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A Study on Generalized Jacobsthal-Padovan Numbers

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Abstract

In this paper, we investigate the generalized Jacobsthal-Padovan sequences and we deal with, in detail, four special cases, namely, Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan and modified Jacobsthal-Padovan sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

1 Introduction

It is the aim of this paper to define and to explore some of the properties of generalized Jacobsthal-Padovan numbers and is to investigate, in details, four particular case, namely sequences of Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan and modified Jacobsthal-Padovan numbers. Before, we recall the generalized Tribonacci sequence and its some properties.

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [1, 2, 3, 6, 7, 11, 12, 13, 14, 18, 19, 20, 21].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

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for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), its characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.2}$$

whose roots are

$$\begin{aligned} \alpha &= \alpha(r,s,t) = \frac{r}{3} + A + B, \\ \beta &= \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B, \\ \gamma &= \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B, \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \\ B &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3}, \\ \Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \\ \omega &= \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3). \end{aligned}$$

Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma &= r, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -s, \\ \alpha\beta\gamma &= t. \end{aligned}$$

If $\Delta(r,s,t) > 0$, then the Equ. (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{b_1 \alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2 \beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3 \gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \tag{1.3}$$

where

$$\begin{aligned} b_1 &= W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0, \\ b_2 &= W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0, \\ b_3 &= W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0. \end{aligned}$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, for a proof of this result see [8]. This result of Howard and Saidak [8] is even true in the case of higher-order recurrence relations.

In this paper we consider the case $r = 0$, $s = 1$, $t = 2$ and in this case we write $V_n = W_n$. A generalized Jacobsthal-Padovan sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-2} + 2V_{n-3} \tag{1.4}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} + \frac{1}{2}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.4) holds for all integer n.

(1.3) can be used to obtain Binet formula of generalized Jacobsthal-Padovan numbers. Binet formula of generalized Jacobsthal-Padovan numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},$$

where

$$\begin{aligned} b_1 &= V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0, \\ b_2 &= V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0, \\ b_3 &= V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \end{aligned} \tag{1.5}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - x - 2 = 0.$$

Moreover

$$\begin{aligned}\alpha &= \sqrt[3]{1+\frac{\sqrt{78}}{9}}+\sqrt[3]{1-\frac{\sqrt{78}}{9}} \simeq 1.521379706804568,\\ \beta &= \omega\sqrt[3]{1+\frac{\sqrt{78}}{9}}+\omega^2\sqrt[3]{1-\frac{\sqrt{78}}{9}},\\ \gamma &= \omega^2\sqrt[3]{1+\frac{\sqrt{78}}{9}}+\omega\sqrt[3]{1-\frac{\sqrt{78}}{9}},\end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}\alpha+\beta+\gamma &= 0,\\ \alpha\beta+\alpha\gamma+\beta\gamma &= -1,\\ \alpha\beta\gamma &= 2.\end{aligned}$$

The first few generalized Jacobsthal-Padovan numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Jacobsthal-Padovan numbers.

n	V_n	V_{-n}
0	V_0	...
1	V_1	$\frac{1}{2}V_2 - \frac{1}{2}V_0$
2	V_2	$-\frac{1}{4}V_2 + \frac{1}{2}V_1 + \frac{1}{4}V_0$
3	$V_1 + 2V_0$	$\frac{1}{8}V_2 - \frac{1}{4}V_1 + \frac{3}{8}V_0$
4	$V_2 + 2V_1$	$\frac{3}{16}V_2 - \frac{7}{16}V_0 + \frac{1}{8}V_1$
5	$2V_2 + V_1 + 2V_0$	$-\frac{7}{32}V_2 + \frac{3}{16}V_1 + \frac{11}{32}V_0$
6	$V_2 + 4V_1 + 4V_0$	$\frac{11}{64}V_2 - \frac{7}{32}V_1 + \frac{1}{64}V_0$
7	$4V_2 + 5V_1 + 2V_0$	$\frac{1}{128}V_2 - \frac{29}{128}V_0 + \frac{11}{64}V_1$
8	$5V_2 + 6V_1 + 8V_0$	$-\frac{29}{256}V_2 + \frac{1}{128}V_1 + \frac{73}{256}V_0$
9	$6V_2 + 13V_1 + 10V_0$	$\frac{73}{512}V_2 - \frac{29}{256}V_1 - \frac{69}{512}V_0$

Now we define four special cases of the sequence $\{V_n\}$. Jacobsthal-Padovan sequence $\{Q_n\}_{n\geq 0}$, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) sequence $\{L_n\}_{n\geq 0}$, adjusted Jacobsthal-Padovan sequence $\{K_n\}_{n\geq 0}$ and modified Jacobsthal-Padovan sequence $\{M_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} Q_{n+3} &= Q_{n+1}+2Q_n, \quad Q_0=1, Q_1=1, Q_2=1, \\ L_{n+3} &= L_{n+1}+2L_n, \quad L_0=3, L_1=0, L_2=2, \\ K_{n+3} &= K_{n+1}+K_n, \quad K_0=0, K_1=1, K_2=0, \\ M_{n+3} &= M_{n+1}+M_n, \quad M_0=3, M_1=1, M_2=3. \end{aligned}$$

The sequences $\{Q_n\}_{n\geq 0}$, $\{L_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{M_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$Q_{-n} = -\frac{1}{2}Q_{-(n-1)}+\frac{1}{2}Q_{-(n-3)}, \tag{1.6}$$

$$L_{-n} = -\frac{1}{2}L_{-(n-1)}+\frac{1}{2}L_{-(n-3)}, \tag{1.7}$$

$$K_{-n} = -\frac{1}{2}K_{-(n-1)}+\frac{1}{2}K_{-(n-3)}, \tag{1.8}$$

$$M_{-n} = -\frac{1}{2}M_{-(n-1)}+\frac{1}{2}M_{-(n-3)}, \tag{1.9}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.6), (1.7), (1.8) and (1.9) hold for all integer n.

For more information on Jacobsthal-Padovan sequence, see [4] and [5].

Q_n is the sequence A159284 in [15] associated with the expansion of $x(1+x)/(1-x^2-2x^3)$ and L_n is the sequence A072328 in [15] and K_n is the sequence A159287 in [15] associated with the expansion of $x^2/(1-x^2-2x^3)$. M_n is not indexed in [15].

Next, we present the first few values of the Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan and modified Jacobsthal-Padovan numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Q_n	1	1	1	3	3	5	9	11	19	29	41	67	99	149
Q_{-n}		0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{8}$	$\frac{5}{16}$	$-\frac{1}{32}$	$-\frac{3}{64}$	$\frac{23}{128}$	$-\frac{27}{256}$	$\frac{15}{512}$	$\frac{77}{1024}$	$-\frac{185}{2048}$	$\frac{245}{4096}$
L_n	3	0	2	6	2	10	14	14	34	42	62	110	146	234
L_{-n}		$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{11}{8}$	$-\frac{15}{16}$	$\frac{19}{32}$	$\frac{25}{64}$	$-\frac{85}{128}$	$\frac{161}{256}$	$-\frac{61}{512}$	$-\frac{279}{1024}$	$\frac{923}{2048}$	$-\frac{1167}{4096}$	$\frac{51}{8192}$
K_n	0	1	0	1	2	1	4	5	6	13	16	25	42	57
K_{-n}		0	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{16}$	$-\frac{7}{32}$	$\frac{11}{64}$	$\frac{1}{128}$	$-\frac{29}{256}$	$\frac{73}{512}$	$-\frac{69}{1024}$	$-\frac{47}{2048}$	$\frac{339}{4096}$
M_n	3	1	3	7	5	13	19	23	45	61	91	151	213	333
M_{-n}		0	$\frac{1}{2}$	$\frac{5}{4}$	$-\frac{5}{8}$	$\frac{9}{16}$	$\frac{11}{32}$	$-\frac{31}{64}$	$\frac{67}{128}$	$-\frac{23}{256}$	$-\frac{101}{512}$	$\frac{369}{1024}$	$-\frac{461}{2048}$	$\frac{57}{4096}$

For all integers n, Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan and modified Jacobsthal-Padovan numbers (using initial conditions in (1.5)) can be expressed using Binet's formulas as

$$\begin{aligned}
Q_n &= \frac{(\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1}+\frac{(\beta+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1}+\frac{(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1},\\
L_n &= \alpha^n+\beta^n+\gamma^n,\\
K_n &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1}+\frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1}+\frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1},\\
M_n &= \frac{(3\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1}+\frac{(3\beta+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1}+\frac{(3\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1},
\end{aligned}$$

respectively.

2 Generating Functions

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1. *Suppose that $f_{V_n}(x)=\sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Jacobsthal-Padovan sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+V_1x+(V_2-V_0)x^2}{1-x^2-2x^3}. \tag{2.1}$$

Proof. Using the definition of generalized Jacobsthal-Padovan numbers, and substracting $x^2\sum_{n=0}^{\infty}V_nx^n$ and $2x^3\sum_{n=0}^{\infty}V_nx^n$ from $\sum_{n=0}^{\infty}V_nx^n$, we obtain

$$\begin{aligned}(1-x^2-2x^3)\sum_{n=0}^{\infty}V_nx^n &= \sum_{n=0}^{\infty}V_nx^n - x^2\sum_{n=0}^{\infty}V_nx^n - 2x^3\sum_{n=0}^{\infty}V_nx^n \\ &= \sum_{n=0}^{\infty}V_nx^n - \sum_{n=0}^{\infty}V_nx^{n+2} - 2\sum_{n=0}^{\infty}V_nx^{n+3} \\ &= \sum_{n=0}^{\infty}V_nx^n - \sum_{n=2}^{\infty}V_{n-2}x^n - 2\sum_{n=3}^{\infty}V_{n-3}x^n \\ &= (V_0+V_1x+V_2x^2) - V_0x^2 + \sum_{n=3}^{\infty}(V_n - V_{n-2} - 2V_{n-3})x^n \\ &= V_0+V_1x+V_2x^2-V_0x^2 = V_0+V_1x+(V_2-V_0)x^2.\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty}V_nx^n = \frac{V_0+V_1x+(V_2-V_0)x^2}{1-x^2-2x^3}.$$

The previous lemma gives the following results as particular examples.

Corollary 2. *Generated functions of Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan and modified Jacobsthal-Padovan numbers are*

$$\begin{aligned}\sum_{n=0}^{\infty}Q_nx^n &= \frac{1+x}{1-x^2-2x^3}, \\ \sum_{n=0}^{\infty}L_nx^n &= \frac{3-x^2}{1-x^2-2x^3}, \\ \sum_{n=0}^{\infty}K_nx^n &= \frac{x}{1-x^2-2x^3}, \\ \sum_{n=0}^{\infty}M_nx^n &= \frac{3+x}{1-x^2-2x^3},\end{aligned}$$

respectively.

3 Obtaining Binet Formula from Generating Function

We next find Binet formula of generalized Jacobsthal-Padovan numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 3. *(Binet formula of generalized Jacobsthal-Padovan numbers)*

$$V_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \tag{3.1}$$

where

$$\begin{aligned} d_1 &= V_0\alpha^2 + V_1\alpha + (V_2 - V_0), \\ d_2 &= V_0\beta^2 + V_1\beta + (V_2 - V_0), \\ d_3 &= V_0\gamma^2 + V_1\gamma + (V_2 - V_0). \end{aligned}$$

Proof. Let

$$h(x) = 1 - x^2 - 2x^3.$$

Then for some α, β and γ, we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x),$$

i.e.,

$$1 - x^2 - 2x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x). \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}$, and $\frac{1}{\gamma}$ are the roots of $h(x)$. This gives α, β, and γ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{1}{x^2} - \frac{2}{x^3} = 0.$$

This implies $x^3 - x - 2 = 0$. Now, by (2.1) and (3.2), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + V_1x + (V_2 - V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Then we write

$$\frac{V_0 + V_1x + (V_2 - V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)} + \frac{A_3}{(1-\gamma x)}. \tag{3.3}$$

So

$$V_0+V_1x+(V_2-V_0)x^2 = A_1(1-\beta x)(1-\gamma x)+A_2(1-\alpha x)(1-\gamma x)+A_3(1-\alpha x)(1-\beta x).$$

If we consider $x = \frac{1}{\alpha}$, we get $V_0 + V_1\frac{1}{\alpha} + (V_2 - V_0)\frac{1}{\alpha^2} = A_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})$. This gives

$$A_1 = \frac{\alpha^2(V_0 + V_1\frac{1}{\alpha} + (V_2 - V_0)\frac{1}{\alpha^2})}{(\alpha-\beta)(\alpha-\gamma)} = \frac{V_0\alpha^2 + V_1\alpha + (V_2 - V_0)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we obtain

$$A_2 = \frac{V_0\beta^2 + V_1\beta + (V_2 - V_0)}{(\beta-\alpha)(\beta-\gamma)}, A_3 = \frac{V_0\gamma^2 + V_1\gamma + (V_2 - V_0)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1-\alpha x)^{-1} + A_2(1-\beta x)^{-1} + A_3(1-\gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} V_n x^n = A_1\sum_{n=0}^{\infty}\alpha^n x^n + A_2\sum_{n=0}^{\infty}\beta^n x^n + A_3\sum_{n=0}^{\infty}\gamma^n x^n = \sum_{n=0}^{\infty}(A_1\alpha^n + A_2\beta^n + A_3\gamma^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n,$$

where

$$\begin{aligned} A_1 &= \frac{V_0\alpha^2 + V_1\alpha + (V_2 - V_0)}{(\alpha-\beta)(\alpha-\gamma)}, \\ A_2 &= \frac{V_0\beta^2 + V_1\beta + (V_2 - V_0)}{(\beta-\alpha)(\beta-\gamma)}, \\ A_3 &= \frac{V_0\gamma^2 + V_1\gamma + (V_2 - V_0)}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

and then we get (3.1).

Note that from (1.5) and (3.1) we have

$$\begin{aligned} V_2-(\beta+\gamma)V_1+\beta\gamma V_0 &= V_0\alpha^2+V_1\alpha+(V_2-V_0), \\ V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0 &= V_0\beta^2+V_1\beta+(V_2-V_0), \\ V_2-(\alpha+\beta)V_1+\alpha\beta V_0 &= V_0\gamma^2+V_1\gamma+(V_2-V_0). \end{aligned}$$

Next, using Theorem 3, we present the Binet formulas of Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan and modified Jacobsthal-Padovan sequences.

Corollary 4. *Binet formulas of Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan and modified Jacobsthal-Padovan sequences are*

$$\begin{aligned} Q_n &= \frac{(\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1}+\frac{(\beta+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1}+\frac{(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}, \\ L_n &= \alpha^n+\beta^n+\gamma^n, \\ K_n &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1}+\frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1}+\frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}, \\ M_n &= \frac{(3\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1}+\frac{(3\beta+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1}+\frac{(3\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}, \end{aligned}$$

respectively.

We can find Binet formulas by using matrix method with a similar technique which is given in [10]. Take $k=i=3$ in Corollary 3.1 in [10]. Let

$$\begin{aligned} \Lambda &= \begin{pmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{pmatrix}, \ \Lambda_1=\begin{pmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{pmatrix}, \\ \Lambda_2 &= \begin{pmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{pmatrix}, \ \Lambda_3=\begin{pmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{pmatrix}. \end{aligned}$$

Then the Binet formula for Jacobsthal-Padovan numbers is

$$\begin{aligned} Q_n &= \frac{1}{\det(\Lambda)}\sum_{j=1}^{3} Q_{4-j}\det(\Lambda_j) = \frac{1}{\Lambda}(Q_3\det(\Lambda_1)+Q_2\det(\Lambda_2)+Q_1\det(\Lambda_3)) \\ &= \frac{1}{\det(\Lambda)}(3\det(\Lambda_1)+\det(\Lambda_2)+\det(\Lambda_3)) \\ &= \left(3\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + \begin{vmatrix} \alpha^{2} & \alpha^{n-1} & 1 \\ \beta^{2} & \beta^{n-1} & 1 \\ \gamma^{2} & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^{2} & \alpha & \alpha^{n-1} \\ \beta^{2} & \beta & \beta^{n-1} \\ \gamma^{2} & \gamma & \gamma^{n-1} \end{vmatrix}\right) \Big/ \begin{vmatrix} \alpha^{2} & \alpha & 1 \\ \beta^{2} & \beta & 1 \\ \gamma^{2} & \gamma & 1 \end{vmatrix}. \end{aligned}$$

Similarly, we obtain the Binet formula for Jacobsthal-Perrin, adjusted Jacobsthal-Padovan and modified Jacobsthal-Padovan as

$$\begin{aligned} L_n &= \frac{1}{\Lambda}(L_3\det(\Lambda_1)+L_2\det(\Lambda_2)+L_1\det(\Lambda_3)) \\ &= \left(2\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 3\begin{vmatrix} \alpha^{2} & \alpha & \alpha^{n-1} \\ \beta^{2} & \beta & \beta^{n-1} \\ \gamma^{2} & \gamma & \gamma^{n-1} \end{vmatrix}\right) \Big/ \begin{vmatrix} \alpha^{2} & \alpha & 1 \\ \beta^{2} & \beta & 1 \\ \gamma^{2} & \gamma & 1 \end{vmatrix}. \end{aligned}$$

and

$$\begin{aligned} K_n &= \frac{1}{\Lambda}(K_3\det(\Lambda_1)+K_2\det(\Lambda_2)+K_1\det(\Lambda_3)) \\ &= \left(\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + \begin{vmatrix} \alpha^{2} & \alpha & \alpha^{n-1} \\ \beta^{2} & \beta & \beta^{n-1} \\ \gamma^{2} & \gamma & \gamma^{n-1} \end{vmatrix}\right) \Big/ \begin{vmatrix} \alpha^{2} & \alpha & 1 \\ \beta^{2} & \beta & 1 \\ \gamma^{2} & \gamma & 1 \end{vmatrix}. \end{aligned}$$

and

$$\begin{aligned} M_n &= \frac{1}{\Lambda}(M_3\det(\Lambda_1)+M_2\det(\Lambda_2)+M_1\det(\Lambda_3)) \\ &= \left(7\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 3\begin{vmatrix} \alpha^{2} & \alpha^{n-1} & 1 \\ \beta^{2} & \beta^{n-1} & 1 \\ \gamma^{2} & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^{2} & \alpha & \alpha^{n-1} \\ \beta^{2} & \beta & \beta^{n-1} \\ \gamma^{2} & \gamma & \gamma^{n-1} \end{vmatrix}\right) \Big/ \begin{vmatrix} \alpha^{2} & \alpha & 1 \\ \beta^{2} & \beta & 1 \\ \gamma^{2} & \gamma & 1 \end{vmatrix} \end{aligned}$$

respectively.

4 Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\left| \begin{array}{cc} F_{n+1} & F_n \\ F_n & F_{n-1} \end{array} \right| = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Jacobsthal-Padovan sequence $\{V_n\}_{n\geq 0}$.

Theorem 5 (Simson Formula of Generalized Jacobsthal-Padovan Numbers)**.** *For all integers n, we have*

$$\left| \begin{array}{ccc} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{array} \right| = 2^n \left| \begin{array}{ccc} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{array} \right|. \tag{4.1}$$

Proof. (4.1) is given in Soykan [17].

The previous theorem gives the following results as particular examples.

Corollary 6. *For all integers n, Simson formula of Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan and modified Jacobsthal-Padovan numbers are given as*

$$\left| \begin{array}{ccc} Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+1} & Q_n & Q_{n-1} \\ Q_n & Q_{n-1} & Q_{n-2} \end{array} \right| = -2^n$$

and

$$\left| \begin{array}{ccc} L_{n+2} & L_{n+1} & L_n \\ L_{n+1} & L_n & L_{n-1} \\ L_n & L_{n-1} & L_{n-2} \end{array} \right| = -26 \times 2^n$$

and

$$\left| \begin{array}{ccc} K_{n+2} & K_{n+1} & K_n \\ K_{n+1} & K_n & K_{n-1} \\ K_n & K_{n-1} & K_{n-2} \end{array} \right| = -2^{n-1}$$

and

$$\left| \begin{array}{ccc} M_{n+2} & M_{n+1} & M_n \\ M_{n+1} & M_n & M_{n-1} \\ M_n & M_{n-1} & M_{n-2} \end{array} \right| = -23 \times 2^n$$

respectively.

5 Some Identities

In this section, we obtain some identities of Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan and modified Jacobsthal-Padovan numbers. First, we can give a few basic relations between $\{Q_n\}$ and $\{L_n\}$.

Lemma 7. *The following equalities are true:*

$$\begin{aligned} 8L_n &= -3Q_{n+4}+14Q_{n+3}-9Q_{n+2}, \\ 4L_n &= 7Q_{n+3}-6Q_{n+2}-3Q_{n+1}, \\ 2L_n &= -3Q_{n+2}+2Q_{n+1}+7Q_n, \\ L_n &= Q_{n+1}+2Q_n-3Q_{n-1}, \\ L_n &= 2Q_n-2Q_{n-1}+2Q_{n-2}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} 26Q_n &= -L_{n+4}+3L_{n+3}+5L_{n+2}, \\ 26Q_n &= 3L_{n+3}+4L_{n+2}-2L_{n+1}, \\ 26Q_n &= 4L_{n+2}+L_{n+1}+6L_n, \\ 26Q_n &= L_{n+1}+10L_n+8L_{n-1}, \\ 26Q_n &= 10L_n+9L_{n-1}+2L_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (5.1). To show (5.1), writing

$$L_n = a\times Q_{n+4}+b\times Q_{n+3}+c\times Q_{n+2}$$

and solving the system of equations

$$\begin{aligned} L_0 &= a\times Q_4+b\times Q_3+c\times Q_2 \\ L_1 &= a\times Q_5+b\times Q_4+c\times Q_3 \\ L_2 &= a\times Q_6+b\times Q_5+c\times Q_4 \end{aligned}$$

we find that $a=-\frac{3}{8}, b=\frac{7}{4}, c=-\frac{9}{8}$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{Q_n\}$ and $\{K_n\}$.

Lemma 8. *The following equalities are true:*

$$\begin{aligned} 4K_n &= -Q_{n+4}+3Q_{n+2}, \\ 2K_n &= Q_{n+2}-Q_{n+1}, \\ 2K_n &= -Q_{n+1}+Q_n+2Q_{n-1}, \\ 2K_n &= Q_n+Q_{n-1}-2Q_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 4Q_n &= K_{n+4}+2K_{n+3}-K_{n+2}, \\ 2Q_n &= K_{n+3}+K_{n+1}, \\ Q_n &= K_{n+1}+K_n, \\ Q_n &= K_n+K_{n-1}+2K_{n-2}. \end{aligned}$$

Now, we give a few basic relations between $\{Q_n\}$ and $\{M_n\}$.

Lemma 9. *The following equalities are true:*

$$\begin{aligned} 4M_n &= -Q_{n+4}+6Q_{n+3}-3Q_{n+2}, \\ 2M_n &= 3Q_{n+3}-2Q_{n+2}-Q_{n+1}, \\ M_n &= -Q_{n+2}+Q_{n+1}+3Q_n, \\ M_n &= Q_{n+1}+2Q_n-2Q_{n-1}, \\ M_n &= 2Q_n-Q_{n-1}+2Q_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 92Q_n &= -7M_{n+4}+10M_{n+3}+19M_{n+2}, \\ 46Q_n &= 5M_{n+3}+6M_{n+2}-7M_{n+1}, \\ 23Q_n &= 3M_{n+2}-M_{n+1}+5M_n, \\ 23Q_n &= -M_{n+1}+8M_n+6M_{n-1}, \\ 23Q_n &= 8M_n+5M_{n-1}-2M_{n-2}. \end{aligned}$$

Next, we present a few basic relations between $\{L_n\}$ and $\{K_n\}$.

Lemma 10. *The following equalities are true*

$$\begin{aligned} 52K_n &= L_{n+4} - 3L_{n+3} + 8L_{n+2}, \\ 52K_n &= -3L_{n+3} + 9L_{n+2} + 2L_{n+1}, \\ 52K_n &= 9L_{n+2} - L_{n+1} - 6L_n, \\ 52K_n &= -L_{n+1} + 3L_n + 18L_{n-1}, \\ 52K_n &= 3L_n + 17L_{n-1} - 2L_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 8L_n &= 11K_{n+4} + 2K_{n+3} - 15K_{n+2}, \\ 4L_n &= K_{n+3} - 2K_{n+2} + 11K_{n+1}, \\ 2L_n &= -K_{n+2} + 6K_{n+1} + K_n, \\ L_n &= 3K_{n+1} - K_{n-1}, \\ L_n &= 2K_{n-1} + 6K_{n-2}. \end{aligned}$$

Next, we give a few basic relations between $\{M_n\}$ and $\{L_n\}$.

Lemma 11. *The following equalities are true*

$$\begin{aligned} 184L_n &= -65M_{n+4} + 106M_{n+3} + 45M_{n+2}, \\ 92L_n &= 53M_{n+3} - 10M_{n+2} - 65M_{n+1}, \\ 46L_n &= -5M_{n+2} - 6M_{n+1} + 53M_n, \\ 23L_n &= -3M_{n+1} + 24M_n - 5M_{n-1}, \\ 23L_n &= 24M_n - 8M_{n-1} - 6M_{n-2} \end{aligned}$$

and

$$\begin{aligned} 26M_n &= -4L_{n+4} + 12L_{n+3} + 7L_{n+2}, \\ 26M_n &= 12L_{n+3} + 3L_{n+2} - 8L_{n+1}, \\ 26M_n &= 3L_{n+2} + 4L_{n+1} + 24L_n, \\ 26M_n &= 4L_{n+1} + 27L_n + 6L_{n-1}, \\ 26M_n &= 27L_n + 10L_{n-1} + 8L_{n-2}. \end{aligned}$$

Now, we present a few basic relations between $\{K_n\}$ and $\{M_n\}$.

Lemma 12. *The following equalities are true*

$$\begin{aligned} 4M_n &= 5K_{n+4} + 2K_{n+3} - 5K_{n+2}, \\ 2M_n &= K_{n+3} + 5K_{n+1}, \\ M_n &= 3K_{n+1} + K_n, \\ M_n &= K_n + 3K_{n-1} + 6K_{n-2} \end{aligned}$$

and

$$\begin{aligned} 92K_n &= M_{n+4} - 8M_{n+3} + 17M_{n+2}, \\ 46K_n &= -3M_{n+1} + M_n + 18M_{n-1}, \\ 46K_n &= 9M_{n+2} - 3M_{n+1} - 8M_n, \\ 46K_n &= -3M_{n+1} + M_n + 18M_{n-1}, \\ 46K_n &= M_n + 15M_{n-1} - 6M_{n-2}. \end{aligned}$$

6 Linear Sums

The following proposition presents some formulas of generalized Jacobsthal-Padovan numbers with positive subscripts.

Proposition 13. *If $r = 0, s = 1, t = 2$, then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k = \frac{1}{2}\left(V_{n+3} + V_{n+2} - V_2 - V_1\right).$

(b) $\sum_{k=0}^{n} V_{2k} = \frac{1}{2}\left(V_{2n+1} + 2V_{2n} - V_1\right).$

(c) $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{2}\left(V_{2n+2} + 2V_{2n+1} - V_2\right).$

Proof. Take $r = 0, s = 1, t = 2$ in Theorem 2.1 in [16].

As special cases of above proposition, we have the following four corollaries. First one presents some summing formulas of Jacobsthal-Padovan numbers (take $V_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

Corollary 14. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{2}\left(Q_{n+3} + Q_{n+2} - 2\right).$

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{2}\left(Q_{2n+1} + 2Q_{2n} - 1\right).$

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{2}\left(Q_{2n+2} + 2Q_{2n+1} - 1\right).$

Second one presents some summing formulas of Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) numbers (take $V_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$).

Corollary 15. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} L_k = \frac{1}{2}(L_{n+3} + L_{n+2} - 2)$.

(b) $\sum_{k=0}^{n} L_{2k} = \frac{1}{2}(L_{2n+1} + 2L_{2n})$.

(c) $\sum_{k=0}^{n} L_{2k+1} = \frac{1}{2}(L_{2n+2} + 2L_{2n+1} - 2)$.

Third one presents some summing formulas of adjusted Jacobsthal-Padovan numbers (take $V_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = 0$).

Corollary 16. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} K_k = \frac{1}{2}(K_{n+3} + K_{n+2} - 1)$.

(b) $\sum_{k=0}^{n} K_{2k} = \frac{1}{2}(K_{2n+1} + 2K_{2n} - 1)$.

(c) $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{2}(K_{2n+2} + 2K_{2n+1})$.

Fourth one presents some summing formulas of modified Jacobsthal-Padovan numbers (take $V_n = M_n$ with $M_0 = 3, M_1 = 1, M_2 = 3$).

Corollary 17. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} M_k = \frac{1}{2}(M_{n+3} + M_{n+2} - 4)$.

(b) $\sum_{k=0}^{n} M_{2k} = \frac{1}{2}(M_{2n+1} + 2M_{2n} - 1)$.

(c) $\sum_{k=0}^{n} M_{2k+1} = \frac{1}{2}(M_{2n+2} + 2M_{2n+1} - 3)$.

The following proposition presents some formulas of generalized Jacobsthal-Padovan numbers with negative subscripts.

Proposition 18. *If $r = 0, s = 1, t = 2$, then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} V_{-k} = \frac{1}{2}(-3V_{-n-1} - 3V_{-n-2} - 2V_{-n-3} + V_2 + V_1)$.

(b) $\sum_{k=1}^{n} V_{-2k} = \frac{1}{2}(-V_{-2n+1} + V_1)$.

(c) $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{2}(-V_{-2n} - 2V_{-2n-1} + V_2)$.

Proof. Take $r = 0, s = 1, t = 2$ in Theorem 3.1 in [16].

From the above proposition, we have the following corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $V_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

Corollary 19. *For $n \geq 1$, Jacobsthal-Padovan numbers have the following properties.*

(a) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{2}\left(-3Q_{-n-1} - 3Q_{-n-2} - 2Q_{-n-3} + 2\right).$

(b) $\sum_{k=1}^{n} Q_{-2k} = \frac{1}{2}\left(-Q_{-2n+1} + 1\right).$

(c) $\sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{2}\left(-Q_{-2n} - 2Q_{-2n-1} + 1\right).$

Taking $V_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Padovan-Lucas numbers.

Corollary 20. *For $n \geq 1$, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) numbers have the following properties.*

(a) $\sum_{k=1}^{n} L_{-k} = \frac{1}{2}\left(-3L_{-n-1} - 3L_{-n-2} - 2L_{-n-3} + 2\right).$

(b) $\sum_{k=1}^{n} L_{-2k} = \frac{-1}{2} L_{-2n+1}.$

(c) $\sum_{k=1}^{n} L_{-2k+1} = \frac{1}{2}\left(-L_{-2n} - 2L_{-2n-1} + 2\right).$

From the above proposition, we have the following corollary which gives sum formulas of adjusted Jacobsthal-Padovan numbers (take $V_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = 0$).

Corollary 21. *For $n \geq 1$, adjusted Jacobsthal-Padovan numbers have the following properties.*

(a) $\sum_{k=1}^{n} K_{-k} = \frac{1}{2}\left(-3K_{-n-1} - 3K_{-n-2} - 2K_{-n-3} + 1\right).$

(b) $\sum_{k=1}^{n} K_{-2k} = \frac{1}{2}\left(-K_{-2n+1} + 1\right).$

(c) $\sum_{k=1}^{n} K_{-2k+1} = \frac{1}{2}\left(-K_{-2n} - 2K_{-2n-1}\right).$

From the above proposition, we have the following corollary which gives sum formulas of modified Jacobsthal-Padovan numbers (take $V_n = M_n$ with $M_0 = 3, M_1 = 1, M_2 = 3$).

Corollary 22. *For $n \geq 1$, modified Jacobsthal-Padovan numbers have the following properties.*

(a) $\sum_{k=1}^{n} M_{-k} = \frac{1}{2}\left(-3M_{-n-1} - 3M_{-n-2} - 2M_{-n-3} + 4\right).$

(b) $\sum_{k=1}^{n} M_{-2k} = \frac{1}{2}\left(-M_{-2n+1} + 1\right).$

(c) $\sum_{k=1}^{n} M_{-2k+1} = \frac{1}{2}\left(-M_{-2n} - 2M_{-2n-1} + 3\right).$

7 Matrices related with Generalized Jacobsthal-Padovan numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{7.1}$$

For matrix formulation (7.1), see [9]. In fact, Kalman gives the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 2$. From (1.4) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = Q$ in (7.2) we have

$$\begin{pmatrix} Q_{n+2} \\ Q_{n+1} \\ Q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Q_{n+1} \\ Q_n \\ Q_{n-1} \end{pmatrix}. \tag{7.3}$$

We also define

$$B_n = \begin{pmatrix} \frac{1}{2}(Q_{n+3}-Q_{n+2}) & \frac{1}{2}(Q_{n+4}-Q_{n+3}) & Q_{n+2}-Q_{n+1} \\ \frac{1}{2}(Q_{n+2}-Q_{n+1}) & \frac{1}{2}(Q_{n+3}-Q_{n+2}) & Q_{n+1}-Q_{n} \\ \frac{1}{2}(Q_{n+1}-Q_{n}) & \frac{1}{2}(Q_{n+2}-Q_{n+1}) & Q_{n}-Q_{n-1} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} Q_{n+3}-Q_{n+2} & Q_{n+4}-Q_{n+3} & 2(Q_{n+2}-Q_{n+1}) \\ Q_{n+2}-Q_{n+1} & Q_{n+3}-Q_{n+2} & 2(Q_{n+1}-Q_{n}) \\ Q_{n+1}-Q_{n} & Q_{n+2}-Q_{n+1} & 2(Q_{n}-Q_{n-1}) \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} \frac{1}{2}(V_{n+3}-V_{n+2}) & \frac{1}{2}(V_{n+4}-V_{n+3}) & V_{n+2}-V_{n+1} \\ \frac{1}{2}(V_{n+2}-V_{n+1}) & \frac{1}{2}(V_{n+3}-V_{n+2}) & V_{n+1}-V_{n} \\ \frac{1}{2}(V_{n+1}-V_{n}) & \frac{1}{2}(V_{n+2}-V_{n+1}) & V_{n}-V_{n-1} \end{pmatrix}.$$

Theorem 23. *For all integers $m, n \geq 0$, we have*

(a) $B_n = A^n$.

(b) $C_1 A^n = A^n C_1$.

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (7.3) to 3-columns and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

NOTE: (a) can be proved by mathematical induction (using directly).

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$AC_{n-1} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(V_{n+2}-V_{n+1}) & \frac{1}{2}(V_{n+3}-V_{n+2}) & V_{n+1}-V_{n} \\ \frac{1}{2}(V_{n+1}-V_{n}) & \frac{1}{2}(V_{n+2}-V_{n+1}) & V_{n}-V_{n-1} \\ \frac{1}{2}(V_{n}-V_{n-1}) & \frac{1}{2}(V_{n+1}-V_{n}) & V_{n-1}-V_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}V_{n}-V_{n-1}+\frac{1}{2}V_{n+1} & \frac{1}{2}V_{n+1}-V_{n}+\frac{1}{2}V_{n+2} & V_{n}+V_{n-1}-2V_{n-2} \\ \frac{1}{2}V_{n+2}-\frac{1}{2}V_{n+1} & \frac{1}{2}V_{n+3}-\frac{1}{2}V_{n+2} & V_{n+1}-V_{n} \\ \frac{1}{2}V_{n+1}-\frac{1}{2}V_{n} & \frac{1}{2}V_{n+2}-\frac{1}{2}V_{n+1} & V_{n}-V_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(V_{n+3}-V_{n+2}) & \frac{1}{2}(V_{n+4}-V_{n+3}) & V_{n+2}-V_{n+1} \\ \frac{1}{2}(V_{n+2}-V_{n+1}) & \frac{1}{2}(V_{n+3}-V_{n+2}) & V_{n+1}-V_{n} \\ \frac{1}{2}(V_{n+1}-V_{n}) & \frac{1}{2}(V_{n+2}-V_{n+1}) & V_{n}-V_{n-1} \end{pmatrix} = C_n,$$

i.e., $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

Note that Theorem 23 is true for all integers m, n.

Some properties of matrix A^n can be given as

$$A^n = A^{n-2} + 2A^{n-3}$$

and

$$A^{n+m} = A^nA^m = A^mA^n$$

and

$$\det(A^n) = 2^n$$

for all integers m and n.

Theorem 24. *For $m, n \geq 0$, we have*

$$\begin{aligned} 2(V_{n+m+2}-V_{n+m+1}) = {} & (V_{n+2}-V_{n+3})(Q_{m+1}-Q_{m+2}) \\ & + (V_{n+1}-V_{n+2})(Q_{m+2}-Q_{m+3}) \\ & + 2(V_{n}-V_{n+1})(Q_{m}-Q_{m+1}). \end{aligned} \tag{7.4}$$

Proof. From the equation $C_{n+m} = C_nB_m = B_mC_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of

C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof.

Remark 25. *By induction, it can be proved that for all integers $m,n \leq 0$, (7.4) holds. So for all integers m,n, (7.4) is true.*

Corollary 26. *For all integers m,n, we have*

$$\begin{aligned}
2(Q_{n+m+2}-Q_{n+m+1}) &= (Q_{n+2}-Q_{n+3})(Q_{m+1}-Q_{m+2}) \\
&\quad +(Q_{n+1}-Q_{n+2})(Q_{m+2}-Q_{m+3}) \\
&\quad +2(Q_n-Q_{n+1})(Q_m-Q_{m+1}), \\
2(L_{n+m+2}-L_{n+m+1}) &= (L_{n+2}-L_{n+3})(Q_{m+1}-Q_{m+2}) \\
&\quad +(L_{n+1}-L_{n+2})(Q_{m+2}-Q_{m+3}) \\
&\quad +2(L_n-L_{n+1})(Q_m-Q_{m+1}), \\
2(K_{n+m+2}-K_{n+m+1}) &= (K_{n+2}-K_{n+3})(Q_{m+1}-Q_{m+2}) \\
&\quad +(K_{n+1}-K_{n+2})(Q_{m+2}-Q_{m+3}) \\
&\quad +2(K_n-K_{n+1})(Q_m-Q_{m+1}), \\
2(M_{n+m+2}-M_{n+m+1}) &= (M_{n+2}-M_{n+3})(Q_{m+1}-Q_{m+2}) \\
&\quad +(M_{n+1}-M_{n+2})(Q_{m+2}-Q_{m+3}) \\
&\quad +2(M_n-M_{n+1})(Q_m-Q_{m+1}).
\end{aligned}$$

Note that using Theorem 23 (a) and the property

$$2K_n = Q_{n+2}-Q_{n+1}$$

we see that

$$A^n = \begin{pmatrix} K_{n+1} & K_{n+2} & 2K_n \\ K_n & K_{n+1} & 2K_{n-1} \\ K_{n-1} & K_n & 2K_{n-2} \end{pmatrix} = B_n.$$

We define

$$E_n = \begin{pmatrix} V_{n+1} & V_{n+2} & 2V_n \\ V_n & V_{n+1} & 2V_{n-1} \\ V_{n-1} & V_n & 2V_{n-2} \end{pmatrix}.$$

In this case, Theorem 23, Theorem 24 and Corollary 26 can be given as follows:

Theorem 27. *For all integer m, n, we have*

(a) $B_n = A^n$.

(b) $E_1 A^n = A^n E_1$.

(c) $E_{n+m} = E_n B_m = B_m E_n$.

Theorem 28. *For all integers m, n, we have*

$$V_{n+m} = V_{n+1}K_m + V_n K_{m+1} + 2V_{n-1}K_{m-1}. \tag{7.5}$$

Corollary 29. *For all integers m, n, we have*

$$\begin{aligned} Q_{n+m} &= Q_{n+1}K_m + Q_n K_{m+1} + 2Q_{n-1}K_{m-1}, \\ L_{n+m} &= L_{n+1}K_m + L_n K_{m+1} + 2L_{n-1}K_{m-1}, \\ K_{n+m} &= K_{n+1}K_m + K_n K_{m+1} + 2K_{n-1}K_{m-1}, \\ M_{n+m} &= M_{n+1}K_m + M_n K_{m+1} + 2M_{n-1}K_{m-1}. \end{aligned}$$

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