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Applications of Certain Operators to the Classes of Analytic Functions Related to the Generalized Janowski Functions

Khalida Inayat Noor and Shujaat Ali Shah*

Abstract

We introduce certain subclasses of analytic functions related to the class of analytic, convex univalent functions. We discuss some results including inclusion relationships and invariance of the classes under convex convolution in terms of certain linear operators. Applications of these results associated with the generalized Janowski functions and conic domains are considered. Also, several radius problems are investigated.

1 Introduction

Let $\mathbf{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disk $E = \{z : |z| < 1\}$. If f and g are analytic in E, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function w in E such that $f(z) = g(w(z))$.

The convolution or Hadamard product of two functions $f, g \in \mathbf{A}$ is denoted by $f * g$ and is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in E. \tag{1.2}$$

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*Corresponding author

Analytic functions p in the class $P[A,B]$ can be defined by using subordination as follows [8].

Let p be analytic in E with $p(0)=1$. Then $p\in P[A,B]$, if and only if,

$$p(z)\prec\frac{1+Az}{1+Bz},\qquad -1\leq B<A\leq 1,\ z\in E.$$

Polatoglu et al. [23] introduced the class $P[A,B,\alpha]$ of analytic functions p, if and only if, there exists $p_1\in P[A,B]$ such that for $z\in E$,

$$p(z)=(1-\alpha)p_1(z)+\alpha,\qquad (0\leq\alpha<1). \tag{1.3}$$

The class $P[A_1,B_1]$ of analytic functions p with $p(z)\prec\frac{1+A_1z}{1+B_1z}$, $z\in E$, by using more general bilinear transformation $h(z)=\frac{1+A_1z}{1+B_1z}$, with $A_1\in\mathbb{C}$ (complex plane), $B_1\in[-1,0]$ and $A_1\neq B_1$.

If $A_1=[(1-\alpha)A+\alpha B]$ and $B_1=B$ for $0\leq\alpha<1$, and $-1\leq B<A\leq 1$, then the class $P[A_1,B_1]$ reduces to the class $P[A,B,\alpha]$, which is denoted by $P[A_1,B]$. See [13].

For $k\geqslant 0$, the conic domains Ω_k, defined as;

$$\Omega_k=\left\{u+iv:u>k\sqrt{(u-1)^2+v^2}\right\}.$$

The domains Ω_k $(k=0)$ represents right half plane, Ω_k $(0<k<1)$ represents hyperbola, Ω_k $(k=1)$ represents a parabola and Ω_k $(k>1)$ represents an ellipse. The extremal functions for these conic regions are given as

$$p_k(z)=\begin{cases}\frac{1+z}{1-z}, & k=0\\ 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k=1\\ 1+\frac{2}{1-k^2}\left[\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], & 0<k<1\\ 1+\frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right)+\frac{1}{k^2-1}, & k>1,\end{cases} \tag{1.4}$$

where $u(z)=\frac{z-\sqrt{t}}{z-\sqrt{tz}}$, $t\in(0,1)$, $z\in E$ and z is chosen such that $k=\cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is Legendre's complete elliptic integral of the first kind and

$R'(t)$ is complementary integral of $R(t)$; see [9, 10] for more information. These conic regions are being studied by several authors. See [1, 15, 17, 21].

Let $f \in \mathbf{A}$ and $D^m : \mathbf{A} \to \mathbf{A}$ be the operator defined by

$$D^m f(z) = \begin{cases} \frac{z}{(1-z)^{m+1}} * f(z); & m > -1 \\ \frac{z\left(z^{m-1} f(z)\right)^m}{m!} & m \in \mathbb{N}_0 = \{0, 1, 2, ...\} \end{cases}.$$

Note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. The operator $D^m f$ is called the Ruscheweyh derivative of mth order of f [24].

We can easily verify the following identity,

$$z(D^m f)' = (m+1)D^{m+1} f - mD^m f. \tag{1.5}$$

Analogous to the Ruscheweyh derivative operator, Noor [16] and Noor and Noor [19] defined an operator as follows:

Let

$$f_m(z) = \frac{z}{(1-z)^{m+1}}, \quad m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$$

and let $f_m^{(-1)}$ be defined such that

$$f_m(z) * f_m^{(-1)}(z) = \frac{z}{(1-z)}.$$

Then, the operator $I_m : \mathbf{A} \to \mathbf{A}$ is defined by

$$I_m f(z) = f_m^{(-1)}(z) * f(z) = \left(\frac{z}{(1-z)^{m+1}}\right)^{(-1)} * f(z).$$

The following identity can easily be verified,

$$z(I_{m+1} f)' = (m+1)I_m f - mI_{m+1} f. \tag{1.6}$$

The multiplier transformation [6] $\mathcal{L}_b^\delta : \mathbf{A} \to \mathbf{A}$ is defined as follows:

$$\mathcal{L}_b^\delta f(z) = \psi(\delta, b; z) * f(z) = z + \sum_{n=2}^{\infty} \left(\frac{b+n}{b+1}\right)^\delta a_n z^n,$$

where $\psi(\delta, b; z) = z + \sum\limits_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^\delta z^n$, ($b > -1$, δ be real, $z \in E$).

One can easily verify the following identity,

$$z(\mathcal{L}_b^{\delta} f(z))' = (1+b)\mathcal{L}_b^{\delta+1} f(z) - b\mathcal{L}_b^{\delta} f(z). \tag{1.7}$$

For further detail about above given operators, one may see [5, 6, 13, 16, 18, 19].

Dziok and Noor [7], introduced the concepts of some general classes as follows:

Let $\mathbf{A_0}$ be the class of functions $f \epsilon \mathbf{A}$ with $f(0) = 1$. Assume that μ, λ, ℓ be real parameters, $\mu \geq 0$, $\ell \geq 2$ and let $\Phi = (\phi, \varphi) \in \mathbf{A} \times \mathbf{A}$, $\xi \in \mathbf{A}$ and $H = (h_1, h_2)$, where h_i $(i = 1, 2)$ are analytic, univalent convex functions with $h_i(0) = 1$ $(i = 1, 2)$. Then

$$P(h) = \left\{q \in \mathbf{A_0} : q \prec h\right\},$$

$$P_\mu(H) = \left\{\mu q_1 + (1-\mu)q_2 : q_1 \prec h_1,\ q_2 \prec h_2\right\},$$
$$P_\mu((h,h)) = P_\mu(h) \text{ and } P_\mu(\frac{1+z}{1-z}) = P_\ell, \qquad \left(\mu = \frac{\ell}{4} + \frac{1}{2}\right)$$

where P_ℓ is the class introduced and studied by Pinchuk [22].

A function $f \in \mathbf{A}$ is said to be in the class $M_\mu^\lambda(\Phi, \xi, H)$, if and only if, $J_\lambda(f(z)) \in P_\mu(H)$, where

$$J_\lambda(f(z)) = (1-\lambda)\frac{\phi * \xi * f}{\varphi * \xi * f} + \lambda\frac{\phi * f}{\varphi * f}.$$

We denote by $W_\mu(\Phi, \xi, H) = M_\mu^0(\Phi, \xi, H)$, the class of functions $f \in \mathbf{A}$ such that

$$\frac{\phi * \xi * f}{\varphi * \xi * f} \in P_\mu(H).$$

Moreover, let us define

$$S^*(\varphi, \xi, h) = W_1((z\varphi', \varphi), \xi, h) \text{ and } C(\varphi, \xi, h) = W_1((\varphi_2, \varphi_1), \xi, h),$$

where $\varphi_1(z) = z\varphi'(z)$ and $\varphi_2(z) = z\varphi_1'$.

Definition 1. A function $f \in \mathbf{A}$ is in the class $S^*(\varphi, \xi, h)$, if and only if,

$$\frac{z(\varphi * \xi * f)'}{(\varphi * \xi * f)} \in P\left(h\right).$$

Definition 2. A function $f \in \mathbf{A}$ is in the class $C(\varphi, \xi, h)$, if and only if,

$$\frac{(z(\varphi * \xi * f)')'}{(\varphi * \xi * f)'} \in P\,(h)\,.$$

For different choices of φ, ξ and h, we will obtain well-known classes, see [4, 13, 14, 20, 27].

It is noted that

$$f \in C(\varphi, \xi, h) \Leftrightarrow zf' \in S^*(\varphi, \xi, h). \tag{1.8}$$

2 Preliminary Results

Lemma 1 ([12]). *Let h be analytic, univalent convex function in E with $h(0) = 1$ and $Re\,(\gamma h(z) + \sigma) > 0$, $\sigma, \gamma \in \mathbb{C}$ and $\gamma \neq 0$. If $p(z)$ is analytic in E and $p(0) = h(0)$, then*

$$\left\{p(z) + \frac{zp'(z)}{\gamma p(z) + \sigma}\right\} \prec h(z), \tag{2.1}$$

implies $p(z) \prec q(z) \prec h(z)$, where $q(z)$ is best dominant and is given as,

$$q(z) = \left[\left\{\int_0^1 \left(\exp \int_t^{tz} \frac{h(u)-1}{u} du\right) dt\right\}^{-1} - \frac{\sigma}{\gamma}\right].$$

Lemma 2 ([26]). *Let p be an analytic function in E with $p(0) = 1$ and $Re\{p(z)\} > 0$, $z \in E$. Then, for $\vartheta > 0$ and $\nu \neq -1$ (complex),*

$$Re\left\{p(z) + \frac{\vartheta zp'(z)}{p(z) + \nu}\right\} > 0, for\ |z| < r_0, \ \textit{where } r_0 \textit{ is given by}$$

$$r_0 = \frac{|\nu + 1|}{\sqrt{s + \sqrt{s^2 - |\nu^2 - 1|^2}}}, \quad s = 2\,(\vartheta + 1)^2 + |\nu|^2 - 1 \tag{2.2}$$

and this radius is best possible.

Lemma 3 ([25]). *If $f \in C, g \in S^*$, then for each h analytic in E with $h(0) = 1$,*

$$\frac{(f * hg)\,(E)}{(f * g)\,(E)} \subset \overline{Co}h(E), \tag{2.3}$$

where $\overline{Co}h(E)$ denotes the convex hull of $h(E)$.

3 Main Results

We assume throughout this paper $k \geq 0$, $b > -1$, δ be a real, $m \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$, $A_1 = [(1-\alpha)A + \alpha B]$, $B_1 = B$, $(-1 \leq B < A \leq 1)$ and $z \in E$, unless otherwise stated.

Theorem 1. *Let* $Re\{h(z) + b\} > 0$ *and* $Re\{h(z) + m\} > 0$. *Then*

$$\begin{aligned} S^*(\psi(\delta+1, b; z), f_m^{(-1)}, h) &\subset S^*(\psi(\delta, b; z), f_m^{(-1)}, h) \\ &\subset S^*(\psi(\delta, b; z), f_{m+1}^{(-1)}, h). \end{aligned}$$

Proof. Let $f \in S^*(\psi(\delta+1, b; z), f_m^{(-1)}, h)$.

For $I_m f(z) = f_m^{(-1)}(z) * f(z)$, we set

$$\frac{z\left[\mathcal{L}_a^\delta(I_m f(z))\right]'}{\left[\mathcal{L}_a^\delta(I_m f(z))\right]} = p(z), \tag{3.1}$$

where $p(z)$ is analytic with $p(0) = 1$.

Using identity (1.7) and (3.1), we have

$$(1+b)\frac{\mathcal{L}_b^{\delta+1}(I_m f(z))}{\mathcal{L}_b^\delta(I_m f(z))} = p(z) + b.$$

Logarithmic differentiating both sides, to get

$$\frac{z\left[\mathcal{L}_b^{\delta+1}(I_m f(z))\right]'}{\mathcal{L}_b^{\delta+1}(I_m f(z))} = p(z) + \frac{zp'(z)}{p(z)+b}. \tag{3.2}$$

Since $f \in S^*(\psi(\delta+1, b; z), f_m^{(-1)}, h)$, so from (3.2) we have

$$p(z) + \frac{zp'(z)}{p(z)+b} \prec h(z). \tag{3.3}$$

By applying Lemma 1, it conclude that, $p(z) \prec h(z)$ and consequently,

$$\frac{z\left[\mathcal{L}_a^\delta(I_m f(z))\right]'}{\left[\mathcal{L}_a^\delta(I_m f(z))\right]} \prec h(z).$$

This implies $f(z) \in S^*(\psi(\delta, b; z), f_m^{(-1)}, h)$.

Now suppose that $f \in S^*(\psi(\delta, b; z), f_m^{(-1)}, h)$.

For $\mathcal{L}_b^\delta f(z) = \psi(\delta, b; z) * f(z)$, we set

$$\frac{z\left[I_{m+1}\left(\mathcal{L}_a^\delta f(z)\right)\right]'}{I_{m+1}\left(\mathcal{L}_a^\delta f(z)\right)} = Q(z), \tag{3.4}$$

where $Q(z)$ is analytic with $Q(0) = 1$.

From (1.6) and (3.4), we have

$$(1+m)\frac{I_m\left(\mathcal{L}_b^\delta f(z)\right)}{I_{m+1}\left(\mathcal{L}_b^\delta f(z)\right)} = Q(z) + m.$$

Logarithmic differentiating both sides, to get

$$\frac{z\left[I_m\left(\mathcal{L}_a^\delta f(z)\right)\right]'}{I_m\left(\mathcal{L}_a^\delta f(z)\right)} = Q(z) + \frac{zQ'(z)}{Q(z)+m}. \tag{3.5}$$

Since $f \in S^*(\psi(\delta, b; z), f_m^{(-1)}, h)$, so from (3.5) we have

$$Q(z) + \frac{zQ'(z)}{Q(z)+m} \prec h(z). \tag{3.6}$$

By applying Lemma 1, it conclude that, $Q(z) \prec h(z)$ and consequently,

$$\frac{z\left[I_{m+1}\left(\mathcal{L}_a^\delta f(z)\right)\right]'}{I_{m+1}\left(\mathcal{L}_a^\delta f(z)\right)} \prec h(z).$$

This implies $f(z) \in S^*(\psi(\delta, b; z), f_{m+1}^{(-1)}, h)$. □

Theorem 2. *Let* $Re\,\{h(z)+b\} > 0$ *and* $Re\,\{h(z)+m\} > 0$. *Then*

$$\begin{aligned} C(\psi(\delta+1, b; z), f_m^{(-1)}, h) &\subset C(\psi(\delta, b; z), f_m^{(-1)}, h) \\ &\subset C(\psi(\delta, b; z), f_{m+1}^{(-1)}, h). \end{aligned}$$

Proof. Let

$$\begin{aligned} & f \in C(\psi(\delta+1, b; z), f_m^{(-1)}, h). \\ & \quad \Leftrightarrow zf' \in S^*(\psi(\delta+1, b; z), f_m^{(-1)}, h), \qquad \text{(by (1.8))} \\ & \quad \Rightarrow zf' \in S^*(\psi(\delta, b; z), f_m^{(-1)}, h), \qquad \text{(by Theorem 1)} \\ & \quad \Leftrightarrow f \in C(\psi(\delta, b; z), f_m^{(-1)}, h). \qquad \text{(by (1.8))} \end{aligned}$$

Similarly, we can prove $C(\psi(\delta, b; z), f_m^{(-1)}, h) \subset C(\psi(\delta, b; z), f_{m+1}^{(-1)}, h)$. □

We can deduce some main results as corollaries of the above theorems for different values of $h(z)$ given as bellow;

(a) $h(z) = \frac{1+A_1 z}{1+Bz}$.

(b) $h(z) = p_k(z)$, where $p_k(z)$ is defined by (1.4).

Theorem 3. *Let $f \in S^*(\varphi, \xi, h)$ and g be any convex univalent functions in E. Then $f * g \in S^*(\varphi, \xi, h)$.*

Proof. Let $f \in S^*(\varphi, \xi, h)$. Then for $F = \xi * f$.

Consider

$$\begin{aligned} \frac{z\left(\varphi * \xi * (g * f)\right)'}{\left(\varphi * \xi * (g * f)\right)} &= \frac{z\left(g * (\varphi * F)\right)'}{\left(g * (\varphi * F)\right)} \\ &= \frac{g * z\left(\varphi * F\right)'}{\left(g * (\varphi * F)\right)} \\ &= \frac{g * \frac{z(\varphi * F)'}{(\varphi * F)}\left(\varphi * F\right)}{\left(g * (\varphi * F)\right)}. \end{aligned}$$

Since $\varphi * F \in S^*(h) \subset S^*$, so by lemma 3, it follows that $f * g \in S^*(\varphi, \xi, h)$. □

Theorem 4. *Let $f \in C(\varphi, \xi, h)$ and g be any convex univalent functions in E. Then $f * g \in C(\varphi, \xi, h)$.*

Proof. We can easily prove this result by using Theorem 3 along with relation (1.8). □

We can deduce some special cases for the Theorem 3 and Theorem 4, for different choices of φ , ξ and h as given below:

(i) If $\varphi(z) = \psi(\delta, b; z)$, $\xi(z) = f_m^{(-1)}(z)$ and h is analytic, univalent convex function.

(ii) If $\varphi(z) = f_m(z)$, $\xi(z) = \frac{z}{(1-z)}$ and $h(z)$ is analytic, univalent convex function.

(iii) If $\varphi, \xi \in \mathbf{A}$ and $h(z) = \frac{1+A_1 z}{1+Bz}$.

(iv) $\varphi, \xi \in \mathbf{A}$ and $h(z) = p_k(z)$, where $p_k(z)$ is defined by (1.4).

We can apply Theorem 3 and Theorem 4 to prove Integral preserving properties for the classes $S^*(\varphi, \xi, h)$ and $C(\varphi, \xi, h)$.

Corollary 1. *The class $S^*(\varphi, \xi, h)$ and $C(\varphi, \xi, h)$ is closed under the following operators.*

(i) $f_1(z) = \int_0^z \frac{f(t)}{t} dt.$
(ii) $f_2(z) = \frac{2}{z} \int_0^z f(t)dt,$ (*Libera's operator* [11]).
(iii) $f_3(z) = \int_0^z \frac{f(t)-f(xt)}{t-xt} dt, \quad |x| \leq 1,\ x \neq 1.$
(iv) $f_4(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t), \quad Re(c) \geq 0,$ (*Generalized Bernardi operator* [3]).

Proof. We may write, $f_i(z) = f(z) * \phi_i(z)$, where $\phi_i(z)$, $i = 1, 2, 3, 4$, are convex and given by

$$\phi_1(z) = -\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n,$$
$$\phi_2(z) = \frac{-2[z - \log(1-z)]}{z} = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n,$$
$$\phi_3(z) = \frac{1}{1-x} \log\left(\frac{1-xz}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1-x^n}{(1-x)^n} z^n, \quad |x| \leq 1,\ x \neq 1,$$
$$\phi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \quad Re(c) \geq 0.$$

The proof follows easily by using Theorem 3 and Theorem 4. □

Now, we discuss some radius problems as follows:

Theorem 5. *Let $f \in S^*(\psi(\delta, b; z), f_m^{(-1)}, \frac{1+A_1 z}{1+Bz})$. Then*

$$f \in S^*(\psi(\delta+1, b; z), f_m^{(-1)}, \frac{1+A_1 z}{1+Bz}), \text{ for } |z| < r_\delta, \text{ where}$$

$$r_\delta = \frac{2(1+\delta)}{L + \sqrt{L^2 - 4M}}, \tag{3.7}$$

where $L = 3A_1^2 + \delta(A_1 + B) - B, \quad M = (1+\delta)\left(A_1^2 + \delta A_1 B\right)$. The value of r_δ is sharp.

Proof. Let $f \in S^*(\psi(\delta, b; z), f_m^{(-1)}, \frac{1+A_1 z}{1+Bz})$. Then, for $I_m f(z) = f_m^{(-1)}(z) * f(z)$

$$\frac{z\left[\mathcal{L}_a^\delta(I_m f(z))\right]'}{\left[\mathcal{L}_a^\delta(I_m f(z))\right]} = p(z), \tag{3.8}$$

where $p(z)$ is analytic with $p(0) = 1$. Using identity (1.7) and (3.8), we have

$$(1+b)\frac{\mathcal{L}_b^{\delta+1}\left(I_m f(z)\right)}{\mathcal{L}_b^{\delta}\left(I_m f(z)\right)} = p(z) + b.$$

Logarithmic differentiating both sides, to get

$$\frac{z\left[\mathcal{L}_b^{\delta+1}\left(I_m f(z)\right)\right]'}{\mathcal{L}_b^{\delta+1}\left(I_m f(z)\right)} = p(z) + \frac{zp'(z)}{p(z)+b}. \tag{3.9}$$

Since $\frac{1-A_1 r}{1-Br} \leq Rep(z) \leq \frac{1+A_1 r}{1+Br}$ and $\left|\frac{zp'(z)}{p(z)}\right| \geq \frac{(A_1-B)r}{(1-A_1 r)(1-Br)}$, (see [2, 23]), then (3.9) implies,

$$Re\left(p(z) + \frac{zp'(z)}{p(z)+b}\right) \geq Rep(z)\left[\frac{(1-A_1 r)\left\{(1-A_1 r)+\delta\left(1-Br\right)\right\} - \left(A_1 - B\right)r}{(1-A_1 r)\left\{(1-A_1 r)+\delta\left(1-Br\right)\right\}}\right].$$

The right hand side of above inequality is positive, for $|z| < r_\delta$, where r_δ is given by (3.7). $\square$

Corollary 2. *Let* $f \in S^*(\psi(\delta, b; z), f_m^{(-1)}, \frac{1+Az}{1+Bz})$. *Then*

$f \in S^*(\psi(\delta{+}1, b; z), f_m^{(-1)}, \dfrac{1+Az}{1+Bz})$, *for* $|z| < r_\delta$, *where*

$$r_\delta = \frac{2\left(1+\delta\right)}{L + \sqrt{L^2 - 4M}},$$

where $L = 3A^2 + \delta\left(A+B\right) - B$, $\quad M = \left(1+\delta\right)\left(A^2 + \delta AB\right)$. *The value of* r_δ *is sharp.*

Corollary 3. *Let* $f \in S^*(\psi(\delta, b; z), f_m^{(-1)}, \frac{1+z}{1-z})$. *Then*

$f \in S^*(\psi(\delta{+}1, b; z), f_m^{(-1)}, \dfrac{1+z}{1-z})$, *for* $|z| < r_\delta$, *where*

$$r_\delta = \frac{\left(1+\delta\right)}{2 + \sqrt{3 + \delta^2}}.$$

The value of r_δ *is sharp.*

Theorem 6. *Let $F_c \in S^*(\psi(\delta,b;z), f_m, \frac{1+A_1z}{1+Bz})$, where F_c is defined by*

$$F_c = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t).$$

Then

$\mathcal{L}_a^\delta(D^m f(z)) \in S^*\left(\frac{1+(1-2\beta)z}{1-z}\right)$, *for* $|z| < r_\beta$, *where*

$$r_\beta = \frac{|\nu+1|}{\sqrt{s+\sqrt{s^2-|\nu^2-1|^2}}}, \tag{3.10}$$

where $s = 2(\vartheta+1)^2 + |\nu|^2 - 1$, $\vartheta = \frac{1}{1-\beta}$, $\nu = \frac{\beta+c}{1-\beta}$, $\beta = \frac{1-A_1}{1-B}$. *The value of* r_β *is sharp.*

Proof. Let $F_c \in S^*(\psi(\delta,b;z), f_m, \frac{1+A_1z}{1+Bz})$. Then for $D^m F_c(z) = f_m(z) * F_c(z)$,

$$p(z) = \frac{z\left[\mathcal{L}_a^\delta(D^m F_c(z))\right]'}{\left[\mathcal{L}_a^\delta(D^m F_c(z))\right]} \in P\left(\frac{1+A_1z}{1+Bz}\right), \tag{3.11}$$

where $p(z)$ is analytic with $p(0)=1$.

Since $F_c = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)$, so by differentiating, we have

$$(z^c F_c(z))' = (1+c)z^{c-1}f(z),$$

$$zF_c'(z) + cF_c(z) = (1+c)f(z).$$

Taking convolution both sides of the above equation by $\psi(\delta,b;z) * f_m$, to get

$$z\left[\mathcal{L}_a^\delta(D^m F_c(z))\right]' + c\left[\mathcal{L}_a^\delta(D^m F_c(z))\right] = (1+c)\mathcal{L}_a^\delta(D^m f(z)),$$

$$\frac{z\left[\mathcal{L}_a^\delta(D^m F_c(z))\right]'}{\mathcal{L}_a^\delta(D^m F_c(z))} + c = (1+c)\frac{\mathcal{L}_a^\delta(D^m f(z))}{\mathcal{L}_a^\delta(D^m F_c(z))},$$

$$(1+c)\frac{\mathcal{L}_a^\delta(D^m f(z))}{\mathcal{L}_a^\delta(D^m F_c(z))} = p(z) + c. \qquad \text{(by (3.11))}$$

Logarithmic differentiating both sides, we get

$$\frac{z\left[\mathcal{L}_a^\delta(I_m f(z))\right]'}{\mathcal{L}_a^\delta(I_m f(z))} = p(z) + \frac{zp'(z)}{p(z)+c}. \tag{3.12}$$

Since $p(z) \in P\left(\frac{1+A_1 z}{1+Bz}\right) \subset P\left(\frac{1+(1-2\beta)z}{1-z}\right)$, so

$$p(z) = (1-\beta)\, p_1(z) + \beta, \qquad p_1 \in P, \tag{3.13}$$

where $\beta = \frac{1-A_1}{1-B}$.

From (3.12) and (3.13), we have

$$\frac{1}{(1-\beta)}\left[\frac{z\left[\mathcal{L}_a^{\delta}(D^m f(z))\right]'}{\mathcal{L}_a^{\delta}(D^m f(z))} - \beta\right] = p_1(z) + \frac{\frac{1}{(1-\beta)} z p_1'(z)}{p_1(z) + \frac{(c+\beta)}{(1-\beta)}}.$$

Take $\vartheta = \frac{1}{1-\beta} > 0$ and $\nu = \frac{\beta+c}{1-\beta} \neq -1$, then by applying Lemma 2, we conclude that

$$\frac{1}{(1-\beta)}\left[\frac{z\left[\mathcal{L}_a^{\delta}(D^m f(z))\right]'}{\mathcal{L}_a^{\delta}(D^m f(z))} - \beta\right] > 0, \quad \text{for } |z| < r_\beta \tag{3.14}$$

where r_β is given by (3.10).

Hence $\mathcal{L}_a^{\delta}(D^m f(z) \in S^*\left(\frac{1+(1-2\beta)z}{1-z}\right)$, for $|z| < r_\beta$, where r_β is given by (3.10). □

4 Conclusion

In the present work, we have introduced subclasses of analytic functions, which are generalization of many well-known classes. Several inclusion and inverse inclusion properties in terms of certain linear operators are discussed. We have proved preserving properties for these subclasses under convex convolution. Also, various applications of these results are deduced by using generalized Janowski functions and conic domains.

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Khalida Inayat Noor
COMSATS University, Islamabad, Pakistan
e-mail: khalida@gmail.com

Shujaat Ali Shah
COMSATS University, Islamabad, Pakistan
and
Quaid-e-Awam University of Engineering, Science and Technology,
Nawabshah, Pakistan
e-mail: shahglike@yahoo.com