Some Properties for Fuzzy Differential Subordination Defined by Wanas Operator

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Abstract

In this article, we establish some interesting geometric properties for fuzzy differential subordination associated with Wanas operator which defined in the open unit disk.

1. Introduction

Let the notation $\mathcal{H}(\mathcal{U})$ stand for the family of holomorphic functions in the unit disk $\mathcal{U} = \{ z \in \mathbb{C} : |z| < 1 \}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, we indicate by

$$\mathcal{H}[a, n] = \{ f \in \mathcal{H}(\mathcal{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \ z \in \mathcal{U} \}$$

and

$$\mathcal{A}_n = \{ f \in \mathcal{H}(\mathcal{U}) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots, \ z \in \mathcal{U} \},$$

with $\mathcal{A}_1 = \mathcal{A}$.

Definition 1.1 [13]. Let $X$ be a non-empty set. An application $F : X \to \{0, 1\}$ is called fuzzy subset. An alternate definition, more precise, would be the following:
A pair \((S, F_S)\), where \(F_S : X \to [0, 1]\) and \(\text{supp}(S, F_S) = \{x \in X : 0 < F_S(x) \leq 1\}\) is called fuzzy subset. The function \(F_S\) is called membership function of the fuzzy subset \((S, F_S)\).

**Definition 1.2** [6]. Let two fuzzy subsets of \(X\), \((M, F_M)\) and \((N, F_N)\). We say that the fuzzy subsets \(M\) and \(N\) are *equal* if and only if \(F_M(x) = F_N(x), x \in X\) and we denote this by \((M, F_M) = (N, F_N)\). The fuzzy subset \((M, F_M)\) is contained in the fuzzy subset \((N, F_N)\) if and only if \(F_M(x) \leq F_N(x), x \in X\) and we denote the inclusion relation by \((M, F_M) \subseteq (N, F_N)\).

Assume that \(D\) is a set in \(\mathbb{C}\) and \(f, g\) are holomorphic functions. We indicate by

\[
f(D) = \text{supp}(f(D), F_{f(D)}) = \{f(z) : 0 < F_{f(D)}(f(z)) \leq 1, z \in D\}
\]

and

\[
g(D) = \text{supp}(g(D), F_{g(D)}) = \{g(z) : 0 < F_{g(D)}(g(z)) \leq 1, z \in D\}.
\]

**Definition 1.3** [6]. Suppose that \(D\) is a set in \(\mathbb{C}\), \(z_0 \in D\) is a fixed point and let the functions \(f, g \in \mathcal{H}(D)\). The function \(f\) is named a fuzzy subordinate to \(g\) and write \(f \prec_F g\) or \(f(z) \prec_F g(z)\) if satisfies the following:

1. \(f(z_0) = g(z_0)\).
2. \(F_{f(D)}(f(z)) \leq F_{g(D)}(g(z)), z \in D\).

**Definition 1.4** [7]. Let \(h\) be univalent in \(U\) and \(\psi : \mathbb{C}^3 \times U \to \mathbb{C}\). If \(\mathcal{P}\) is holomorphic in \(U\) satisfies the fuzzy differential subordination:

\[
F_{\psi(\mathbb{C}^3 \times U)}(\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z)) \leq F_h(U)(h(z)),
\]  

(1.1)

i.e.,

\[
\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z) \prec_F h(z), z \in U,
\]

then \(\mathcal{P}\) is called a fuzzy solution of the fuzzy differential subordination. The univalent function \(q\) is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if \(\mathcal{P}(z) \prec_F q(z), z \in U\) for all \(\mathcal{P}\).
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satisfying (1.1). A fuzzy dominant \( \bar{q} \) that satisfies \( \bar{q}(z) \prec_F q(z) \), \( z \in \mathcal{U} \) for all fuzzy dominant \( q \) of (1.1) is said to be the fuzzy best dominant of (1.1).

For \( \alpha \in \mathbb{R}, \beta \geq 0 \) with \( \alpha + \beta > 0 \), \( m, \lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( f \in A \), we consider the differential operator \( W_{\alpha, \beta}^{k, \lambda} : A \rightarrow A \), introduced by Wanas [11], where

\[
W_{\alpha, \beta}^{k, \lambda} f(z) = z + \sum_{n=2}^{\infty} \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^\lambda a_n z^n. \tag{1.2}
\]

By making use of (1.2), it is evident that

\[
z(W_{\alpha, \beta}^{k, \lambda} f(z))' = \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right] W_{\alpha, \beta}^{k, \lambda+1} f(z)
- \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m}{\beta^m} \right) \right] W_{\alpha, \beta}^{k, \lambda} f(z). \tag{1.3}
\]

We will need the following lemmas in investigating our main results.

**Lemma 1.1** [5]. Suppose that the convex function \( h \) satisfies \( h(0) = a \), let \( \mu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) such that \( \text{Re}(\mu) \geq 0 \). If \( \mathcal{P} \in \mathcal{H}[a, n] \) with \( \mathcal{P}(0) = a \) and \( \psi : \mathbb{C}^2 \times \mathcal{U} \rightarrow \mathbb{C} \),

\[
\psi(\mathcal{P}(z), z\mathcal{P}'(z)) = \mathcal{P}(z) + \frac{1}{\mu} z\mathcal{P}'(z)
\]

is holomorphic in \( \mathcal{U} \), then

\[
F_{\psi(\mathbb{C}^2 \times \mathcal{U})}\left[ \mathcal{P}(z) + \frac{1}{\mu} z\mathcal{P}'(z) \right] \leq F_{h(\mathcal{U})} h(z),
\]

implies

\[
F_{\mathcal{P}(\mathcal{U})}\mathcal{P}(z) \leq F_{q(\mathcal{U})} q(z) \leq F_{h(\mathcal{U})} h(z), \quad z \in \mathcal{U},
\]

i.e.,

\[
\mathcal{P}(z) \prec_F q(z) \prec_F h(z),
\]

where

\[
q(z) = \frac{\mu}{\mu - m} \int_0^z h(t)t^{n-1} dt
\]

is convex and is the fuzzy best dominant.
Lemma 1.2 [5]. Suppose that $q$ is a convex function in $\mathcal{U}$, let $h(z) = q(z) + \nu vz^q(z), \nu > 0$ and $n \in \mathbb{N}$. If $p \in \mathcal{H}(q(0), n)$ and $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$, $\psi(p(z), zp'(z)) = \mathcal{P}(z) + \nu vz^p(z)$ is holomorphic in $\mathcal{U}$, then

$$F_{\psi(\mathbb{C}^2 \times \mathcal{U})} \left[ \mathcal{P}(z) + \nu vz^p(z) \right] \leq F_{h(\mathcal{U})} h(z),$$

implies

$$F_{\mathcal{P}(\mathcal{U})} \mathcal{P}(z) \leq F_q(\mathcal{U}) q(z), \quad z \in \mathcal{U},$$

i.e.,

$$\mathcal{P}(z) \prec_F q(z)$$

and $q$ is the fuzzy best dominant.

Recently, Oros and Oros [7, 8], Lupaş [2-5], Haydar [1] and Wanas and Majeed [10, 11, 12] have obtained fuzzy differential subordination results for certain classes of holomorphic functions.

2. Main Results

Theorem 2.1. Suppose that convex function $h$ satisfies $h(0) = 1$. Let $f \in \mathcal{A}$ and

$$\frac{1}{z} \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m + 1 \right] W_{\alpha, \beta}^k, \lambda + 1 f(z) - \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right] W_{\alpha, \beta}^k, \lambda f(z)$$

$$+ z(W_{\alpha, \beta}^k, \lambda f(z))''$$

is holomorphic in $\mathcal{U}$.

If

$$F_{\psi(\mathbb{C}^2 \times \mathcal{U})} \left[ \frac{1}{z} \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m + 1 \right] W_{\alpha, \beta}^k, \lambda + 1 f(z)$$

$$- \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right] W_{\alpha, \beta}^k, \lambda f(z) + z(W_{\alpha, \beta}^k, \lambda f(z))'' \right] \leq F_{h(\mathcal{U})} h(z),$$

(2.1)
then
\[ F_{(W_{\alpha,\beta}^{k,\lambda} f)'(t)}((W_{\alpha,\beta}^{k,\lambda} f(z))') \leq F_{q(t)} q(z) \leq F_{h(t)} h(z), \]
i.e.
\[ (W_{\alpha,\beta}^{k,\lambda} f(z))' \prec_F q(z) \prec_F h(z), \]
where \( q(z) = \frac{1}{z} \int_0^z h(t) \) is convex and is the fuzzy best dominant.

**Proof.** Assume that
\[ P(z) = (W_{\alpha,\beta}^{k,\lambda} f(z))'. \] (2.2)
Then \( P \in H[1, 1] \) and \( P(0) = 1 \). Therefore, in view of (1.3) and (2.2), we have
\[
P(z) + zP'(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1} \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m + n \beta^m}{\alpha^m + \beta^m} \right) \right]^\lambda a_n z^{n-1}
\]
\[
= \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m}{\beta} + 1 \right)
\]
\[
\times \left[ 1 + \sum_{n=2}^{\infty} \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m + n \beta^m}{\alpha^m + \beta^m} \right)^{\lambda+1} a_n z^{n-1} \right]
\]
\[
- \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \frac{\alpha^m}{\beta}
\]
\[
\times \left[ 1 + \sum_{n=2}^{\infty} \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m + n \beta^m}{\alpha^m + \beta^m} \right)^\lambda a_n z^{n-1} \right]
\]
\[
+ \sum_{n=2}^{\infty} (n-1) \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m + n \beta^m}{\alpha^m + \beta^m} \right)^\lambda a_n z^{n-1}
\]
According to (2.1) and (2.3), we deduce that

$$F_{\mathcal{U}}(\mathbb{C}^2 \times \mathcal{U}) [P(z) + zP'(z)] \leq F_{h(\mathcal{U})}h(z).$$

Thus applying Lemma 1.1 with $\mu = 1$, we obtain

$$F_{P(\mathcal{U})}P(z) \leq F_{q(\mathcal{U})}q(z) \leq F_{h(\mathcal{U})}h(z).$$

From (2.2), we find that

$$F_{(W_{k,\lambda}^k f)(\mathcal{U})}((W_{k,\lambda}^k f(z)))' \leq F_{q(\mathcal{U})}q(z) \leq F_{h(\mathcal{U})}h(z),$$

i.e.,

$$(W_{k,\lambda}^k f(z))' \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z} \int_0^z h(t) \, dt$ is convex and is the fuzzy best dominant.

Putting $\lambda = 0$ and $h(z) = \frac{1 + (2\rho - 1)z}{1 + z}$ ($0 \leq \rho < 1$) in Theorem 2.1, we obtain the following corollary:

**Corollary 2.1.** Let $f \in \mathcal{A}$ and $zf''(z) + f'(z)$ is holomorphic in $\mathcal{U}$. If

$$zf''(z) + f'(z) \prec_F \frac{1 + (2\rho - 1)z}{1 + z},$$

then

$$f'(z) \prec_F q(z) \prec_F \frac{1 + (2\rho - 1)z}{1 + z},$$

where $q(z) = 2\rho - 1 + \frac{2(1 - \rho)}{z} \ln(1 + z)$ is convex and is the fuzzy best dominant.
Theorem 2.2. Suppose that the convex function $h$ satisfies $h(0) = 1$. Let $f \in \mathcal{A}$ and $(W_{\alpha, \beta}^{k, \lambda} f(z))'$ is holomorphic in $\mathcal{U}$. If
\[
F_{\psi(C^2 \times \mathcal{U})}[(W_{\alpha, \beta}^{k, \lambda} f(z))'] \leq F_{h(\mathcal{U})} h(z),
\]
then
\[
F_{(W_{\alpha, \beta}^{k, \lambda} f)(\mathcal{U})}\left(\frac{W_{\alpha, \beta}^{k, \lambda} f(z)}{z}\right) \leq F_{q(\mathcal{U})} q(z) \leq F_{h(\mathcal{U})} h(z),
\]
i.e.,
\[
\frac{W_{\alpha, \beta}^{k, \lambda} f(z)}{z} \prec_F q(z) \prec_F h(z),
\]
where $q(z) = \frac{1}{z} \int_0^z h(t) dt$ is convex and is the fuzzy best dominant.

**Proof.** Assume that
\[
\mathcal{P}(z) = \frac{W_{\alpha, \beta}^{k, \lambda} f(z)}{z}.
\]
It is clear that $\mathcal{P} \in \mathcal{H}[1, 1]$ and $\mathcal{P}(0) = 1$.

We find
\[
\mathcal{P}(z) + z\mathcal{P}'(z) = (W_{\alpha, \beta}^{k, \lambda} f(z))'.
\]
In view of (2.6), the fuzzy differential subordination (2.4) becomes
\[
F_{\psi(C^2 \times \mathcal{U})}[\mathcal{P}(z) + z\mathcal{P}'(z)] \leq F_{h(\mathcal{U})} h(z).
\]
Thus applying Lemma 1.1 with $\mu = 1$, we obtain
\[
F_{\mathcal{P}(\mathcal{U})}\mathcal{P}(z) \leq F_{q(\mathcal{U})} q(z) \leq F_{h(\mathcal{U})} h(z).
\]
From (2.5), we get
\[
F_{(W_{\alpha, \beta}^{k, \lambda} f)(\mathcal{U})}\left(\frac{W_{\alpha, \beta}^{k, \lambda} f(z)}{z}\right) \leq F_{q(\mathcal{U})} q(z) \leq F_{h(\mathcal{U})} h(z),
\]
i.e.,
\[
\frac{W_{\alpha, \beta}^{k, \lambda} f(z)}{z} \preceq_F q(z) \preceq_F h(z),
\]
where \( q(z) = \frac{1}{z} \int_{0}^{z} h(t) \, dt \) is convex and is the fuzzy best dominant.

Putting \( \lambda = 0 \) and \( h(z) = e^{b z} \), \( |b| \leq 1 \) in Theorem 2.2, we obtain the following corollary:

**Corollary 2.2.** If \( f \in A \), \( f'(z) \) is holomorphic in \( U \) and \( f'(z) \preceq_F e^{b z} \), then
\[
\frac{f(z)}{z} \preceq_F q(z) \preceq_F e^{b z},
\]
where \( q(z) = \frac{e^{b z} - 1}{b z} \) is convex and is the fuzzy best dominant.

**Theorem 2.3.** Suppose that \( q \) is a convex function in \( U \) such that \( q(0) = 1 \),
\[
h(z) = q(z) + \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \frac{\beta^m}{\alpha^m + \beta^m} \right] z q'(z).\]
Let \( f \in A \) and \( (W_{\alpha, \beta}^{k, \lambda+1} f(z))' \) is holomorphic in \( U \). If
\[
F_{\psi(C^2 \times U)}([(W_{\alpha, \beta}^{k, \lambda+1} f(z))']) \leq F_{h(U)} h(z), \tag{2.7}
\]
then
\[
F_{(W_{\alpha, \beta}^{k, \lambda} f(z))'} ((W_{\alpha, \beta}^{k, \lambda} f(z))') \leq F_{q(U)} q(z),
\]
i.e.,
\[
(W_{\alpha, \beta}^{k, \lambda} f(z))' \preceq_F q(z)
\]
and \( q \) is fuzzy best dominant.

**Proof.** Assume that
\[
\mathcal{P}(z) = (W_{\alpha, \beta}^{k, \lambda} f(z))'. \tag{2.8}
\]
It is clear that \( p \in \mathcal{H}[1, 1] \).
By simple computations of (2.8), we find that
\[
\mathcal{P}(z) + \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\beta^m}{\alpha^m + \beta^m} \right) \right] z\mathcal{P}'(z)
\]
\[
= (W_{\frac{\alpha}{\beta}}^{k,\lambda} f(z))' + \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\beta^m}{\alpha^m + \beta^m} \right) \right] z(W_{\frac{\alpha}{\beta}}^{k,\lambda} f(z))''.
\] (2.9)

Using (1.3) and differentiating with respect to \( z \), we obtain
\[
(W_{\frac{\alpha}{\beta}}^{k,\lambda} f(z))' = (W_{\frac{\alpha}{\beta}}^{k,\lambda} f(z))' + \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\beta^m}{\alpha^m + \beta^m} \right) \right] z(W_{\frac{\alpha}{\beta}}^{k,\lambda} f(z))''.
\] (2.10)

In the light of (2.9) and (2.10), (2.7) becomes
\[
F_{\psi(\mathbb{C}^2 \times \mathcal{U})} \left[ \mathcal{P}(z) + \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\beta^m}{\alpha^m + \beta^m} \right) \right] z\mathcal{P}'(z) \right] \leq F_{h(\mathcal{U})} h(z).
\]

Thus applying Lemma 1.2 with \( v = \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\beta^m}{\alpha^m + \beta^m} \right) \right] \), we obtain
\[
F_{(W_{\frac{\alpha}{\beta}}^{k,\lambda} f)(\mathcal{U})} ((W_{\frac{\alpha}{\beta}}^{k,\lambda} f(z))') \leq F_{q(\mathcal{U})} q(z),
\]
i.e.,
\[
(W_{\frac{\alpha}{\beta}}^{k,\lambda} f(z))' \prec_{F} q(z)
\]
and \( q \) is fuzzy best dominant.

**Theorem 2.4.** Suppose that \( q \) is a convex function in \( \mathcal{U} \) such that \( q(0) = 1 \),
\[
h(z) = q(z) + zq'(z). \text{ Let } f \in \mathcal{A} \text{ and } \left( \frac{zW_{\frac{\alpha}{\beta}}^{k,\lambda} f(z)'}{W_{\frac{\alpha}{\beta}}^{k,\lambda} f(z)} \right) \text{ is holomorphic in } \mathcal{U}. \text{ If }
\]
\[
F_{\psi(\mathbb{C}^2 \times \mathcal{U})} \left[ \left( \frac{zW_{\frac{\alpha}{\beta}}^{k,\lambda} f(z)'}{W_{\frac{\alpha}{\beta}}^{k,\lambda} f(z)} \right) \right] \leq F_{h(\mathcal{U})} h(z),
\] (2.11)
then

\[
F\left[\frac{W^{k,\lambda+1}_{\alpha,\beta}f}{W^{k,\lambda}_{\alpha,\beta}f}\right]_{(L)}\left[\frac{W^{k,\lambda+1}_{\alpha,\beta}f(z)}{W^{k,\lambda}_{\alpha,\beta}f(z)}\right] \leq F_{q(L)}q(z),
\]

i.e.,

\[
\frac{W^{k,\lambda+1}_{\alpha,\beta}f(z)}{W^{k,\lambda}_{\alpha,\beta}f(z)} \preceq_F q(z)
\]

and \( q \) is fuzzy best dominant.

**Proof.** Assume that

\[
\mathcal{P}(z) = \frac{W^{k,\lambda+1}_{\alpha,\beta}f(z)}{W^{k,\lambda}_{\alpha,\beta}f(z)}.
\]  

(2.12)

Therefore, we note that \( \mathcal{P} \in \mathcal{H}[1, 1] \).

Differentiating both sides of (2.12) with respect to \( z \), it yields

\[
\mathcal{P}'(z) = \frac{(W^{k,\lambda+1}_{\alpha,\beta}f(z))'}{W^{k,\lambda}_{\alpha,\beta}f(z)} - \mathcal{P}(z) \frac{(W^{k,\lambda}_{\alpha,\beta}f(z))'}{W^{k,\lambda}_{\alpha,\beta}f(z)}.
\]

Then

\[
\mathcal{P}(z) + z\mathcal{P}'(z) = \frac{W^{k,\lambda}_{\alpha,\beta}f(z)(W^{k,\lambda+1}_{\alpha,\beta}f(z))'}{W^{k,\lambda}_{\alpha,\beta}f(z)} - zW^{k,\lambda+1}_{\alpha,\beta}f(z)W^{k,\lambda}_{\alpha,\beta}f(z) + zW^{k,\lambda}_{\alpha,\beta}f(z)W^{k,\lambda+1}_{\alpha,\beta}f(z)
\]

\[
= \left(\frac{zW^{k,\lambda+1}_{\alpha,\beta}f(z)}{W^{k,\lambda}_{\alpha,\beta}f(z)}\right)'.
\]  

(2.13)

Utilizing (2.13) in (2.11), we can get

\[
F_{\mathcal{W}(\mathbb{C}^2 \times L)}[\mathcal{P}(z) + z\mathcal{P}'(z)] \leq F_{h(L)}h(z).
\]
Thus applying Lemma 1.2 with \( v = 1 \), we obtain

\[
F_{\left( \begin{array}{c} W^{k,\lambda+1}_{\alpha,\beta} f \\ W^{k,\lambda}_{\alpha,\beta} f \end{array} \right)} (z) \leq F_{q(z)}(z),
\]

i.e.,

\[
\frac{W^{k,\lambda+1}_{\alpha,\beta} f(z)}{W^{k,\lambda}_{\alpha,\beta} f(z)} \preceq_F q(z)
\]

and \( q \) is fuzzy best dominant.

References


