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Some Properties for Fuzzy Differential Subordination Defined by Wanas Operator

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Abstract

In this article, we establish some interesting geometric properties for fuzzy differential subordination associated with Wanas operator which defined in the open unit disk.

1. Introduction

Let the notation $\mathcal{H}(\mathcal{U})$ stand for the family of holomorphic functions in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, we indicate by

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathcal{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in \mathcal{U}\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}(\mathcal{U}) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots, z \in \mathcal{U}\},$$

with $\mathcal{A}_1 = \mathcal{A}$.

Definition 1.1 [13]**.** Let X be a non-empty set. An application $F : X \to [0, 1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:

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A pair (S, F_S), where $F_S : X \to [0, 1]$ and $supp(S, F_S) = \{x \in X : 0 < F_S(x) \leq 1\}$ is called fuzzy subset. The function F_S is called membership function of the fuzzy subset (S, F_S).

Definition 1.2 [6]**.** Let two fuzzy subsets of X, (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are *equal* if and only if $F_M(x) = F_N(x)$, $x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \leq F_N(x)$, $x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.

Assume that D is a set in $\mathbb{C}$ and f, g are holomorphic functions. We indicate by

$$f(D) = supp(f(D), F_{f(D)}) = \{f(z) : 0 < F_{f(D)}(f(z)) \leq 1, z \in D\}$$

and

$$g(D) = supp(g(D), F_{g(D)}) = \{g(z) : 0 < F_{g(D)}(g(z)) \leq 1, z \in D\}.$$

Definition 1.3 [6]**.** Suppose that D is a set in $\mathbb{C}$, $z_0 \in D$ is a fixed point and let the functions $f, g \in \mathcal{H}(D)$. The function f is named a fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if satisfies the following:

(1) $f(z_0) = g(z_0)$,

(2) $F_{f(D)}(f(z)) \leq F_{g(D)}(g(z))$, $z \in D$.

Definition 1.4 [7]**.** Let h be univalent in $\mathcal{U}$ and $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. If $\mathcal{P}$ is holomorphic in $\mathcal{U}$ satisfies the fuzzy differential subordination:

$$F_{\psi(\mathbb{C}^3 \times \mathcal{U})}(\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z)) \leq F_{h(\mathcal{U})}(h(z)), \tag{1.1}$$

i.e.,

$$\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z) \prec_F h(z), z \in \mathcal{U},$$

then $\mathcal{P}$ is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $\mathcal{P}(z) \prec_F q(z)$, $z \in \mathcal{U}$ for all $\mathcal{P}$

satisfying (1.1). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z),\ z \in \mathcal{U}$ for all fuzzy dominant q of (1.1) is said to be the fuzzy best dominant of (1.1).

For $\alpha \in \mathbb{R},\ \beta \geq 0$ with $\alpha + \beta > 0,\ m,\ \lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$, we consider the differential operator $W_{\alpha,\beta}^{k,\lambda} : \mathcal{A} \to \mathcal{A}$, introduced by Wanas [11], where

$$W_{\alpha,\beta}^{k,\lambda} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\lambda} a_n z^n. \tag{1.2}$$

By making use of (1.2), it is evident that

$$z(W_{\alpha,\beta}^{k,\lambda} f(z))' = \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \right] W_{\alpha,\beta}^{k,\lambda+1} f(z)$$

$$- \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right] W_{\alpha,\beta}^{k,\lambda} f(z). \tag{1.3}$$

We will need the following lemmas in investigating our main results.

Lemma 1.1 [5]**.** *Suppose that the convex function h satisfies $h(0) = a$, let $\mu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ such that $\mathrm{Re}(\mu) \geq 0$. If $\mathcal{P} \in \mathcal{H}[a, n]$ with $\mathcal{P}(0) = a$ and $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$, $\psi(\mathcal{P}(z),\ z\mathcal{P}'(z)) = \mathcal{P}(z) + \frac{1}{\mu} z\mathcal{P}'(z)$ is holomorphic in $\mathcal{U}$, then*

$$F_{\psi(\mathbb{C}^2 \times \mathcal{U})}\left[\mathcal{P}(z) + \frac{1}{\mu} z\mathcal{P}'(z) \right] \leq F_{h(\mathcal{U})} h(z),$$

implies

$$F_{\mathcal{P}(\mathcal{U})}\mathcal{P}(z) \leq F_{q(\mathcal{U})} q(z) \leq F_{h(\mathcal{U})} h(z), \quad z \in \mathcal{U},$$

i.e.,

$$\mathcal{P}(z) \prec_F q(z) \prec_F h(z),$$

where

$$q(z) = \frac{\mu}{n z^{\frac{\mu}{n}}} \int_0^z h(t) t^{\frac{\mu}{n} - 1} dt$$

is convex and is the fuzzy best dominant.

Lemma 1.2 [5]. *Suppose that* q *is a convex function in* $\mathcal{U}$, *let* $h(z) = q(z) + nvzq'(z)$, $v > 0$ *and* $n \in \mathbb{N}$. *If* $\mathcal{P} \in \mathcal{H}[q(0), n]$ *and* $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$, $\psi(\mathcal{P}(z), z\mathcal{P}'(z)) = \mathcal{P}(z) + vz\mathcal{P}'(z)$ *is holomorphic in* $\mathcal{U}$, *then*

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}[\mathcal{P}(z) + vz\mathcal{P}'(z)] \leq F_{h(\mathcal{U})}h(z),$$

implies

$$F_{\mathcal{P}(\mathcal{U})}\mathcal{P}(z) \leq F_{q(\mathcal{U})}q(z), \quad z \in \mathcal{U},$$

i.e.,

$$\mathcal{P}(z) \prec_F q(z)$$

and q *is the fuzzy best dominant.*

Recently, Oros and Oros [7, 8], Lupaş [2-5], Haydar [1] and Wanas and Majeed [10, 11, 12] have obtained fuzzy differential subordination results for certain classes of holomorphic functions.

2. Main Results

Theorem 2.1. *Suppose that convex function* h *satisfies* $h(0) = 1$. *Let* $f \in \mathcal{A}$ *and*

$$\frac{1}{z}\left(\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\right]W_{\alpha,\beta}^{k,\lambda+1}f(z) - \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right]W_{\alpha,\beta}^{k,\lambda}f(z)\right)$$
$$+ z(W_{\alpha,\beta}^{k,\lambda}f(z))''$$

is holomorphic in $\mathcal{U}$.

If

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{1}{z}\left(\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\right]W_{\alpha,\beta}^{k,\lambda+1}f(z)\right.\right.$$
$$\left.\left.- \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right]W_{\alpha,\beta}^{k,\lambda}f(z)\right) + z(W_{\alpha,\beta}^{k,\lambda}f(z))''\right] \leq F_{h(\mathcal{U})}h(z), \tag{2.1}$$

then

$$F_{(W_{\alpha,\beta}^{k,\lambda} f)'(\mathcal{U})}((W_{\alpha,\beta}^{k,\lambda} f(z))') \leq F_{q(\mathcal{U})} q(z) \leq F_{h(\mathcal{U})} h(z),$$

i.e.

$$(W_{\alpha,\beta}^{k,\lambda} f(z))' \prec_F \; q(z) \prec_F \; h(z),$$

where $q(z) = \frac{1}{z} \int_0^z h(t)$ *is convex and is the fuzzy best dominant.*

Proof. Assume that

$$\mathcal{P}(z) = (W_{\alpha,\beta}^{k,\lambda} f(z))'. \tag{2.2}$$

Then $\mathcal{P} \in \mathcal{H}[1, 1]$ and $\mathcal{P}(0) = 1$. Therefore, in view of (1.3) and (2.2), we have

$$\mathcal{P}(z) + z\mathcal{P}'(z) = 1 + \sum_{n=2}^{\infty} n^2 \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\lambda} a_n z^{n-1}$$

$$= \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right)$$

$$\times \left[1 + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\lambda+1} a_n z^{n-1} \right]$$

$$- \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m$$

$$\times \left[1 + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\lambda} a_n z^{n-1} \right]$$

$$+ \sum_{n=2}^{\infty} n(n-1) \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\lambda} a_n z^{n-1}$$

$$= \frac{1}{z}\left(\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]W_{\alpha,\beta}^{k,\lambda+1}f(z)\right.$$

$$\left.-\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right]W_{\alpha,\beta}^{k,\lambda}f(z)\right)+z(W_{\alpha,\beta}^{k,\lambda}f(z))''. \tag{2.3}$$

According to (2.1) and (2.3), we deduce that

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}[\mathcal{P}(z)+z\mathcal{P}'(z)] \le F_{h(\mathcal{U})}h(z).$$

Thus applying Lemma 1.1 with $\mu = 1$, we obtain

$$F_{\mathcal{P}(\mathcal{U})}\mathcal{P}(z) \le F_{q(\mathcal{U})}q(z) \le F_{h(\mathcal{U})}h(z).$$

From (2.2), we find that

$$F_{(W_{\alpha,\beta}^{k,\lambda}f)'(\mathcal{U})}((W_{\alpha,\beta}^{k,\lambda}f(z))') \le F_{q(\mathcal{U})}q(z) \le F_{h(\mathcal{U})}h(z),$$

i.e.,

$$(W_{\alpha,\beta}^{k,\lambda}f(z))' \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant.

Putting $\lambda = 0$ and $h(z) = \frac{1+(2\rho-1)z}{1+z}$ $(0 \le \rho < 1)$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. *Let* $f \in \mathcal{A}$ *and* $zf''(z)+f'(z)$ *is holomorphic in* $\mathcal{U}$. *If*

$$zf''(z)+f'(z) \prec_F \frac{1+(2\rho-1)z}{1+z},$$

then

$$f'(z) \prec_F q(z) \prec_F \frac{1+(2\rho-1)z}{1+z},$$

where $q(z) = 2\rho - 1 + \frac{2(1-\rho)}{z}\ln(1+z)$ *is convex and is the fuzzy best dominant.*

Theorem 2.2. *Suppose that the convex function h satisfies* $h(0) = 1$. *Let* $f \in \mathcal{A}$ *and* $(W_{\alpha,\beta}^{k,\lambda} f(z))'$ *is holomorphic in* $\mathcal{U}$. *If*

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}[(W_{\alpha,\beta}^{k,\lambda} f(z))'] \leq F_{h(\mathcal{U})}h(z), \tag{2.4}$$

then

$$F_{(W_{\alpha,\beta}^{k,\lambda} f)(\mathcal{U})}\left(\frac{W_{\alpha,\beta}^{k,\lambda} f(z)}{z}\right) \leq F_{q(\mathcal{U})}q(z) \leq F_{h(\mathcal{U})}h(z),$$

i.e.,

$$\frac{W_{\alpha,\beta}^{k,\lambda} f(z)}{z} \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ *is convex and is the fuzzy best dominant.*

Proof. Assume that

$$\mathcal{P}(z) = \frac{W_{\alpha,\beta}^{k,\lambda} f(z)}{z}. \tag{2.5}$$

It is clear that $\mathcal{P} \in \mathcal{H}[1, 1]$ and $\mathcal{P}(0) = 1$.

We find

$$\mathcal{P}(z) + z\mathcal{P}'(z) = (W_{\alpha,\beta}^{k,\lambda} f(z))'. \tag{2.6}$$

In view of (2.6), the fuzzy differential subordination (2.4) becomes

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}[\mathcal{P}(z) + z\mathcal{P}'(z)] \leq F_{h(\mathcal{U})}h(z).$$

Thus applying Lemma 1.1 with $\mu = 1$, we obtain

$$F_{\mathcal{P}(\mathcal{U})}\mathcal{P}(z) \leq F_{q(\mathcal{U})}q(z) \leq F_{h(\mathcal{U})}h(z).$$

From (2.5), we get

$$F_{(W_{\alpha,\beta}^{k,\lambda} f)(\mathcal{U})}\left(\frac{W_{\alpha,\beta}^{k,\lambda} f(z)}{z}\right) \leq F_{q(\mathcal{U})}q(z) \leq F_{h(\mathcal{U})}h(z),$$

i.e.,

$$\frac{W_{\alpha,\beta}^{k,\lambda} f(z)}{z} \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant.

Putting $\lambda = 0$ and $h(z) = e^{bz}, |b| \le 1$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.2. *If $f \in \mathcal{A}$, $f'(z)$ is holomorphic in $\mathcal{U}$ and $f'(z) \prec_F e^{bz}$, then*

$$\frac{f(z)}{z} \prec_F q(z) \prec_F e^{bz},$$

where $q(z) = \frac{e^{bz} - 1}{bz}$ is convex and is the fuzzy best dominant.

Theorem 2.3. *Suppose that q is a convex function in $\mathcal{U}$ such that $q(0) = 1$, $h(z) = q(z) + \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\beta^m}{\alpha^m + \beta^m}\right)\right] zq'(z)$. Let $f \in \mathcal{A}$ and $(W_{\alpha,\beta}^{k,\lambda+1} f(z))'$ is holomorphic in $\mathcal{U}$. If*

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}[(W_{\alpha,\beta}^{k,\lambda+1} f(z))'] \le F_{h(\mathcal{U})}h(z), \tag{2.7}$$

then

$$F_{(W_{\alpha,\beta}^{k,\lambda} f)'(\mathcal{U})}((W_{\alpha,\beta}^{k,\lambda} f(z))') \le F_{q(\mathcal{U})}q(z),$$

i.e.,

$$(W_{\alpha,\beta}^{k,\lambda} f(z))' \prec_F q(z)$$

and q is fuzzy best dominant.

Proof. Assume that

$$\mathcal{P}(z) = (W_{\alpha,\beta}^{k,\lambda} f(z))'. \tag{2.8}$$

It is clear that $p \in \mathcal{H}[1, 1]$.

By simple computations of (2.8), we find that

$$\mathcal{P}(z) + \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\beta^m}{\alpha^m + \beta^m}\right)\right] z\mathcal{P}'(z)$$

$$= (W_{\alpha,\beta}^{k,\lambda} f(z))' + \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\beta^m}{\alpha^m + \beta^m}\right)\right] z(W_{\alpha,\beta}^{k,\lambda} f(z))''. \tag{2.9}$$

Using (1.3) and differentiating with respect to z, we obtain

$$(W_{\alpha,\beta}^{k,\lambda+1} f(z))' = (W_{\alpha,\beta}^{k,\lambda} f(z))' + \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\beta^m}{\alpha^m + \beta^m}\right)\right] z(W_{\alpha,\beta}^{k,\lambda} f(z))''. \tag{2.10}$$

In the light of (2.9) and (2.10), (2.7) becomes

$$F_{\psi(\mathbb{C}^2 \times \mathcal{U})}\left[\mathcal{P}(z) + \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\beta^m}{\alpha^m + \beta^m}\right)\right] z\mathcal{P}'(z)\right] \leq F_{h(\mathcal{U})} h(z).$$

Thus applying Lemma 1.2 with $v = \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\beta^m}{\alpha^m + \beta^m}\right)\right]$, we obtain

$$F_{(W_{\alpha,\beta}^{k,\lambda} f)'(\mathcal{U})}((W_{\alpha,\beta}^{k,\lambda} f(z))') \leq F_{q(\mathcal{U})} q(z),$$

i.e.,

$$(W_{\alpha,\beta}^{k,\lambda} f(z))' \prec_F q(z)$$

and q is fuzzy best dominant.

Theorem 2.4. *Suppose that q is a convex function in $\mathcal{U}$ such that $q(0) = 1$, $h(z) = q(z) + zq'(z)$. Let $f \in \mathcal{A}$ and $\left(\frac{zW_{\alpha,\beta}^{k,\lambda+1} f(z)}{W_{\alpha,\beta}^{k,\lambda} f(z)}\right)'$ is holomorphic in $\mathcal{U}$. If*

$$F_{\psi(\mathbb{C}^2 \times \mathcal{U})}\left[\left(\frac{zW_{\alpha,\beta}^{k,\lambda+1} f(z)}{W_{\alpha,\beta}^{k,\lambda} f(z)}\right)'\right] \leq F_{h(\mathcal{U})} h(z), \tag{2.11}$$

then

$$F_{\left(\frac{W_{\alpha,\beta}^{k,\lambda+1}f}{W_{\alpha,\beta}^{k,\lambda}f}\right)(\mathcal{U})}\left(\frac{W_{\alpha,\beta}^{k,\lambda+1}f(z)}{W_{\alpha,\beta}^{k,\lambda}f(z)}\right) \leq F_{q(\mathcal{U})}q(z),$$

i.e.,

$$\frac{W_{\alpha,\beta}^{k,\lambda+1}f(z)}{W_{\alpha,\beta}^{k,\lambda}f(z)} \prec_F q(z)$$

and q is fuzzy best dominant.

Proof. Assume that

$$\mathcal{P}(z) = \frac{W_{\alpha,\beta}^{k,\lambda+1}f(z)}{W_{\alpha,\beta}^{k,\lambda}f(z)}. \tag{2.12}$$

Therefore, we note that $\mathcal{P} \in \mathcal{H}[1, 1]$.

Differentiating both sides of (2.12) with respect to z, it yields

$$\mathcal{P}'(z) = \frac{(W_{\alpha,\beta}^{k,\lambda+1}f(z))'}{W_{\alpha,\beta}^{k,\lambda}f(z)} - \mathcal{P}(z)\frac{(W_{\alpha,\beta}^{k,\lambda}f(z))'}{W_{\alpha,\beta}^{k,\lambda}f(z)}.$$

Then

$$\mathcal{P}(z) + z\mathcal{P}'(z) = \frac{W_{\alpha,\beta}^{k,\lambda}f(z)(z(W_{\alpha,\beta}^{k,\lambda+1}f(z))' + W_{\alpha,\beta}^{k,\lambda+1}f(z)) - zW_{\alpha,\beta}^{k,\lambda+1}f(z)(W_{\alpha,\beta}^{k,\lambda}f(z))'}{(W_{\alpha,\beta}^{k,\lambda}f(z))^2}$$

$$= \left(\frac{zW_{\alpha,\beta}^{k,\lambda+1}f(z)}{W_{\alpha,\beta}^{k,\lambda}f(z)}\right)'. \tag{2.13}$$

Utilizing (2.13) in (2.11), we can get

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}[\mathcal{P}(z) + z\mathcal{P}'(z)] \leq F_{h(\mathcal{U})}h(z).$$

Thus applying Lemma 1.2 with $v = 1,$ we obtain

$$F_{\left(\frac{W_{\alpha,\beta}^{k,\lambda+1} f}{W_{\alpha,\beta}^{k,\lambda} f}\right)(\mathcal{U})}\left(\frac{W_{\alpha,\beta}^{k,\lambda+1} f(z)}{W_{\alpha,\beta}^{k,\lambda} f(z)}\right) \leq F_{q(\mathcal{U})} q(z),$$

i.e.,

$$\frac{W_{\alpha,\beta}^{k,\lambda+1} f(z)}{W_{\alpha,\beta}^{k,\lambda} f(z)} \prec_F q(z)$$

and q is fuzzy best dominant.

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