{n-1})^{\frac{1}{2}}. \end{aligned}$$

Taking limits in the above, and using the continuity of T, we deduce that

$$0 < d(a, T^2 a) \leq 0$$

which implies that $d(a, T^2 a) = 0$, which is a contradiction. So a is also a fixed point of T^2, and the proof is finished. □

7 A Result in the Sense of Wardowski on Metric Spaces

In [7], they introduced the notion of F-contraction, and used it to give a new proof of the Banach contraction mapping theorem. Inspired by this development, this section introduces a notion of F-interpolative Berinde weak contraction, and uses it to give a new proof of the Interpolative Berinde weak contraction mapping theorem (Section 1). We begin with the following

Definition 7.1. A mapping $T : X \mapsto X$ will be called an *F-interpolative Berinde weak contraction* if there exists $\tau > 0$ such that for all $x, y \in X$, $x, y \notin Fix(T)$, the following implication holds

$$d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}}),$$

where F is a mapping satisfying Definition 2.1[7].

Example 7.2. Let $F : [0, \infty) \mapsto (-\infty, \infty)$ be given by $F(\alpha) = ln(\alpha)$, then F satisfies Definition 2.1 [7] for any $\lambda \in (0, 1)$. Now each mapping satisfying the Definition 7.1 is an F-interpolative Berinde weak contraction such that

$$d(Tx, Ty) \leq e^{-\tau} d(x, y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}}$$

for all $x, y \in X$, $x, y \notin Fix(T)$, $Tx \neq Ty$.

Remark 7.3. For $x, y \in X$, $x, y \notin Fix(T)$, such that $Tx = Ty$, the inequality

$$d(Tx, Ty) \leq e^{-\tau} d(x, y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}}.$$

In particular, T is an interpolative Berinde weak operator in the sense of Section 1

Now our main result is as follows

Theorem 7.4. *Let (X, d) be a complete metric space, and let $T : X \mapsto X$ be a continuous F-interpolative Berinde weak contraction. Then T has a fixed point $x^* \in X$ and for every $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^**

Proof. Let $x_0 \in X$ be arbitrary and fixed. Define the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$. Put $\gamma_n = d(x_{n+1}, x_n)$, $n = 0, 1, 2, \cdots$. If there exists

$n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$, and the proof is finished. So let us assume that $x_{n+1} \neq x_n$ for every $n \in \mathbb{N}$. Now observe we have the following

$$\begin{aligned}\tau + F(d(x_{n+1}, x_n)) &= \tau + F(Tx_n, Tx_{n-1}) \\ &\leq F(d(x_n, x_{n-1})^{\frac{1}{2}} d(x_n, Tx_n)^{\frac{1}{2}}) \\ &= F(d(x_n, x_{n-1})^{\frac{1}{2}} d(x_n, x_{n+1})^{\frac{1}{2}}).\end{aligned}$$

Since

$$d(x_n, x_{n-1})^{\frac{1}{2}} d(x_n, x_{n+1})^{\frac{1}{2}} < d(x_n, x_{n-1}) \iff d(x_{n+1}, x_n) < d(x_n, x_{n-1})$$

and F is strictly increasing, the above implies the following

$$\tau + F(d(x_{n+1}, x_n)) \leq F(d(x_n, x_{n-1}))$$

or equivalently

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau.$$

By induction, we obtain

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-1}) - 2\tau \leq \cdots \leq F(\gamma_0) - n\tau.$$

Taking limits in the above we deduce that

$$\lim_{n\to\infty} F(\gamma_n) = -\infty.$$

From (F2) of Definition 2.1[7], we have $\lim_{n\to\infty} \gamma_n = 0$. From (F3) of Definition 2.1[7], there is $\lambda \in (0, 1)$ such that

$$\lim_{n\to\infty} \gamma_n^\lambda F(\gamma_n) = 0.$$

Since $F(\gamma_n) \leq F(\gamma_0) - n\tau$ holds for all $n \in \mathbb{N}$, we deduce the following for all $n \in \mathbb{N}$

$$\begin{aligned}\gamma_n^\lambda F(\gamma_n) - \gamma_n^\lambda F(\gamma_0) &\leq \gamma_n^\lambda (F(\gamma_0) - n\tau) - \gamma_n^\lambda F(\gamma_0) \\ &= -\gamma_n^\lambda n\tau \\ &\leq 0.\end{aligned}$$

Since $\lim_{n\to\infty} \gamma_n = 0$, and $\lim_{n\to\infty} \gamma_n^\lambda F(\gamma_n) = 0$, if we take limits in the inequality immediately above we deduce that

$$\lim_{n\to\infty} n\gamma_n^\lambda = 0.$$

From the above, there is $n_1 \in \mathbb{N}$ such that $n\gamma_n^\lambda \leq 1$ for all $n_1 \geq n$. It follows we have the following

$$\gamma_n \leq \frac{1}{n^{\frac{1}{\lambda}}}$$

for all $n \geq n_1$. Now we show the sequence $\{x_n\}$ is Cauchy. For this, let $m, n \in \mathbb{N}$ with $m > n \geq n_1$. From definition of metric and above inequality we deduce the following

$$\begin{aligned} d(x_m, x_n) &\leq \gamma_{m-1} + \gamma_{m-2} + \cdots + \gamma_n \\ &\leq \sum_{i=n}^{\infty} \gamma_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}. \end{aligned}$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}$ is convergent, it follows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there is $a \in X$ such that $\lim_{n\to\infty} x_n = a$. Since T is continuous, we deduce the following

$$d(Ta, a) = \lim_{n\to\infty} d(Tx_n, x_n) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0$$

and the proof is finished. □

8 An Expanding Mapping Theorem in Partial Metric Spaces

This section is inspired by [8] in which expanding mappings in the setting of partial metric spaces analogous to expanding mappings in metric spaces are defined, and some fixed point theorems for expanding mappings are obtained. In this section, we obtain a so-called *Expanding Interpolative Berinde Weak Mapping Theorem* in the setting of partial metric spaces. We begin with the following

Definition 8.1. Let (X, ρ) be a partial metric space. We say a continuous mapping $T : X \mapsto X$ is an interpolative Berinde weak expanding operator if it satisfies

$$\rho(Tx, Ty) \geq \lambda\ \rho(x, y)^{\frac{1}{2}}\ \rho(x, Tx)^{\frac{1}{2}}$$

for all $x, y \in X,\ x, y \notin Fix(T)$, where $\lambda > 1$.

Our main result is as follows

Theorem 8.2. *Let (X, ρ) be a complete partial metric space, and the continuous mapping $T : X \mapsto X$ be a surjection. Suppose $T : X \mapsto X$ satisfies*

$$\rho(Tx, Ty) \geq \lambda\ \rho(x, y)^{\frac{1}{2}}\ \rho(x, Tx)^{\frac{1}{2}}$$

for all $x, y \in X,\ x, y \notin Fix(T)$, where $\lambda > 1$, and $x \neq y$. Then T has a fixed point in X.

Proof. Let $x_0 \in X$. Since T is surjective, choose $x_1 \in X$ such that $Tx_1 = x_0$. By induction we have $x_{n-1} = Tx_n$ for $n = 1, 2, \cdots$. If there exists n_0 such that $x_{n_0-1} = x_{n_0}$, then x_{n_0} is a fixed point of T, so we assume $x_{n-1} \neq x_n$ for $n = 1, 2, \cdots$. Now observe we have the following

$$\begin{aligned} \rho(x_{n-1}, x_n) &= \rho(Tx_n, Tx_{n+1}) \\ &\geq \lambda \rho(x_n, x_{n+1})^{\frac{1}{2}}\ \rho(x_n, Tx_n)^{\frac{1}{2}} \\ &= \lambda \rho(x_n, x_{n+1})^{\frac{1}{2}}\ \rho(x_n, x_{n-1})^{\frac{1}{2}}. \end{aligned}$$

The above implies

$$\rho(x_n, x_{n+1})^{\frac{1}{2}} \leq \frac{1}{\lambda} \rho(x_{n-1}, x_n)^{\frac{1}{2}}$$

or equivalently

$$\rho(x_n, x_{n+1}) \leq \left(\frac{1}{\lambda}\right)^2 \rho(x_{n-1}, x_n).$$

Since $\left(\frac{1}{\lambda}\right)^2 < 1$, by Lemma 2.1 [6], $\{x_n\}$ is a Cauchy sequence in X. Since (X, ρ) is complete, then Lemma 1.1 [6], implies (X, ρ^s) is complete, and so $\{x_n\}$ converges in the metric space (X, ρ^s), that is, there exists $z \in X$ such that

$$\lim_{n\to\infty} \rho^s(x_n, z) = 0.$$

Consequently, we can find $u \in X$ such that $z = Tu$. Again from Lemma 1.1[6], we have,

$$\rho(z, z) = \lim_{n\to\infty} \rho(x_n, z) = \lim_{n,m\to\infty} \rho(x_n, x_m).$$

Moreover, since $\{x_n\}$ is Cauchy in (X, ρ^s), we have,

$$\lim_{n,m\to\infty} \rho^s(x_n, x_m) = 0.$$

On the other hand since

$$\max\{\rho(x_n, x_n), \rho(x_{n+1}, x_{n+1})\} \leq \rho(x_n, x_{n+1}),$$

using $\rho(x_n, x_{n+1}) \leq \left(\frac{1}{\lambda}\right)^2 \rho(x_{n-1}, x_n)$, by induction, we have

$$\max\{\rho(x_n, x_n), \rho(x_{n+1}, x_{n+1})\} \leq \left(\frac{1}{\lambda}\right)^{2n} \rho(x_1, x_0).$$

It now follows that

$$\lim_{n\to\infty} \rho(x_n, x_n) = 0.$$

The definition of ρ^s implies

$$\lim_{n,m\to\infty} \rho^s(x_n, x_m) = 0.$$

Since $\rho(z, z) = \lim_{n\to\infty} \rho(x_n, z) = \lim_{n,m\to\infty} \rho(x_n, x_m)$, we have

$$\rho(z, z) = \lim_{n\to\infty} \rho(x_n, z) = \lim_{n,m\to\infty} \rho(x_n, x_m) = 0.$$

Now we show $u = z$. Using the inequality of the Theorem, we have

$$\begin{aligned}\rho(x_n, z) &= \rho(Tx_{n+1}, Tu) \\ &\geq \lambda \rho(x_{n+1}, u)^{\frac{1}{2}} \rho(x_{n+1}, Tx_{n+1})^{\frac{1}{2}}.\end{aligned}$$

Taking limits in above as $m \to \infty$, and using continuity of T, we have

$$0 \geq \lambda \rho(z, u)^{\frac{1}{2}} \rho(z, Tz)^{\frac{1}{2}}.$$

Since $Tz = u$, the above is equivalent to

$$0 \geq \lambda \rho(z, u).$$

Since $\frac{1}{\lambda} \neq 0$, the above implies $\rho(z, u) = 0$, that is, $z = u$. Since $Tz = u$, then we have

$$Tz = u = z.$$

So z is a fixed point of T, and the proof is finished. □

9 Concluding Remarks

In this work we introduced the Interpolative Berinde weak contraction, and obtained some fixed point theory results for operators satisfying the inequality and its various extensions in the setting of metric spaces, cone metric space over Banach algebras, partial cone metric spaces, and partial metric spaces. The interpolative weak contraction was extended to accomodate non-self maps leading to a best proximity result, it was also extended to accomodate convexity conditions leading to a convex type contraction mapping theorem. We also gave an implicit characterization leading to a Wardowski type mapping theorem, finally, the expanding counterpart of the inequality was also presented leading to an expansion type theorem.

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