



# On Commutators of Fuzzy Multigroups

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## Abstract

Fuzzy multigroup is an application of fuzzy multiset to group theory. Although, a lots have been done on the theory of fuzzy multigroups, some group's theoretic notions could still be investigated in fuzzy multigroup context. Certainly, the idea of commutator is one of such group's theoretic notions yet to be studied in the environment of fuzzy multigroups. Hence, the aim of this article is to establish the notion of commutator in fuzzy multigroup setting. A number of some related results are obtained and characterized. Among several results that are obtained, it is established that, if  $A$  and  $B$  are fuzzy submultigroups of a fuzzy multigroup  $C$ , then  $[A, B] \subseteq A \cup B$  holds. Some homomorphic properties of commutator in fuzzy multigroup context are discussed. The notion of admissible fuzzy submultisets  $A$  and  $B$  of  $C \in FMG(X)$  under an operator domain  $\mathcal{D}$  is explicated, and it is shown that  $(A, B)$  and  $[A, B]$  are  $\mathcal{D}$ -admissible.

## 1 Introduction

Fuzzy set theory proposed by Zadeh [26] (although with heated disagreement as at then) has been widely studied with applications ranging from engineering and computer science to medical diagnosis and social behavior, etc. In a way of extending the application of fuzzy sets to group theory, Rosenfeld [22] proposed the idea of fuzzy groups as an application of fuzzy sets to group theory and some number of results were obtained. Numerous studies have been carried out on some group theoretic notions in fuzzy group setting (cf. [1, 4, 13, 18, 19, 20]).

By synthesizing the concepts of fuzzy sets and multisets (cf. [14]), the idea of fuzzy multisets or fuzzy bags was proposed in [25] as a generalization of fuzzy sets

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in multiset framework. Many researches have been carried out on the theory of fuzzy multisets (cf. [3, 5, 11, 15, 16, 17, 24]). In recent times, the concept of fuzzy multigroups was introduced as an application of fuzzy multisets to group theory [23]. The ideas of abelian fuzzy multigroups and order of fuzzy multigroups have been studied with some results [2, 6], and the notions of center and centralizer in fuzzy multigroup context were established [6]. In the same vein, the concept of fuzzy multigroupoids was introduced and the idea of fuzzy submultigroups was explored with a number of results [7]. The concept of normal fuzzy submultigroups has been established and some of its properties were explicated [8]. In [9], the idea of homomorphism in the environment of fuzzy multigroups was defined and some homomorphic properties of fuzzy multigroups were elaborated. Some group analog concepts were established in fuzzy multigroup context, as seen in [10, 12]. The idea of fuzzy multigroups was extended to t-norms and some results were established [21].

The notion of commutator occupied a loft height in crisp group theory. Notwithstanding, this idea has not been investigated in fuzzy multigroup setting. Thus, an attempt to establish commutator in fuzzy multigroup setting is not only necessary but overdue. This constitutes the motivation of the article. In recap, this paper assayed to introduce commutator in fuzzy multigroups with some analog results. The remaining part of this article is thus presented: Section 2 provides some preliminaries on fuzzy multisets and fuzzy multigroups. In Section 3, the idea of commutator in fuzzy multigroups is proposed and some of its properties are discussed. Section 4 discusses some homomorphic properties of commutator in fuzzy multigroup context. Lastly, Section 5 concludes the paper and provides direction for future studies.

## 2 Preliminaries

**Definition 2.1** ([25]). Assume  $X$  is a set of elements. Then, a fuzzy bag/multiset  $A$  drawn from  $X$  is an object of the form

$$A = \{ \langle \frac{CM_A(x)}{x} \rangle \mid x \in X \}$$

defined by a count membership function

$$CM_A : X \rightarrow Q,$$

where  $Q$  is the set of all crisp bags or multisets from the unit interval  $I = [0, 1]$  and

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), \dots, \mu_A^n(x)\},$$

where  $\mu_A^1(x), \mu_A^2(x), \dots, \mu_A^n(x) \in [0, 1]$  such that  $\mu_A^1(x) \geq \mu_A^2(x) \geq \dots \geq \mu_A^n(x)$ .

From [24], a fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset  $A$  can be characterized by a function

$$CM_A : X \rightarrow N^I \text{ or } CM_A : X \rightarrow [0, 1] \rightarrow N,$$

where  $I = [0, 1]$  and  $N = \mathbb{N} \cup \{0\}$ .

We denote the set of all fuzzy multisets by  $FMS(X)$ .

**Definition 2.2** ([15]). Let  $A$  and  $B$  be fuzzy multisets of  $X$ . Then

- (i)  $A \subseteq B \Leftrightarrow CM_A(x) \leq CM_B(x) \forall x \in X$ .
- (ii)  $A = B \Leftrightarrow CM_A(x) = CM_B(x) \forall x \in X$ .
- (iii)  $A \subset B \Leftrightarrow A \subseteq B$  and  $A \neq B$ .
- (iv)  $A \cap B \Rightarrow CM_A(x) \wedge CM_B(x) \forall x \in X$ .
- (v)  $A \cup B \Rightarrow CM_A(x) \vee CM_B(x) \forall x \in X$ .

Note that  $\wedge$  and  $\vee$  denote minimum and maximum operations.

**Definition 2.3** ([23]). Suppose  $X$  is a group. Then, a fuzzy multiset  $A$  of  $X$  is called a fuzzy multigroup of  $X$  if the following conditions are satisfied:

- (i)  $CM_A(xy) \geq CM_A(x) \wedge CM_A(y) \forall x, y \in X$ ,
- (ii)  $CM_A(x^{-1}) \geq CM_A(x) \forall x \in X$ .

By implication, a fuzzy multiset  $A$  over  $X$  is called a fuzzy multigroup of a group  $X$  if

$$CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y), \forall x, y \in X.$$

It follows immediately from the definition that,

$$CM_A(e) \geq CM_A(x) \forall x \in X,$$

where  $e$  is the identity element of  $X$ . The second condition above is strictly  $CM_A(x^{-1}) = CM_A(x) \forall x \in X$ , since  $CM_A(x) = CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1})$ .

We denote the set of all fuzzy multigroups of  $X$  by  $FMG(X)$ .

**Definition 2.4** ([8]). Let  $A, B \in FMG(X)$ . Then, the product  $A \circ B$  is defined to be a fuzzy multiset of  $X$  as follows:

$$CM_{A \circ B}(x) = \begin{cases} \bigvee_{x=yz} [CM_A(y) \wedge CM_B(z)], & \text{if } \exists y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition 2.5** ([23]). Let  $A \in FMS(X)$ . Then  $A \in FMG(X)$  if and only if  $A \circ A = A$ .

**Definition 2.6** ([7]). Let  $A \in FMG(X)$ . A fuzzy submultiset  $B$  of  $A$  is called a fuzzy submultigroup of  $A$  denoted by  $B \subseteq A$  if  $B$  is a fuzzy multigroup. A fuzzy submultigroup  $B$  of  $A$  is a proper fuzzy submultigroup denoted by  $B \subset A$ , if  $B \subseteq A$  and  $A \neq B$ .

**Proposition 2.7** ([7]). Let  $A \in FMG(X)$ . Then, the sets  $A_*$  and  $A^*$  defined by

$$A_* = \{x \in X \mid CM_A(x) > 0\}$$

and

$$A^* = \{x \in X \mid CM_A(x) = CM_A(e)\}$$

are subgroups of  $X$ .

**Proposition 2.8.** Let  $A \in FMG(X)$ . Then, the set  $A_{[\alpha]}$  defined by

$$A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha, \alpha \in [0, 1]\}$$

is a subgroup of  $X$  for  $\alpha \leq CM_A(e)$  and  $A^{[\alpha]}$  defined by

$$A^{[\alpha]} = \{x \in X \mid CM_A(x) > \alpha, \alpha \in [0, 1]\}$$

is a subgroup of  $X$  for  $\alpha \geq CM_A(e)$ .

**Definition 2.9** ([8]). Let  $A, B \in FMG(X)$  such that  $A \subseteq B$ . Then,  $A$  is called a normal fuzzy submultigroup of  $B$  if for all  $x, y \in X$ ,

$$CM_A(xyx^{-1}) = CM_A(y).$$

**Definition 2.10** ([6]). Let  $A \in FMG(X)$ . Then,  $A$  is said to be commutative if for all  $x, y \in X$ ,

$$CM_A(xy) = CM_A(yx).$$

**Definition 2.11** ([9]). Let  $X$  and  $Y$  be groups and let  $f : X \rightarrow Y$  be a homomorphism. Suppose  $A$  and  $B$  are fuzzy multigroups of  $X$  and  $Y$ , respectively. Then,  $f$  induces a homomorphism from  $A$  to  $B$  which satisfies

$$(i) \quad CM_A(f^{-1}(y_1y_2)) \geq CM_A(f^{-1}(y_1)) \wedge CM_A(f^{-1}(y_2)) \quad \forall y_1, y_2 \in Y,$$

$$(ii) \quad CM_B(f(x_1x_2)) \geq CM_B(f(x_1)) \wedge CM_B(f(x_2)) \quad \forall x_1, x_2 \in X,$$

where

(i) the image of  $A$  under  $f$ , denoted by  $f(A)$ , is a fuzzy multiset over  $Y$  defined by

$$CM_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} CM_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each  $y \in Y$ .

(ii) the inverse image of  $B$  under  $f$ , denoted by  $f^{-1}(B)$ , is a fuzzy multiset over  $X$  defined by

$$CM_{f^{-1}(B)}(x) = CM_B(f(x)) \quad \forall x \in X.$$

**Theorem 2.12** ([9]). Let  $X, Y$  be two groups and  $f : X \rightarrow Y$  be a homomorphism. If  $A \in FMG(X)$  and  $B \in FMG(Y)$ , respectively, then  $f(A) \in FMG(Y)$  and  $f^{-1}(B) \in FMG(X)$ .

**Definition 2.13** ([12]). Let  $A$  be a fuzzy submultigroup of  $B \in FMG(X)$ . Then,  $A$  is called a characteristic (f-invariant) fuzzy submultigroup of  $B$  if

$$CM_{A^\theta}(x) = CM_A(x) \forall x \in X$$

for every automorphism,  $\theta$  of  $X$ . That is,  $\theta(A) \subseteq A$  for every  $\theta \in Aut(X)$ .

It follows from [12] that, every characteristic fuzzy submultigroup of a fuzzy multigroup is normal.

### 3 Commutator of fuzzy multigroups

Recall that the commutator of two elements  $x$  and  $y$  of a group  $X$  is the element  $[x, y] = x^{-1}y^{-1}xy \in X$ . If  $H$  and  $K$  are subgroups of  $X$ , then the commutator subgroup or derived subgroup  $[H, K]$  of  $X$  is generated by  $\{[x, y] | x \in H, y \in K\}$ . Now, the idea of commutator in fuzzy multigroup context is introduced.

**Definition 3.1.** Let  $A$  and  $B$  be fuzzy submultigroups of  $C \in FMG(X)$ . Then,  $(A, B)$  is a fuzzy multiset of  $X$  defined as follows:  $\forall x \in X$

$$CM_{(A,B)}(x) = \begin{cases} \bigvee_{x=[a,b]} [CM_A(a) \wedge CM_B(b)], & \text{if } x \text{ is a commutator in } X \\ 0, & \text{otherwise.} \end{cases}$$

The commutator of  $A$  and  $B$  is a fuzzy multigroup  $[A, B]$  of  $X$  generated by  $(A, B)$ .

Now, we present some properties of commutator in fuzzy multigroup context as follow:

**Proposition 3.2.** Suppose  $A \in FMG(X)$  and  $[x, y]$  is a commutator of  $x$  and  $y$  in  $A$ . Then  $CM_A([x, y]) = CM_A(e)$  if and only if  $A$  is commutative.

*Proof.* Assume  $CM_A([x, y]) = CM_A(e)$ , where  $e$  is the identity element in  $A$ . Then  $CM_A(x^{-1}y^{-1}xy) = CM_A(e) \Rightarrow CM_A((yx)^{-1}xy) = CM_A(e) \Rightarrow CM_A(xy) = CM_A(yx)$ . Thus,  $A$  is commutative.

Conversely, if  $CM_A(xy) = CM_A(yx)$ . Then, we have

$$\begin{aligned} CM_A((yx)^{-1}xy) &= CM_A((yx)^{-1}yx) \\ &= CM_A((x^{-1}x)(y^{-1}y)) \\ &\geq CM_A(x^{-1}x) \wedge CM_A(y^{-1}y) \\ &= CM_A(e), \end{aligned}$$

Thus,  $CM_A(x^{-1}y^{-1}xy) = CM_A([x, y]) \geq CM_A(e)$ . Again, we have  $CM_A(e) = CM_A((yx)^{-1}yx) \geq CM_A([x, y])$ , since  $CM_A(e) \geq CM_A(x) \forall x \in X$ . Thus,  $CM_A([x, y]) = CM_A(e)$ . The result follows.  $\square$

**Proposition 3.3.** *Suppose  $A \in FMG(X)$ . If  $A$  is commutative, then  $CM_A([x, y]^{-1}) = CM_A([y, x]) \forall x, y \in X$ .*

*Proof.* Suppose  $A$  is commutative. Then, we get

$$\begin{aligned} CM_A([x, y]^{-1}) \geq CM_A([x, y]) &= CM_A(x^{-1}y^{-1}xy) \\ &= CM_A(y^{-1}x^{-1}yx) \\ &= CM_A([y, x]), \end{aligned}$$

$\Rightarrow CM_A([x, y]^{-1}) \geq CM_A([y, x])$ . Similarly,

$$\begin{aligned} CM_A([y, x]) = CM_A((([y, x]^{-1})^{-1}) &\geq CM_A([y, x]^{-1}) \\ &= CM_A((y^{-1}x^{-1}yx)^{-1}) \\ &= CM_A((x^{-1}y^{-1}xy)^{-1}) \\ &= CM_A([x, y]^{-1}), \end{aligned}$$

$\Rightarrow CM_A([y, x]) \geq CM_A([x, y]^{-1})$ . Hence,  $CM_A([x, y]^{-1}) = CM_A([y, x]) \forall x, y \in X$ .  $\square$

**Remark 3.4.** If a fuzzy multigroup  $A$  of  $X$  is commutative, then it is easy to see that  $CM_A(z[x, y]z^{-1}) = CM_A([zxxz^{-1}, zyyz^{-1}]) \forall x, y, z \in X$ .

**Proposition 3.5.** *Let  $x, y, z \in X$  and  $A$  be a commutative fuzzy multigroup of  $X$ . Then*

$$(i) \quad CM_A([xy, z]) = CM_A([x, z]^y[y, z]),$$

$$(ii) \quad CM_A([x, yz]) = CM_A([x, z][x, y]^z).$$

*Proof.* For  $x, y, z \in X$ , we have

(i)

$$\begin{aligned} CM_A([xy, z]) &= CM_A((xy)^{-1}z^{-1}xyz) \\ &= CM_A(y^{-1}x^{-1}z^{-1}xyz) \\ &= CM_A(y^{-1}x^{-1}z^{-1}xzy) \\ &= CM_A([x, z]^y). \end{aligned}$$

Similarly,

$$\begin{aligned} CM_A([x, y]^y[y, z]) &= CM_A((y^{-1}x^{-1}y^{-1}xyy)[y, z]) \\ &= CM_A([x, z]^y). \end{aligned}$$

Hence, the result.

(ii)

$$\begin{aligned} CM_A([x, yz]) &= CM_A(x^{-1}(yz)^{-1}xyz) \\ &= CM_A(x^{-1}z^{-1}y^{-1}xyz) \\ &= CM_A([x, y]^z) \end{aligned}$$

and similarly, we have

$$\begin{aligned} CM_A([x, z][x, y]^z) &= CM_A(x^{-1}z^{-1}y^{-1}xyz) \\ &= CM_A([x, y]^z). \end{aligned}$$

The result follows. □

**Lemma 3.6.** *If  $x, y, z \in X$  and  $A$  is a commutative fuzzy multigroup of  $X$ , we have*



$$(i) \quad CM_A([xy, z]) = CM_A([x, z][x, z, y][y, z]),$$

$$(ii) \quad CM_A([x, yz]) = CM_A([x, z][x, y][x, y, z]).$$

*Proof.* Straightforward from Proposition 3.5. □

**Proposition 3.7.** *Let  $x, y, z \in X$ , and  $A$  be a fuzzy multigroup of  $X$ . Then, we have  $CM_A([x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x) = CM_A(e)$ , where  $e$  is the identity of  $X$ .*

*Proof.* For  $x, y, z \in X$ , we have

$$\begin{aligned} CM_A([x, y^{-1}, z]^y) &= CM_A(y^{-1}[x^{-1}yxy^{-1}, z]y) \\ &= CM_A(y^{-1}(yx^{-1}y^{-1}x)z^{-1}(x^{-1}yxy^{-1})zy) \\ &= CM_A(x^{-1}y^{-1}xz^{-1}x^{-1}yxy^{-1}zy). \end{aligned}$$

Setting  $a = xzx^{-1}yx$ ,  $b = yxy^{-1}zy$  and  $c = zyz^{-1}xz$ . By observation,  $b$  and  $c$  can be obtained by cyclic permutation of  $x, y, z$ . Furthermore, we see that  $CM_A([x, y^{-1}, z]^y) = CM_A(a^{-1}b)$ . Deducibly, it follows that  $CM_A([y, z^{-1}, x]^z) = CM_A(b^{-1}c)$  and  $CM_A([z, x^{-1}, y]^x) = CM_A(c^{-1}a)$ . Since

$$CM_A((a^{-1}b)(b^{-1}c)(c^{-1}a)) = CM_A(e),$$

the result follows. □

**Theorem 3.8.** *Let  $x, y \in X$ ,  $z = [x, y]$  commutes with both  $x$  and  $y$ . If  $A$  is a fuzzy multigroup of  $X$ , then  $CM_A([x^i, y^j]) = CM_A(z^{ij}) \forall i, j$ .*

*Proof.* By a given hypothesis,  $CM_A(z) = CM_A(x^{-1}y^{-1}xy)$ , and so  $CM_A(y^{-1}xy) = CM_A(xz)$ , where

$$\begin{aligned} CM_A(y^{-1}x^i y) = CM_A((y^{-1}xy)^i) &= CM_A((xz)^i) \\ &= CM_A(x^i z^i) \end{aligned}$$

as  $x$  and  $z$  commute. Conjugating by  $y$  gives

$$\begin{aligned} CM_A(y^{-2}x^i y^2) = CM_A(y^{-1}x^i z^i y) &= CM_A(y^{-1}x^i y z^i) \\ &= CM_A((x^i z^i) z^i) \\ &= CM_A(x^i z^{2i}) \end{aligned}$$

as  $y$  and  $z$  commute. Repeating this argument  $j$  times, we have  $CM_A(y^{-j}x^iy^j) = CM_A(x^iz^{ij})$ , hence  $CM_A([x^i, y^j]) = CM_A(z^{ij})$ .  $\square$

**Theorem 3.9.** *Let  $A \in FMG(X)$  and  $x, y, z \in X$ . If  $y$  commutes with  $z$  in  $A$  and  $X$  is abelian, then  $CM_A([x, y, z]) = CM_A([x, z, y])$ .*

*Proof.* To start with, we have

$$\begin{aligned} CM_A([x, y, z]) &= CM_A([[x, y], z]) \\ &= CM_A([x, y]^{-1}z^{-1}[x, y]z) \\ &= CM_A(y^{-1}x^{-1}yxz^{-1}x^{-1}y^{-1}xyz) \\ &= CM_A(x^{-1}(xy^{-1}x^{-1}y)(xz^{-1}x^{-1}z)z^{-1}y^{-1}xyz). \end{aligned}$$

We observe that  $xy^{-1}x^{-1}y$  and  $xz^{-1}x^{-1}z$  lies in  $X$ , and thus commute. It follows that

$$CM_A([x, y, z]) = CM_A(x^{-1}(xz^{-1}x^{-1}z)(xy^{-1}x^{-1}y)z^{-1}y^{-1}xyz).$$

Since  $y$  and  $z$  commute, then

$$\begin{aligned} CM_A([x, y, z]) &= CM_A(z^{-1}x^{-1}zxy^{-1}x^{-1}z^{-1}xzy) \\ &= CM_A([x, z, y]). \end{aligned}$$

This completes the proof.  $\square$

**Theorem 3.10.** *Let  $A \in FMG(X)$  and  $x, y, z \in X$ . If  $[x, y]$  commutes with both  $x$  and  $y$ , then*

$$CM_A([x, y]^{-1}) = CM_A([x^{-1}, y]) = CM_A([x, y^{-1}]).$$

*Proof.* By synthesizing Lemma 3.6, we have

$$CM_A(e) = CM_A([xx^{-1}, y]) = CM_A([x, y][x, y, x^{-1}][x^{-1}, y]).$$

But,  $[x, y, x^{-1}] = [[x, y], x^{-1}]$  and  $[x, y]$  commutes with  $x$  by hypothesis. It follows that  $CM_A([x, y, x^{-1}]) = CM_A(e) \Rightarrow [x, y, x^{-1}] = e$ , where  $e$  is the identity of  $X$ . Thus,

$$CM_A(e) = CM_A([x, y][x^{-1}, y]) \Rightarrow CM_A([x, y]^{-1}) = CM_A([x^{-1}, y]).$$

Similarly, we have  $CM_A([x, y]^{-1}) = CM_A([x, y^{-1}])$ . Hence, the equality holds.  $\square$

**Definition 3.11.** Let  $A, B \in FMG(X)$  such that  $A \subseteq B$ . Define  $A^{(0)} = A$ . For  $n \in \mathbb{N}$  and suppose  $A^{(n-1)}$  can be defined for  $n \geq 1$ . Then,  $A^{(n)}$  can be defined by

$$A^{(n)} = (A^{(n-1)}, A^{(n-1)}).$$

**Theorem 3.12.** If  $A$  is a fuzzy multigroup of  $X$ , then  $A^{(n)} \subseteq A^{(n-1)}$  for all  $n \in \mathbb{N}$ .

*Proof.* We establish the proof by induction on  $n$ . Let  $x \in X$ . If  $x$  is not a commutator in  $X$ . Then

$$CM_{(A,A)} = 0 \text{ and } CM_A(x) \geq 0 \Rightarrow CM_{(A,A)} = 0 \leq CM_A(x).$$

Suppose  $x$  is a commutator in  $X$ . Then, for  $a, b \in X$  we have

$$\begin{aligned} CM_{(A,A)} &= \bigvee_{x=[a,b]} [CM_A(a) \wedge CM_A(b)] \\ &= \bigvee_{x=[a,b]} [CM_A(a^{-1}) \wedge CM_A(b^{-1}) \wedge CM_A(a) \wedge CM_A(b)] \\ &\leq \bigvee_{x=[a,b]} [CM_A(a^{-1}b^{-1}ab)] \\ &= CM_A(x), \end{aligned}$$

and so  $(A, A) \subseteq A$ . Since  $A \in FMG(X)$ , then

$$CM_{A^{(1)}}(x) = CM_{(A^{(0)}, A^{(0)})}(x) \leq CM_A(x) = CM_{A^{(0)}}(x),$$

and so the result follows for  $k = 1$ . Since  $CM_{A^{(n)}}(x) \leq CM_{A^{(n-1)}}(x)$  for  $n \in \mathbb{N}$ , we have

$$\begin{aligned} CM_{A^{(k+1)}}(x) &= CM_{(A^{(k)}, A^{(k)})}(x) \\ &\leq CM_{(A^{(k-1)}, A^{(k-1)})}(x) \\ &= CM_{A^k}(x). \end{aligned}$$

Thus,  $A^{(k+1)} \subseteq A^{(k)} \Rightarrow A^{(k)} \subseteq A^{(k-1)}$ . Hence, the result follows for  $n = k$ .

□

**Theorem 3.13.** *If  $A$  and  $B$  are normal (characteristic, fully invariant) fuzzy submultigroups of a fuzzy multigroup  $C$  of  $X$ . Then  $[A, B]$  is a normal (characteristic, fully invariant) fuzzy submultigroup of  $C$  contained in  $A \cap B$ .*

*Proof.* Suppose  $A$  and  $B$  are normal (characteristic, fully invariant) fuzzy submultigroups of  $C$ , then  $[A, B]$  is a normal (characteristic, fully invariant) fuzzy submultigroups of  $C$ . Because  $A$  and  $B$  are normal and so, they assume constant value on the conjugacy classes of  $X$ .

Since  $A \cap B$  is a fuzzy multigroup of  $X$ , it suffices to prove that  $(A, B) \subseteq A \cap B$ . Let  $x \in X$ . If  $x$  is not a commutator, then

$$CM_{(A,B)}(x) = 0 \leq CM_{A \cap B}(x).$$

Suppose that  $x = aba^{-1}b^{-1}$  for some  $a, b \in X$ . Then

$$\begin{aligned} CM_{A \cap B}(x) &= CM_A(x) \wedge CM_B(x) \\ &= CM_A(aba^{-1}b^{-1}) \wedge CM_B(aba^{-1}b^{-1}) \\ &\geq [CM_A(a) \wedge CM_A(ba^{-1}b^{-1})] \wedge [CM_B(aba^{-1}) \wedge CM_B(b^{-1})] \\ &\geq [CM_A(a) \wedge CM_A(a^{-1})] \wedge [CM_B(b) \wedge CM_B(b^{-1})] \\ &= CM_A(a) \wedge CM_B(b). \end{aligned}$$

This implies that

$$\begin{aligned} CM_{A \cap B}(x) &\geq \bigvee_{x=aba^{-1}b^{-1}} CM_A(a) \wedge CM_B(b) \\ &= CM_{(A,B)}(x). \end{aligned}$$

Thus,  $(A, B) \subseteq A \cap B$ . Consequently,  $[A, B] \subseteq A \cap B$ . □

**Corollary 3.14.** *Using the same hypothesis in Theorem 3.13, then  $[A, B]$  is a normal (characteristic, fully invariant) fuzzy submultigroup of  $C$  contained in  $A \cup B$ .*

*Proof.* By Definition 2.2, we get  $A \cap B \subseteq A \cup B$ . Hence, the result follows by transitivity because  $[A, B] \subseteq A \cap B$ . □

**Theorem 3.15.** *Suppose  $A$  and  $B$  are fuzzy submultigroups of  $C \in FMG(X)$ . Then  $[A, B] = [B, A]$ .*

*Proof.* Let  $x \in X$ . Since  $[A, B]$  is a fuzzy multigroup generated by  $(A, B)$ , if we prove that  $(A, B) = (B, A)$  we are done. Assume  $x$  is not a commutator in  $X$ , then  $x^{-1}$  is not a commutator and consequently,

$$CM_{(A,B)}(x) = 0 = CM_{(B,A)}(x^{-1}).$$

Suppose  $x = [a, b]$  for some  $a, b \in X$ . Then

$$\begin{aligned} CM_{(A,B)}(x) &= \bigvee_{x=[a,b]} [CM_A(a) \wedge CM_B(b)] \\ &= \bigvee_{x^{-1}=[b,a]} [CM_B(b) \wedge CM_A(a)] \\ &= CM_{(B,A)}(x^{-1}). \end{aligned}$$

Hence,  $(A, B) = (B, A)$  and the result follows. □

**Corollary 3.16.** *If  $A$  and  $B$  are fuzzy submultisets of  $C \in FMG(X)$ . Then*

$$(i) [A, B]_* = [B, A]_*,$$

$$(ii) [A, B]^* = [B, A]^*.$$

*Proof.* Using Proposition 2.7 and Theorem 3.15, the results follow. □

**Corollary 3.17.** *If  $A$  and  $B$  are fuzzy submultisets of  $C \in FMG(X)$ . Then, for  $\alpha \in [0, 1]$ ,*

$$(i) [A, B]_{[\alpha]} = [B, A]_{[\alpha]},$$

$$(ii) [A, B]^{[\alpha]} = [B, A]^{[\alpha]}.$$

*Proof.* Combining Proposition 2.8 and Theorem 3.15, the results follow. □

**Theorem 3.18.** *If  $A \in FMG(X)$ , then  $[A, A] \subseteq A$ .*

*Proof.* If  $x$  is not a commutator in  $X$ , then  $CM_{(A,A)}(x) = 0$ . Suppose  $x$  is a commutator, then

$$\begin{aligned}
 CM_{(A,A)}(x) &= \bigvee_{x=[a,b]} [CM_A(a) \wedge CM_A(b)] \text{ for some } a, b \in X \\
 &= \bigvee_{x=[a,b]} [CM_A(a^{-1}) \wedge CM_A(b^{-1}) \wedge CM_A(a) \wedge CM_A(b)] \\
 &\leq \bigvee_{x=[a,b]} [CM_A(a^{-1}b^{-1}ab)] \\
 &= CM_A(x).
 \end{aligned}$$

Thus,  $(A, A) \subseteq A \Rightarrow [A, A] \subseteq A$ . □

**Theorem 3.19.** *If  $A, B, C, D \in FMG(X)$  such that  $A \subseteq B$  and  $C \subseteq D$ , then  $[A, C] \subseteq [B, D]$ .*

*Proof.* Recall that  $[A, C] = \langle\langle A, C \rangle\rangle$ . If  $x$  is not a commutator in  $X$ , then  $CM_{(A,B)}(x) = 0$  and therefore there is nothing to prove. Suppose  $x$  is a commutator, then

$$\begin{aligned}
 CM_{(A,C)}(x) &= \bigvee_{x=[a,b]} [CM_A(a) \wedge CM_C(b)] \text{ for some } a, b \in X \\
 &\leq \bigvee_{x=[a,b]} [CM_B(a) \wedge CM_D(b)] \\
 &= CM_{(B,D)}(x).
 \end{aligned}$$

Thus,  $[A, C] = \langle\langle A, C \rangle\rangle \subseteq \langle\langle B, D \rangle\rangle = [B, D]$ . Hence,  $[A, C] \subseteq [B, D]$ . □

**Corollary 3.20.** *Let  $A, B, C, D, E, F \in FMG(X)$ . If  $[C, D] \subseteq [A, B]$  and  $[E, F] \subseteq [A, B]$ , then  $[[C, D], [E, F]] \subseteq [A, B]$ .*

*Proof.* Suppose  $[C, D] \subseteq [A, B]$  and  $[E, F] \subseteq [A, B]$ . Then  $[[C, D], [E, F]] \subseteq [[A, B], [A, B]] \subseteq [A, B]$ . Hence,  $[[C, D], [E, F]] \subseteq [A, B]$ . □

**Proposition 3.21.** *Let  $A, B \in FMG(X)$ . Then  $[A, B] \circ [B, A] = [A, B]$ .*

*Proof.* Combining Proposition 2.5 and Theorem 3.15, we get  $[A, B] \circ [B, A] \subseteq [A, B]$ . Again, if  $x = 0$ , we have  $CM_{[A,B] \circ [B,A]}(x) = 0$ . Otherwise,

$$\begin{aligned} CM_{[A,B] \circ [B,A]}(x) &= \bigvee_{x=ab} [CM_{[A,B]}(a) \wedge CM_{[B,A]}(b)] \\ &\geq CM_{[A,B]}(x) \wedge CM_{[A,B]}(e) \\ &= CM_{[A,B]}(x), \end{aligned}$$

and so,  $[A, B] \subseteq [A, B] \circ [B, A]$ . Hence,  $[A, B] \circ [B, A] = [A, B]$ . □

**Proposition 3.22.** *Let  $A, B, C, D, E, F \in FMG(X)$  such that  $[C, D] \subseteq [A, B]$  and  $[E, F] \subseteq [A, B]$ . Then  $[C, D] \circ [E, F] \subseteq [A, B]$ .*

*Proof.* Let  $x \in X$ . If  $x = 0$ , we have  $CM_{[C,D] \circ [E,F]}(x) = 0$ . Otherwise,

$$\begin{aligned} CM_{[C,D] \circ [E,F]}(x) &= \bigvee_{x=ab} [CM_{[C,D]}(a) \wedge CM_{[E,F]}(b)] \\ &\leq \bigvee_{x=ab} [CM_{[A,B]}(a) \wedge CM_{[A,B]}(b)] \\ &= CM_{[A,B] \circ [A,B]}(x) \\ &= CM_{[A,B]}(x). \end{aligned}$$

Hence,  $[C, D] \circ [E, F] \subseteq [A, B]$ . □

**Lemma 3.23.** *Suppose  $A$  and  $B$  are fuzzy submultigroups of  $D \in FMG(X)$  such that  $A \subseteq B$ , then  $[A, C] \subseteq [B, C]$  for any fuzzy submultigroup  $C$  of  $D$ .*

*Proof.* Given  $A \subseteq B$ , then  $CM_A(x) \leq CM_B(x) \forall x \in X$ . Let  $C$  be any fuzzy submultigroup of  $D$ . If  $x$  is not a commutator in  $X$ , then

$$CM_{(A,C)}(x) = 0 = CM_{(B,C)}(x).$$

Suppose  $x = [a, b]$  for some  $a, b \in X$ . Then, we have

$$\begin{aligned} CM_{(A,C)}(x) &= \bigvee_{x=[a,b]} [CM_A(a) \wedge CM_C(b)] \\ &\leq \bigvee_{x=[a,b]} [CM_B(a) \wedge CM_C(b)] \\ &= CM_{(B,C)}(x), \end{aligned}$$

implies that  $(A, C) \subseteq (B, C)$  and so  $[A, C] \subseteq [B, C]$ . □

**Theorem 3.24.** *Let  $A$  and  $B$  be normal fuzzy submultigroups of  $D \in FMG(X)$  and let  $C$  be any fuzzy submultigroup of  $D$ . Then  $[A \circ C, B] \subseteq [A, B] \circ [C, B]$  with equality holding if  $CM_A(e) = CM_C(e)$ , where  $e$  is the identity of  $X$ .*

*Proof.* Firstly, we show that  $(A \circ C, B) \subseteq [A, B] \circ [C, B]$ . Let  $x \in X$ . If  $x$  is not a commutator in  $X$ , then  $CM_{(A \circ C, B)}(x) = 0$  and the result is trivial. Suppose  $x$  is a commutator in  $X$ . Then

$$\begin{aligned}
 CM_{(A \circ C, B)}(x) &= \bigvee \{ CM_{A \circ C}(a) \wedge CM_B(b) \mid x = [a, b], a, b \in X \} \\
 &= \bigvee \{ \bigvee \{ CM_A(u) \wedge CM_C(v) \mid a = uv, u, v \in X \} \wedge CM_B(b) \mid x = [a, b], a, b \in X \} \\
 &= \bigvee \{ \bigvee \{ [CM_A(u) \wedge CM_C(v)] \wedge CM_B(b) \mid a = uv, u, v \in X \} \mid x = [a, b], a, b \in X \} \\
 &= \bigvee \{ \bigvee \{ [CM_A(u) \wedge CM_B(b)] \wedge [CM_C(v) \wedge CM_B(b)] \mid a = uv, u, v \in X \} \mid x = [a, b], a, b \in X \} \\
 &= \bigvee \{ [CM_A(u) \wedge CM_B(b)] \wedge [CM_C(v) \wedge CM_B(b)] \mid x = [uv, b], uv, b \in X \} \\
 &\leq \bigvee \{ CM_{[A, B]}([u, b]) \wedge CM_{[C, B]}([v, b]) \mid x = [uv, b], uv, b \in X \} \\
 &= \bigvee \{ CM_{[A, B]}([u, b]^v) \wedge CM_{[C, B]}([v, b]) \mid x = [uv, b], uv, b \in X \} \\
 &\leq \bigvee \{ CM_{[A, B]}(y) \wedge CM_{[C, B]}(z) \mid x = yz, y, z \in X \} \\
 &= CM_{[A, B] \circ [C, B]}(x),
 \end{aligned}$$

where  $y = [u, b]^v$ ,  $z = [v, b]$  and  $x = yz = [uv, b] = [u, b]^v[v, b]$ . Since  $[A, B]$  is normal,  $[A, B] \circ [C, B]$  is a fuzzy multigroup of  $X$ . Hence,  $[A \circ C, B] \subseteq [A, B] \circ [C, B]$ .

Finally, suppose  $CM_A(e) = CM_C(e)$ . Then,  $A \subseteq A \circ C$  and  $C \subseteq A \circ C$ . Thus,  $[A, B] \subseteq [A \circ C, B]$  and  $[C, B] \subseteq [A \circ C, B]$  by Lemma 3.23. Hence,  $[A, B] \circ [C, B] \subseteq [A \circ C, B]$  since  $[A \circ C, B]$  is a fuzzy multigroup of  $X$ . Consequently, the desired equality holds. □

## 4 Some homomorphic properties of commutator in fuzzy multigroup context

Now, we consider homomorphic images and preimages of commutator of fuzzy multigroups.

**Lemma 4.1.** *Let  $f$  be a homomorphism of a group  $X$  into a group  $Y$ . If  $A$  and  $B$  are fuzzy submultisets of  $C \in FMG(X)$ , then  $f([A, B])$  is a fuzzy multigroup of  $Y$  generated by  $f((A, B))$ .*



*Proof.* Since  $[A, B]$  is generated by  $(A, B)$ , then  $f([A, B])$  is a fuzzy multigroup of  $Y$  generated by  $f((A, B))$ , which is a direct consequent of Theorem 2.12.  $\square$

**Lemma 4.2.** *Let  $f$  be an isomorphism of a group  $X$  into a group  $Y$ . If  $A$  and  $B$  are fuzzy submultisets of  $C \in FMG(Y)$ , then  $f^{-1}([A, B])$  is a fuzzy multigroup of  $X$  generated by  $f^{-1}((A, B))$ .*

*Proof.* Similar to Lemma 4.1.  $\square$

**Theorem 4.3.** *If  $f$  is a homomorphism of a group  $X$  into a group  $Y$ . Then  $[f(A), f(B)] = f([A, B])$  for fuzzy multigroups  $A$  and  $B$  of  $X$ .*

*Proof.* Foremostly, we show that  $f((A, B)) \subseteq [f(A), f(B)]$ . Let  $y \in Y$ . If  $f^{-1}(y) = \emptyset$ , then

$$CM_{f(A,B)}(y) = 0 \leq CM_{[f(A),f(B)]}(y).$$

Assume  $y = f(x)$  for some  $x \in X$ . Then, we have

$$\begin{aligned} CM_{(A,B)}(x) &= \bigvee \{CM_A(a) \wedge CM_B(b) \mid x = [a, b], a, b \in X\} \\ &\leq \bigvee \{CM_{f(A)}(f(a)) \wedge CM_{f(B)}(f(b)) \mid y = [f(a), f(b)], a, b \in X\} \\ &\leq \bigvee \{CM_{f(A)}(c) \wedge CM_{f(B)}(d) \mid y = [c, d], c, d \in X\} \\ &= C_{(f(A),f(B))}(y) \\ &\leq C_{[f(A),f(B)]}(y). \end{aligned}$$

Hence,

$$\begin{aligned} CM_{(f(A,B))}(y) &= \bigvee \{CM_{(A,B)}(x) \mid y = f(x), x \in X\} \\ &\leq C_{[f(A),f(B)]}(y). \end{aligned}$$

Thus,  $f((A, B)) \subseteq [f(A), f(B)]$ . By Lemma 4.1,  $f([A, B]) \subseteq [f(A), f(B)]$ .

Now, we proof that  $(f(A), f(B)) \subseteq f([A, B])$ . Let  $y \in Y$ . If  $y$  is not a commutator of  $Y$ , then

$$CM_{(f(A),f(B))}(y) = 0 \leq C_{f([A,B])}(y).$$

Suppose  $y$  is a commutator in  $Y$ . Then,  $y = [u, v]$  for some  $u, v \in Y$ . If either  $f^{-1}(u) = \emptyset$  or  $f^{-1}(v) = \emptyset$ , then

$$CM_{f(A)}(u) \wedge CM_{f(B)}(v) = 0.$$

Otherwise, we get

$$\begin{aligned} CM_{f(A)}(u) \wedge CM_{f(B)}(v) &= (\bigvee\{CM_A(s)|u = f(s)\}) \wedge (\bigvee\{CM_B(t)|v = f(t)\}) \\ &= \bigvee\{CM_A(s) \wedge CM_B(t)|u = f(s), v = f(t)\} \\ &\leq \bigvee\{CM_{(A,B)}([s, t])|y = f([s, t])\} \\ &\leq \bigvee\{CM_{[A,B]}([s, t])|y = f([s, t])\} \\ &\leq \bigvee\{CM_{[A,B]}(x)|y = f(x)\} \\ &= CM_{f([A,B])}(y). \end{aligned}$$

Thus,  $(f(A), f(B)) \subseteq f([A, B]) \Rightarrow [f(A), f(B)] \subseteq f([A, B])$ . Hence, the equality holds.  $\square$

**Theorem 4.4.** *Let  $f : X \rightarrow Y$  be a homomorphism such that  $A, B \in FMG(X)$ . Then  $[f^{-1}(A), f^{-1}(B)] \subseteq f^{-1}([A, B])$  in  $FMG(X)$ .*

*Proof.* Let  $x \in X$ . If  $x$  is not a commutator in  $X$ , then

$$CM_{(f^{-1}(A), f^{-1}(B))}(x) = 0 \leq CM_{[f^{-1}(A), f^{-1}(B)]}(x).$$

Assume that  $x$  is a commutator in  $X$ . Then, we have

$$\begin{aligned} CM_{(f^{-1}(A), f^{-1}(B))}(x) &= \bigvee\{CM_A(f(a)) \wedge CM_B(f(b))|x = [a, b], a, b \in X\} \\ &= \bigvee\{CM_A(f(a)) \wedge CM_B(f(b))|f(x) = [f(a), f(b)], a, b \in X\} \\ &\leq \bigvee\{CM_A(c) \wedge CM_B(d)|f(x) = [c, d], c, d \in X\} \\ &= CM_{(A,B)}(f(x)) \\ &\leq CM_{[A,B]}(f(x)). \end{aligned}$$

Hence,  $(f^{-1}(A), f^{-1}(B)) \subseteq f^{-1}([A, B])$  and consequently,  $[f^{-1}(A), f^{-1}(B)] \subseteq f^{-1}([A, B])$ .  $\square$

**Definition 4.5.** A non-empty collection  $\mathcal{D}$  of endomorphisms of a group  $X$  is an operator domain on  $X$ . A fuzzy multiset  $A$  of  $X$  is admissible under  $\mathcal{D}$  or  $\mathcal{D}$ -admissible if for every  $f \in \mathcal{D}$ ,  $f(A) \subseteq A$ , where  $f$  is a function. Thus,  $f(A) \subseteq A$  if and only if  $A \subseteq f^{-1}(A)$ .

**Theorem 4.6.** Suppose  $A$  and  $B$  are admissible fuzzy submultisets of  $C \in FMG(X)$  under an operator domain  $\mathcal{D}$ , then  $(A, B)$  and  $[A, B]$  are  $\mathcal{D}$ -admissible.

*Proof.* The fact that  $A$  and  $B$  are  $\mathcal{D}$ -admissible, we have

$$CM_{f^{-1}(A)}(x) \geq CM_A(x) \text{ and } CM_{f^{-1}(B)}(x) \geq CM_B(x) \forall f \in \mathcal{D}.$$

Let  $f \in \mathcal{D}$  and  $x \in X$ . If  $x$  is not a commutator in  $X$ , then

$$CM_{(A,B)}(x) = 0 \leq CM_{(A,B)}(f(x)).$$

Suppose  $x = [a, b]$  for some  $a, b \in X$ . Then

$$\begin{aligned} CM_{(A,B)}(x) &= \bigvee_{x=[a,b]} [CM_A(a) \wedge CM_B(b)] \\ &\leq \bigvee_{x=[a,b]} [CM_A(f(a)) \wedge CM_B(f(b))] \\ &= \bigvee_{f(x)=[f(a),f(b)]} [CM_A(f(a)) \wedge CM_B(f(b))] \\ &\leq \bigvee_{f(x)=[c,d]} [CM_A(c) \wedge CM_B(d)], \ c, d \in X \\ &= CM_{(A,B)}(f(x)). \end{aligned}$$

Thus,  $(A, B) \subseteq f^{-1}((A, B))$ . Hence,  $(A, B)$  is  $\mathcal{D}$ -admissible. Since  $[A, B]$  is generated by  $(A, B)$ , it follows that  $(A, B) \subseteq f^{-1}([A, B])$ . Thus,  $[A, B] \subseteq f^{-1}([A, B])$ . Hence,  $[A, B]$  is  $\mathcal{D}$ -admissible. □

## 5 Conclusion

This work further the study of fuzzy multigroup theory parallel to crisp group theory. The ideas of commutator of fuzzy multigroups and commutator fuzzy

submultigroups of fuzzy multigroups were proposed and characterized with a number of some related results. Some homomorphic images and preimages of commutator of fuzzy multigroups were considered with some results. The notion of admissible fuzzy submultisets  $A$  and  $B$  of  $C \in FMG(X)$  under an operator domain  $\mathcal{D}$  was explicated, and it was shown that  $(A, B)$  and  $[A, B]$  are  $\mathcal{D}$ -admissible. However, more properties of commutator in fuzzy multigroup setting could be exploited in future investigation.

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