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Some Subordination and Superordination Results for Normalized Analytic Functions Defined by Convolution Structure Associated with Wanas Differential Operator

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Abstract

The purpose of this paper is to establish some subordination and superordination results involving Hadamard product for certain normalized analytic functions associated with Wanas differential operator defined in the open unit disk and obtain sandwich results. Our results extend corresponding previously known results.

1. Introduction and Preliminaries

Denote by $\mathcal{H}$ the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, assume that $\mathcal{H}[a, n]$ is the subclass of $\mathcal{H}$ consisting of functions that have the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots. \tag{1.1}$$

Also, let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

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$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

For the functions $f \in \mathcal{A}$ given by (1.2) and $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

With a view to recalling the principal of subordination between analytic functions, let $f, g \in \mathcal{H}$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. In such a case we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. Furthermore, if g is univalent in U, then we have the following equivalent (see [9]), $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p, h \in \mathcal{H}$ and $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2 p''(z); z)$ are univalent functions in U and if p satisfies the second-order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z), \tag{1.3}$$

then p is called a solution of the differential superordination (1.3). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinant of (1.3), if $q \prec p$ for all the functions p satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.3) is called the best subordinant.

For $\alpha \in \mathbb{R}, \beta \geq 0$ with $\alpha + \beta > 0$, $m, \eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$, the Wanas differential operator $W_{\alpha,\beta}^{k,\eta} : \mathcal{A} \to \mathcal{A}$ (see [16]) is defined by

$$W_{\alpha,\beta}^{k,\eta} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\eta} a_n z^n. \tag{1.4}$$

It is easily verified from (1.2) that

$$z(W_{\alpha,\beta}^{k,\eta}f(z))' = \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]W_{\alpha,\beta}^{k,\eta+1}f(z)$$

$$-\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right]W_{\alpha,\beta}^{k,\eta}f(z). \tag{1.5}$$

Special cases of this operator can be found in [1, 2, 4, 6, 7, 8, 11, 13, 14, 15]. For more details see [19].

Very recently, Rahrovi [10], Attiya and Yassen [3], Seoudy [12] and Wanas and Lupas [18] have studied differential subordinations and superordinations for different conditions of analytic functions.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions *f* in *U* such that $(f * \Psi)(z) \neq 0$ and *f* to satisfy

$$q_1(z) \prec \frac{W_{\alpha,\beta}^{k,\eta+1}(f * \Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f * \Psi)(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in *U* and $\Phi(z) = z + \sum_{n=2}^{\infty} t_n z^n$, $\Psi(z) = z + \sum_{n=2}^{\infty} \delta_n z^n$ are analytic functions in *U* with $t_n \geq 0$, $\delta_n \geq 0$ and $t_n \geq \delta_n$. Also, we obtain the number of results as their special cases.

To establish our main results, we need the following definition and lemmas:

Definition 1.1 [9]**.** Denote by *Q* the set of all functions *f* that are analytic and injective on $\overline{U}\setminus E(f)$, where

$$E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1.1 [9]**.** *Let q be univalent in the unit disk U and let* θ *and* ϕ *be analytic in a domain D containing* $q(U)$ *with* $\phi(w) \neq 0$ *when* $w \in q(U)$. *Set* $Q(z) = zq'(z)\phi(q(z))$ *and* $h(z) = \theta(q(z)) + Q(z)$. *Suppose that*

(1) $Q(z)$ *is starlike univalent in U,*

(2) $\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in U$.

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.6}$$

then $p \prec q$ *and q is the best dominant of* (1.6).

Lemma 1.2 [5]**.** *Let q be convex univalent in the unit disk U and let* θ *and* ϕ *be analytic in a domain D containing* $q(U)$. *Suppose that*

(1) $\mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$,

(2) $Q(z) = zq'(z)\phi(q(z))$ *is starlike univalent in U.*

If $p \in \mathcal{H}[q(0), 1] \cap Q$, *with* $p(U) \subset D$, $\theta(p(z)) + zp'(z)\phi(p(z))$ *is univalent in U and*

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{1.7}$$

then $q \prec p$ *and q is the best subordinant of* (1.7).

2. Main Results

Theorem 2.1. *Let* $\Phi, \Psi \in \mathcal{A}$ *and q be univalent in U with* $q(z) \neq 0$, $q(0) = 1$ *and assume that*

$$\mathrm{Re}\left\{1 + \frac{\lambda_2(\gamma - \sigma)}{\lambda_3\sigma} + \frac{\lambda_1\gamma}{\lambda_3\sigma}q(z) + \left(\frac{\gamma}{\sigma} - 2\right)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0, \tag{2.1}$$

where $\lambda_1, \lambda_2, \gamma \in \mathbb{C}, \lambda_3, \sigma \in \mathbb{C}\backslash\{0\}$. *Suppose that* $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ *is starlike univalent in U. If* $f \in \mathcal{A}, \frac{W_{\alpha,\beta}^{k,\eta+1}(f*\Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f*\Psi)(z)} \neq 0, z \in U$, *satisfies the differential subordination*

$$\Upsilon_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z) \prec (q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right)^{\sigma}, \quad (2.2)$$

where

$$\Upsilon_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$$

$$= \left(\frac{W_{\alpha,\beta}^{k,\eta+1}(f*\Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f*\Psi)(z)}\right)^{\gamma}$$

$$\times\left(\lambda_1 + \lambda_2 \frac{W_{\alpha,\beta}^{k,\eta}(f*\Psi)(z)}{W_{\alpha,\beta}^{k,\eta+1}(f*\Phi)(z)} + \lambda_3 \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\frac{W_{\alpha,\beta}^{k,\eta}(f*\Psi)(z)}{W_{\alpha,\beta}^{k,\eta+1}(f*\Phi)(z)}\right.$$

$$\left.\times\left(\frac{W_{\alpha,\beta}^{k,\eta+2}(f*\Phi)(z)}{W_{\alpha,\beta}^{k,\eta+1}(f*\Phi)(z)} - \frac{W_{\alpha,\beta}^{k,\eta+1}(f*\Psi)(z)}{W_{\alpha,\beta}^{k,\eta}(f*\Psi)(z)}\right)\right)^{\sigma}, \quad (2.3)$$

then

$$\frac{W_{\alpha,\beta}^{k,\eta+1}(f*\Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function p by

$$p(z) = \frac{W_{\alpha,\beta}^{k,\eta+1}(f*\Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f*\Psi)(z)}, \quad z \in U. \quad (2.4)$$

Differentiating (2.4) with respect to z and using (1.5), we obtain

$$(p(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{p(z)} + \lambda_3 \frac{zp'(z)}{(p(z))^2}\right)^{\sigma}$$

$$= \left(\frac{W_{\alpha,\beta}^{k,\eta+1}(f * \Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f * \Psi)(z)}\right)^{\gamma}\left(\lambda_1 + \lambda_2 \frac{W_{\alpha,\beta}^{k,\eta}(f * \Psi)(z)}{W_{\alpha,\beta}^{k,\eta+1}(f * \Phi)(z)}\right.$$

$$\left. + \frac{\lambda_3(\alpha+\beta)}{\beta} \frac{W_{\alpha,\beta}^{k,\eta}(f * \Psi)(z)}{W_{\alpha,\beta}^{k,\eta+1}(f * \Phi)(z)}\left(\frac{W_{\alpha,\beta}^{k,\eta+2}(f * \Phi)(z)}{W_{\alpha,\beta}^{k,\eta+1}(f * \Phi)(z)} - \frac{W_{\alpha,\beta}^{k,\eta+1}(f * \Psi)(z)}{W_{\alpha,\beta}^{k,\eta}(f * \Psi)(z)}\right)\right)^{\sigma}. \quad (2.5)$$

In view of (2.2) and (2.5), we have

$$(p(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{p(z)} + \lambda_3 \frac{zp'(z)}{(p(z))^2}\right)^{\sigma} \prec (q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right)^{\sigma}.$$

This equivalently to

$$(p(z))^{\frac{\gamma}{\sigma}}\left(\lambda_1 + \frac{\lambda_2}{p(z)} + \lambda_3 \frac{zp'(z)}{(p(z))^2}\right) \prec (q(z))^{\frac{\gamma}{\sigma}}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right).$$

By setting

$$\theta(w) = (\lambda_1 w + \lambda_2) w^{\frac{\gamma}{\sigma}-1} \quad \text{and} \quad \phi(w) = \lambda_3 w^{\frac{\gamma}{\sigma}-2},$$

it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \lambda_3 z(q(z))^{\frac{\gamma}{\sigma}-2} q'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = (q(z))^{\frac{\gamma}{\sigma}}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right).$$

It is clear that $Q(z)$ is starlike univalent in U,

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{\lambda_2(\gamma - \sigma)}{\lambda_3\sigma} + \frac{\lambda_1\gamma}{\lambda_3\sigma} q(z) + \left(\frac{\gamma}{\sigma} - 2\right)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\}. \quad (2.6)$$

From (2.1) and (2.6), we have

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0.$$

Therefore by an application of Lemma 1.1, we get $p(z) \prec q(z)$. By using (2.4), we obtain the desired result.

Remark 2.1. By taking $k = 1$ in Theorem 2.1, we obtain the results for the operator $I^{\eta}_{\alpha,\beta}$ which was obtained recently by Wanas and Joudah [17, Theorem 3.1].

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. *Let q be univalent in U with $q(z) \neq 0$, $q(0) = 1$ and assume that* (2.1) *holds true. Suppose that $z(q(z))^{\frac{\gamma}{\sigma}-2} q'(z)$ is starlike univalent in U. If $f \in \mathcal{A}$, $\frac{W^{k,\eta+1}_{\alpha,\beta} f(z)}{W^{k,\eta}_{\alpha,\beta} f(z)} \neq 0$, $z \in U$, satisfies the differential subordination*

$$\Upsilon_2(f, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z) \prec (q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right)^{\sigma},$$

where

$$\Upsilon_2(f, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$$

$$= \left(\frac{W^{k,\eta+1}_{\alpha,\beta} f(z)}{W^{k,\eta}_{\alpha,\beta} f(z)}\right)^{\gamma}\left(\lambda_1 + \lambda_2 \frac{W^{k,\eta}_{\alpha,\beta} f(z)}{W^{k,\eta+1}_{\alpha,\beta} f(z)} + \lambda_3 \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m} + 1\right)\right.$$

$$\left. \times \frac{W^{k,\eta}_{\alpha,\beta} f(z)}{W^{k,\eta+1}_{\alpha,\beta} f(z)}\left(\frac{W^{k,\eta+2}_{\alpha,\beta} f(z)}{W^{k,\eta+1}_{\alpha,\beta} f(z)} - \frac{W^{k,\eta+1}_{\alpha,\beta} f(z)}{W^{k,\eta}_{\alpha,\beta} f(z)}\right)\right)^{\sigma}, \quad (2.7)$$

then

$$\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)} \prec q(z)$$

and q is the best dominant.

Theorem 2.2. *Let* $\Phi, \Psi \in \mathcal{A}$ *and* q *be convex univalent in* U *with* $q(z) \neq 0$, $q(0) = 1$ *and assume that*

$$\operatorname{Re}\left\{\frac{\lambda_2(\gamma-\sigma)}{\lambda_3\sigma} + \frac{\lambda_1\gamma}{\lambda_3\sigma}q(z)\right\} > 0, \tag{2.8}$$

where $\lambda_1, \lambda_2, \gamma \in \mathbb{C}, \lambda_3, \sigma \in \mathbb{C}\backslash\{0\}$. *Suppose that* $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ *is starlike univalent in U. Let* $f \in \mathcal{A}$, $\frac{W_{\alpha,\beta}^{k,\eta+1}(f*\Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f*\Psi)(z)} \in \mathcal{H}[q(0), 1] \cap Q$ *with* $\frac{W_{\alpha,\beta}^{k,\eta+1}(f*\Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f*\Psi)(z)} \neq 0$, $z \in U$ *and* $\Upsilon_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$ *be univalent in U, where* $\Upsilon_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$ *is given by* (2.3). *If*

$$(q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3\frac{zq'(z)}{(q(z))^2}\right)^{\sigma}$$

$$\prec \Upsilon_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z), \tag{2.9}$$

then

$$q(z) \prec \frac{W_{\alpha,\beta}^{k,\eta+1}(f*\Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f*\Psi)(z)}$$

and q is the best subordinant.

Proof. Let the function p be defined by (2.4). After simple computation and making use of (1.5), the superordination (2.9) becomes

$$(q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3\frac{zq'(z)}{(q(z))^2}\right)^{\sigma} \prec (p(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{p(z)} + \lambda_3\frac{zp'(z)}{(p(z))^2}\right)^{\sigma}.$$

This equivalently to

$$(q(z))^{\frac{\gamma}{\sigma}}\left(\lambda_1+\frac{\lambda_2}{q(z)}+\lambda_3\frac{zq'(z)}{(q(z))^2}\right)\prec(p(z))^{\frac{\gamma}{\sigma}}\left(\lambda_1+\frac{\lambda_2}{p(z)}+\lambda_3\frac{zp'(z)}{(p(z))^2}\right).$$

By setting

$$\theta(w)=(\lambda_1 w+\lambda_2)w^{\frac{\gamma}{\sigma}-1} \quad \text{and} \quad \phi(w)=\lambda_3 w^{\frac{\gamma}{\sigma}-2},$$

it is easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0,\ w\in\mathbb{C}\setminus\{0\}$. Also, we get

$$Q(z)=zq'(z)\phi(q(z))=\lambda_3 z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z).$$

It is clear that $Q(z)$ is starlike univalent in U,

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}=\operatorname{Re}\left\{\frac{\lambda_2(\gamma-\sigma)}{\lambda_3\sigma}+\frac{\lambda_1\gamma}{\lambda_3\sigma}q(z)\right\}. \tag{2.10}$$

From (2.8) and (2.10), we have

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}>0.$$

Therefore by an application of Lemma 1.2, we get $q(z)\prec p(z)$. By using (2.4), we obtain the desired result.

Remark 2.2. By taking $k=1$ in Theorem 2.2, we obtain the results for the operator $I^{\eta}_{\alpha,\beta}$ which was obtained recently by Wanas and Joudah [17, Theorem 4.1].

By fixing $\Phi(z)=\Psi(z)=\frac{z}{1-z}$ in Theorem 4.1, we obtain the following corollary:

Corollary 2.2. *Let q be convex univalent in U with* $q(z)\neq 0,\ q(0)=1$ *and assume that* (2.8) *holds true. Suppose that* $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ *is starlike univalent in U. Let* $f\in\mathcal{A},\ \frac{W^{k,\eta+1}_{\alpha,\beta}f(z)}{W^{k,\eta}_{\alpha,\beta}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ *with* $\frac{W^{k,\eta+1}_{\alpha,\beta}f(z)}{W^{k,\eta}_{\alpha,\beta}f(z)}\neq 0,\ z\in U$, *and* $\Upsilon_2(f,\lambda_1,$

$\lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$ *be univalent in U, where* $\Upsilon_2(f, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$ *is given by* (2.7). *If*

$$(q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right)^{\sigma} \prec \Upsilon_2(f, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z),$$

then

$$q(z) \prec \frac{W_{\alpha,\beta}^{k,\eta+1} f(z)}{W_{\alpha,\beta}^{k,\eta} f(z)}$$

and q is the best subordinant.

Concluding the results of differential subordination and superordination, we arrive at the following "sandwich results".

Theorem 2.3. *Let* $\Phi, \Psi \in \mathcal{A}$. *Let* q_1 *and* q_2 *be convex univalent in U with* $q_1(0) = q_2(0) = 1$. *Suppose* q_2 *satisfies* (2.1) *and* q_1 *satisfies* (2.8). *Let* $f \in \mathcal{A}$, $\frac{W_{\alpha,\beta}^{k,\eta+1}(f * \Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f * \Psi)(z)} \in \mathcal{H}[1, 1] \cap Q$ *with* $\frac{W_{\alpha,\beta}^{k,\eta+1}(f * \Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f * \Psi)(z)} \neq 0$, $z \in U$ *and* $\Upsilon_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$ *be univalent in U, where* $\Upsilon_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$ *is given by* (2.3). *If*

$$(q_1(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q_1(z)} + \lambda_3 \frac{zq_1'(z)}{(q_1(z))^2}\right)^{\sigma} \prec \Upsilon_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$$

$$\prec (q_2(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q_2(z)} + \lambda_3 \frac{zq_2'(z)}{(q_2(z))^2}\right)^{\sigma},$$

then

$$q_1(z) \prec \frac{W_{\alpha,\beta}^{k,\eta+1}(f * \Phi)(z)}{W_{\alpha,\beta}^{k,\eta}(f * \Psi)(z)} \prec q_2(z)$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant.*

Remark 2.3. By taking $k = 1$ in Theorem 2.3, we obtain the results for the operator $I_{\alpha,\beta}^{\eta}$ which was obtained recently by Wanas and Joudah [17, Theorem 5.1].

By making use of Corollaries 2.1 and 2.2, we obtain the following corollary:

Corollary 2.3. *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0) = q_2(0) = 1$. *Suppose* q_2 *satisfies* (2.1) *and* q_1 *satisfies* (2.8). *Let* $f \in \mathcal{A}$, $\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)} \in \mathcal{H}[1, 1] \cap Q$ *with* $\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)} \neq 0$, $z \in U$ *and* $\Upsilon_2(f, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$ *be univalent in U, where* $\Upsilon_2(f, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$ *is given by* (2.7). *If*

$$(q_1(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q_1(z)} + \lambda_3 \frac{zq_1'(z)}{(q_1(z))^2}\right)^{\sigma} \prec \Upsilon_2(f, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta, k, \eta; z)$$

$$\prec (q_2(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q_2(z)} + \lambda_3 \frac{zq_2'(z)}{(q_2(z))^2}\right)^{\sigma},$$

then

$$q_1(z) \prec \frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)} \prec q_2(z)$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant.*

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