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Fuzzy Differential Subordinations Results for λ-pseudo Starlike and λ-pseudo Convex Functions with Respect to Symmetrical Points

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Abstract

In the present work, we establish some fuzzy differential subordination results for λ-pseudo starlike and λ-pseudo convex functions with respect to symmetrical points in the open unit disk.

1. Introduction and Preliminaries

Let T indicate the family of functions f and has the series form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

For functions $f_j \in T$ $(j = 1, 2)$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2),$$

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the Hadamard product of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z).$$

A function $f \in T$ is called starlike with respect to symmetrical points, if (see [8])

$$\operatorname{Re}\left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in U.$$

The family of all such functions is denote by S_S^*.

The family of starlike functions with respect to symmetrical points obviously includes the family of convex functions with respect to symmetrical points, C_s, satisfying the following condition:

$$\operatorname{Re}\left\{ \frac{(zf'(z))'}{(f(z) - f(-z))} \right\} > 0, \quad z \in U.$$

Recently, Babalola [1] defined the class $\mathcal{L}_\lambda$ of λ-pseudo-starlike functions as follows:

Let $f \in T$ and $\lambda \geq 1$. Then $f \in \mathcal{L}_\lambda$ of λ-pseudo-starlike functions in U if and only if

$$\operatorname{Re}\left\{ \frac{z(f'(z))^\lambda}{f(z)} \right\} > 0, \quad z \in U.$$

Definition 1.1 [9]**.** Let X be a non-empty set. An application $F : X \to [0, 1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:

A pair (A, F_A), where $F_A : X \to [0, 1]$ and $A = \{x \in X : 0 < F_A(x) \leq 1\} = supp(A, F_A)$ is called fuzzy subset. The function F_A is called membership function of the fuzzy subset (A, F_A).

Definition 1.2 [5]**.** Let two fuzzy subsets of X, (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x)$, $x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \le F_N(x),\ x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.

Let $D \subseteq \mathbb{C}$ and f, g analytic functions. We denote by

$$f(D) = supp(f(D), F_{f(D)}) = \{f(z) : 0 < F_{f(D)}(f(z)) \le 1,\ z \in D\}$$

and

$$g(D) = supp(g(D), F_{g(D)}) = \{g(z) : 0 < F_{g(D)}(g(z)) \le 1,\ z \in D\}.$$

Definition 1.3 [5]**.** Let $D \subseteq \mathbb{C},\ z_0 \in D$ be a fixed point, and let the functions $f, g \in \mathcal{H}(D)$. The function *f* is said to be *fuzzy subordinate to* g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if the following conditions are satisfied:

1. $f(z_0) = g(z_0)$,

2. $F_{f(D)}(f(z)) \le F_{g(D)}(g(z)),\ z \in D.$

Definition 1.4 [6]**.** Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let *h* be univalent in *U*. If *p* is analytic in *U* and satisfies the (second-order) fuzzy differential subordination:

$$F_{\psi(\mathbb{C}^3 \times U)}(\psi(p(z), zp'(z), z^2 p''(z); z)) \le F_{h(U)}(h(z)), \tag{1.2}$$

i.e.,

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec_F h(z), \quad z \in U,$$

then *p* is called a *fuzzy solution* of the fuzzy differential subordination. The univalent function is *q* called a *fuzzy dominant* of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $p(z) \prec_F q(z),\ z \in U$ for all *p* satisfying (1.2). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z),\ z \in U$ for all fuzzy dominant *q* of (1.2) is said to be the fuzzy best dominant of (1.2).

Lemma 1.1 [2]**.** *Let q be univalent in U and let* θ *and* ϕ *be analytic in a domain D containing* $q(U)$ *with* $\phi(w) \ne 0$ *when* $w \in q(U)$. *Set* $Q(z) = zq'(z)\phi(q(z))$ *and* $h(z) = \theta(q(z)) + Q(z)$. *Suppose that*

(1) $Q(z)$ *is starlike in U*,

(2) $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in U$.

If p *is analytic in* U, *with* $p(0) = q(0)$, $p(U) \subset D$ *and* $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z)) = \theta(p(z)) + zp'(z) \cdot \phi(p(z))$ *is analytic in* U, *then*

$$F_{\psi(\mathbb{C}^2 \times U)}[\theta(p(z)) + zp'(z) \cdot \phi(p(z))] \le F_{h(U)}h(z),$$

implies

$$F_{p(U)}p(z) \le F_{q(U)}q(z),$$

i.e., $p(z) \prec_F q(z)$ *and q is the fuzzy best dominant*, *where*

$$\psi(\mathbb{C}^2 \times U) = supp(\mathbb{C}^2 \times U,\ F_{\psi(\mathbb{C}^2 \times U)}\psi(p(z),\ zp'(z)))$$

$$= \{z \in \mathbb{C} : 0 < F_{\psi(\mathbb{C}^2 \times U)}\psi(p(z),\ zp'(z)) \le 1\},$$

and

$$h(U) = supp(U,\ F_{h(U)}h(z)) = \{z \in \mathbb{C} : 0 < F_{h(U)}h(z) \le 1\}.$$

Recently, Oros and Oros [6, 7], Lupaş [3, 4] and Wanas and Majeed [10, 11] have obtained fuzzy differential subordination results for certain classes of analytic functions.

2. Main Results

Theorem 2.1. *Suppose that* $\alpha, \beta, \gamma \in \mathbb{C}$, $\delta > 0$, $\lambda \ge 1$, $t \in \mathbb{C}\backslash\{0\}$ *and q be univalent function in U such that* $q(0) = 1$, $q(z) \ne 0$ *and*

$$\operatorname{Re}\left\{1 + \frac{\beta}{t}(\gamma - 1) + \frac{\alpha\gamma}{t}q(z) + (\gamma - 2)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{2.1}$$

Assume that $z(q(z))^{\gamma-2}q'(z)$ *is starlike in U. If* $f \in T$ *and* $\Phi(\alpha, \beta, \gamma, \lambda, \delta, t; z)$ *is analytic in* U, *where*

$$\Phi(\alpha, \beta, \gamma, \lambda, \delta, t;\ z) = \left(\frac{2z(f'(z))^{\lambda}}{f(z) - f(-z)}\right)^{\gamma\delta}\left[\alpha + \beta\left(\frac{f(z) - f(-z)}{2z(f'(z))^{\lambda}}\right)^{\delta}\right.$$

$$+ t\delta\left(\frac{f(z)-f(-z)}{2z(f'(z))^{\lambda}}\right)^{\delta}$$

$$\times\left(1+\frac{\lambda z f''(z)}{f'(z)}-\frac{2(f(z)-f(-z))'}{f(z)-f(-z)}\right)\Bigg], \tag{2.2}$$

then

$$F_{\psi(\mathbb{C}^2\times U)}[\Phi(\alpha, \beta, \gamma, \lambda, \delta, t; z)] \le F_{\psi(\mathbb{C}^2\times U)}\left[(q(z))^{\gamma}\left(\alpha+\frac{\beta}{q(z)}+t\frac{zq'(z)}{(q(z))^2}\right)\right]$$

$$= F_{h(U)}h(z), \tag{2.3}$$

implies

$$F_{\left(\frac{(f'(U))^{\lambda}}{f(U)-f(-U)}\right)^{\delta}}\left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\delta} \le F_{q(U)}q(z),$$

i.e.,

$$\left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\delta} \prec_F q(z), \quad (z \in U)$$

and q is the fuzzy best dominant.

Proof. For given $f \in T$, define p by

$$p(z) = \left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\delta}. \tag{2.4}$$

It is clear that the function p is analytic in U and $p(0) = 1$. Simple calculations show that

$$(p(z))^{\gamma}\left(\alpha+\frac{\beta}{p(z)}+t\frac{zp'(z)}{(p(z))^2}\right) = \Phi(\alpha, \beta, \gamma, \lambda, \delta, t; z), \tag{2.5}$$

where $\Phi(\alpha, \beta, \gamma, \lambda, \delta, t; z)$ is given by (2.2).

In the light of (2.3) and (2.5), we conclude that

$$F_{\psi(\mathbb{C}^2\times U)}\left[(p(z))^{\gamma}\left(\alpha+\frac{\beta}{p(z)}+t\frac{zp'(z)}{(p(z))^2}\right)\right]$$

$$\leq F_{\psi(\mathbb{C}^2\times U)}\left[(q(z))^{\gamma}\left(\alpha+\frac{\beta}{q(z)}+t\frac{zq'(z)}{(q(z))^2}\right)\right].$$

Define the functions θ and ϕ by

$$\theta(w)=(\alpha w+\beta)w^{\gamma-1} \quad \text{and} \quad \phi(w)=tw^{\gamma-2}.$$

Evidently, the functions θ and ϕ are analytic in $D=\mathbb{C}\backslash\{0\}$ and $\phi(w)\neq 0$, $w\in D$. Also, we find that

$$Q(z)=zq'(z)\phi(q(z))=tz(q(z))^{\gamma-2}q'(z)$$

and

$$h(z)=\theta(q(z))+Q(z)=(q(z))^{\gamma}\left(\alpha+\frac{\beta}{q(z)}+t\frac{zq'(z)}{(q(z))^2}\right).$$

Since $z(q(z))^{\gamma-2}q'(z)$ is starlike univalent in U, we observe that Q is starlike univalent in U.

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}=\operatorname{Re}\left\{1+\frac{\beta}{t}(\gamma-1)+\frac{\alpha\gamma}{t}q(z)+(\gamma-2)\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right\}. \tag{2.6}$$

Now from (2.1) and (2.6), it is evident that

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}>0.$$

On the application of Lemma 1.1, yields $F_{p(U)}p(z)\leq F_{q(U)}q(z)$. By using (2.4), we obtain

$$F_{\left(\frac{(f'(U))^{\lambda}}{f(U)-f(-U)}\right)^{\delta}}\left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\delta}\leq F_{q(U)}q(z),$$

i.e. $\left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\delta} \prec_F q(z)$ and q is the fuzzy best dominant.

By putting the fuzzy dominant $q(z) = \frac{1+z}{1-z}$, $\gamma = t = 1$ and $\alpha = \beta = 0$ in Theorem 2.1, we obtain the following results:

Corollary 2.1. *Let* $\operatorname{Re}\left\{\frac{1+z^2}{1-z^2}\right\} > 0$. *If* $f \in T$ *and*

$$\delta\left(1 + \frac{\lambda z f''(z)}{f'(z)} - \frac{z(f(z)-f(-z))'}{f(z)-f(-z)}\right)$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2 \times U)}\left[\delta\left(1 + \frac{\lambda z f''(z)}{f'(z)} - \frac{z(f(z)-f(-z))'}{f(z)-f(-z)}\right)\right] \leq F_{\psi(\mathbb{C}^2 \times U)}\left[\frac{2z}{1-z^2}\right],$$

implies

$$\left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\delta} \prec_F \frac{1+z}{1-z}, \quad (z \in U)$$

and $q(z) = \frac{1+z}{1-z}$ *is the fuzzy best dominant.*

Theorem 2.2. *Suppose that* $\alpha, \beta, \gamma \in \mathbb{C}$, $\delta > 0$, $\lambda \geq 1$, $t \in \mathbb{C}\backslash\{0\}$ *and q is univalent function in U such that* $q(0) = 1$, $q(z) \neq 0$ *and let q satisfy* (2.1). *Assume that* $z(q(z))^{\gamma-2} q'(z)$ *is starlike in U. If* $f \in T$ *and* $\Psi(\alpha, \beta, \gamma, \lambda, \delta, t; z)$ *is analytic in U, where*

$$\Psi(\alpha, \beta, \gamma, \lambda, \delta, t; z) = \left(\frac{2((zf'(z))')^{\lambda}}{(f(z)-f(-z))'}\right)^{\gamma\delta}\left[\alpha + \beta\left(\frac{(f(z)-f(-z))'}{2((zf'(z))')^{\lambda}}\right)^{\delta}\right.$$

$$\left. + t\delta\left(\frac{(f(z)-f(-z))'}{2((zf'(z))')^{\lambda}}\right)^{\delta}\right.$$

$$\times\left(\frac{\lambda z(zf'''(z)+2f'(z))}{zf''(z)+f'(z)}-\frac{z(f(z)-f(-z))''}{(f(z)-f(-z))'}\right)\Bigg], \tag{2.7}$$

then

$$F_{\psi(\mathbb{C}^2\times U)}[\Psi(\alpha, \beta, \gamma, \lambda, \delta, t; z)] \leq F_{\psi(\mathbb{C}^2\times U)}\left[(q(z))^{\gamma}\left(\alpha+\frac{\beta}{q(z)}+t\frac{zq'(z)}{(q(z))^2}\right)\right]$$
$$= F_{h(U)}h(z), \tag{2.8}$$

implies

$$F_{\left(\frac{((f'(U))')^{\lambda}}{(f(U)-f(-U))'}\right)^{\delta}}\left(\frac{2((zf'(z))')^{\lambda}}{(f(z)-f(-z))'}\right)^{\delta} \leq F_{q(U)}q(z),$$

i.e.,

$$\left(\frac{2((zf'(z))')^{\lambda}}{(f(z)-f(-z))'}\right)^{\delta} \prec_F q(z), \quad (z\in U)$$

and q is the fuzzy best dominant.

Proof. For given $f \in T$, define p by

$$p(z)=\left(\frac{2((zf'(z))')^{\lambda}}{(f(z)-f(-z))'}\right)^{\delta}. \tag{2.9}$$

It is obvious that the function p is analytic in U and $p(0)=1$. After some computations, we deduce that

$$(p(z))^{\gamma}\left(\alpha+\frac{\beta}{p(z)}+t\frac{zp'(z)}{(p(z))^2}\right)=\Psi(\alpha, \beta, \gamma, \lambda, \delta, t; z), \tag{2.10}$$

where $\Psi(\alpha, \beta, \gamma, \lambda, \delta, t; z)$ is given by (2.7).

By making use of (2.8) and (2.10), it follows that

$$F_{\psi(\mathbb{C}^2\times U)}\left[(p(z))^{\gamma}\left(\alpha+\frac{\beta}{p(z)}+t\frac{zp'(z)}{(p(z))^2}\right)\right]$$

$$\leq F_{\psi(\mathbb{C}^2\times U)}\left[(q(z))^{\gamma}\left(\alpha+\frac{\beta}{q(z)}+t\frac{zq'(z)}{(q(z))^2}\right)\right].$$

The remaining part of Theorem 2.2 is similar to that of Theorem 2.1 and thus we omit it.

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