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Anti Q-fuzzy Subgroups under t-conorms

Rasul Rasuli

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran
e-mail: rasulirasul@yahoo.com

Abstract

In this paper we introduce the notion of anti Q-fuzzy subgroups of G with respect to t-conorm C and study their important properties. Next we define the union, normal and direct product of two anti Q-fuzzy subgroups of G with respect to t-conorm C and we show that the union, normal and direct product of them is again an anti Q-fuzzy subgroup of G with respect to t-conorm C. It is also shown that the homomorphic image and pre image of anti Q-fuzzy subgroup of G with respect to t-conorm C is again an anti Q-fuzzy subgroup of G with respect to t-conorm C.

1. Introduction

Undoubtedly the notion of fuzzy set theory initiated by Zadeh [31] in 1965 in a seminal paper, plays the central role for further development. This notion tries to show that an object corresponds more or less to the particular category we want to assimilate it to; that was how the idea of defining the membership of an element to a set not on the Aristotelian pair $\{0, 1\}$ any more but on the continuous interval $[0, 1]$ was born. The notion of a fuzzy set is completely non-statistical in nature and the concept of fuzzy set provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables. Since the concept of fuzzy group was introduced by Rosenfeld in [27]

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in 1971, the theories and approaches on different fuzzy algebraic structures developed rapidly. Yuan and Lee [30] defined the fuzzy subgroup and fuzzy subring based on the theory of falling shadows. Also Solairaju and Nagarajan [29] introduced the notion of Q-fuzzy groups. The triangular conorm (t-conorm) originated from the studies of probabilistic metric spaces [5, 28] in which triangular inequalities were extended using the theory of t-conorm. The author by using norms, investigated some properties of fuzzy algebraic structures [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. In Section 2, some preliminary definitions and results regarding multisets have been introduced. In Section 3, we have studied anti Q-fuzzy subgroups of G with respect to t-conorm C. Also the union, normal and direct product of them has also been discussed here. In Section 4, the homomorphic image and pre image of them have been investigated under group homomorphisms and anti group homomorphisms.

2. Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief.

Definition 2.1 (See [3])**.** A group is a non-empty set G on which there is a binary operation $(a, b) \to ab$ such that

(1) if a and b belong to G, then ab is also in G (closure),

(2) $a(bc) = (ab)c$ for all $a, b, c \in G$ (associativity),

(3) there is an element $e \in G$ such that $ae = ea = a$ for all $a \in G$ (identity),

(4) if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called *abelian* if the binary operation is commutative, i.e., $ab = ba$ for all $a, b \in G$.

Remark 2.2. There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \to a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \to ab$ for which the identity is denoted by e.

Proposition 2.3 (See [3])**.** *Let G be a group. Let H be a non-empty subset of G. Then the following are equivalent*:

(1) *H is a subgroup of G.*

(2) $x,\ y \in H$ *implies* $xy^{-1} \in H$ *for all* $x,\ y.$

Definition 2.4 (See [3]). Let $(G, \cdot), (H, \cdot)$ be any two groups. The function $f: G \to H$ is called a *homomorphism* (*anti-homomorphism*) if $f(xy) = f(x)f(y)$ $(f(xy) = f(y)f(x))$, for all $x,\ y \in G.$

Definition 2.5 (See [4]). Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0, 1]$. The set of all fuzzy subsets of G is called the $[0, 1]$-*power set* of G and is denoted $[0, 1]^G$.

Definition 2.6 (See [4]). Let $\varphi : A \to B$ be a function such that $\mu \in [0, 1]^{A\times Q}$ and $\nu \in [0, 1]^{B\times Q}$. Then *fuzzy image* $\varphi(\mu)$ of μ under φ is defined by

$$\varphi(\mu)(y, q) = \begin{cases} \inf\{\mu(x, q) \mid (x, q) \in A \times Q,\ \varphi(x) = y\} & \text{if } \varphi^{-1}(y) \neq \varnothing \\ 0 & \text{if } \varphi^{-1}(y) = \varnothing \end{cases}$$

and *fuzzy pre-image* (or *fuzzy inverse image*) of ν under φ is

$$\varphi^{-1}(\nu)(x, q) = \nu(\varphi(x), q)$$

for all $(x, q) \in A \times Q.$

Definition 2.7 (See [2]). A *t-conorm* C is a function $C : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:

(C1) $C(x, 0) = x$

(C2) $C(x,\ y) \le C(x,\ z)$ if $y \le z$

(C3) $C(x,\ y) = C(y,\ x)$

(C4) $C(x,\ C(y,\ z)) = C(C(x,\ y),\ z),$

for all $x,\ y,\ z \in [0, 1].$

Example 2.8. (1) Standard union t-conorm $C_m(x,\ y) = \max\{x,\ y\}.$

(2) Bounded sum t-conorm $C_b(x, y) = \min\{1, x + y\}$.

(3) Algebraic sum t-conorm $C_p(x, y) = x + y - xy$.

(4) Drastic t-conorm

$$C_D(x, y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic t-norm.

(5) Nilpotent maximum t-conorm, dual to the nilpotent minimum t-norm:

$$C_{nM}(x, y) = \begin{cases} \max\{x, y\} & \text{if } x + y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x, y) = \dfrac{x + y}{1 + xy}$ is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x, y) \leq C(x, y) \leq C_D(x, y)$ for any t-conorm C and all $x, y \in [0, 1]$.

Recall that t-conorm C is idempotent if for all $x \in [0, 1]$, we have that $C(x, x) = x$.

Lemma 2.9 (See [1]). *Let C be a t-conorm. Then*

$$C(C(x, y), C(w, z)) = C(C(x, w), C(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

3. Anti Q-fuzzy Subgroups and t-conorms

Definition 3.1. Let $(G, \cdot)$ be a group and Q be a non empty set. Then $\mu \in [0, 1]^{G \times Q}$ is said to be an *anti Q-fuzzy subgroup* of G with respect to t-conorm C if the following conditions are satisfied:

(1) $\mu(xy, q) \leq C(\mu(x, q), \mu(y, q))$,

(2) $\mu(x^{-1}, q) \leq \mu(x, q)$,

for all $x, y \in G$ and $q \in Q$.

Throughout this paper the set of all anti Q-fuzzy subgroups of G with respect to t-conorm C will be denoted by $AQFSC(G)$.

Remark 3.2. The condition (2) of Definition 3.1 implies that

$$\mu(x, q) = \mu((x^{-1})^{-1}, q) \leq \mu(x^{-1}, q) \leq \mu(x, q)$$

and then $\mu(x, q) = \mu(x^{-1}, q)$.

Lemma 3.3. *Let $\mu \in AQFSC(G)$ and C be idempotent t-conorm. Then $\mu(e_G, q) \leq \mu(x, q)$ for all $x \in G$ and $q \in Q$.*

Proof. Since $\mu \in AQFSC(G)$, so

$$\mu(e_G, q) = \mu(xx^{-1}, q) \leq C(\mu(x, q), \mu(x^{-1}, q)) = C(\mu(x, q), \mu(x, q)) = \mu(x, q)$$

for all $x \in G$, $q \in Q$. □

Proposition 3.4. *Let $\mu \in AQFSC(G)$ and C be idempotent t-conorm. If $\mu(xy^{-1}, q) = \mu(e_G, q)$, then $\mu(x, q) = \mu(y, q)$ for all $x, y \in G$ and $q \in Q$.*

Proof. Let $x, y \in G$ and $q \in Q$. Then

$$\begin{aligned}
\mu(x, q) &= \mu(xy^{-1}y, q) \leq C(\mu(xy^{-1}, q), \mu(y, q)) \\
&= C(\mu(e_G, q), \mu(y, q)) \leq C(\mu(y, q), \mu(y, q)) \\
&= \mu(y, q) = \mu(yx^{-1}x, q) \leq C(\mu(yx^{-1}, q), \mu(x, q)) \\
&= C(\mu((xy^{-1})^{-1}, q), \mu(x, q)) = C(\mu(xy^{-1}, q), \mu(x, q)) \\
&= C(\mu(e_G, q), \mu(x, q)) \leq C(\mu(x, q), \mu(x, q)) = \mu(x, q).
\end{aligned}$$

Thus $\mu(x, q) = \mu(y, q)$. □

Proposition 3.5. *Let C be idempotent t-conorm. Then $\mu \in AQFSC(G)$ if and only if*

$$\mu(xy^{-1}, q) \leq C(\mu(x, q), \mu(y, q))$$

for all $x, y \in G$ and $q \in Q$.

Proof. Let $\mu \in AQFSC(G)$ and $x, y \in G, q \in Q$. Then

$$\mu(xy^{-1}, q) \leq C(\mu(x, q), \mu(y^{-1}, q)) \leq C(\mu(x, q), \mu(y, q)).$$

Conversely, let $\mu(xy^{-1}, q) \leq C(\mu(x, q), \mu(y, q))$. Then

$$\mu(x^{-1}, q) = \mu(e_G x^{-1}, q) \leq C(\mu(e_G, q), \mu(x, q)) \leq C(\mu(x, q), \mu(x, q)) = \mu(x, q)$$

and then $\mu(x^{-1}, q) \leq \mu(x, q)$.

Also

$$\mu(xy, q) = \mu(x(y^{-1})^{-1}, q) \leq C(\mu(x, q), \mu(y^{-1}, q)) \leq C(\mu(x, q), \mu(y, q)).$$

Then $\mu \in AQFSC(G)$. □

Proposition 3.6. *Let* $\mu \in [0, 1]^{G\times Q}$ *such that* $\mu(e_G, q) = 0$ *and* $\mu(xy^{-1}, q) \leq C(\mu(x, q), \mu(y, q))$ *for all* $x, y \in G$ *and* $q \in Q$. *Then* $\mu \in AQFSC(G)$.

Proof. Let $x, y \in G$ and $q \in Q$. Then

$$\mu(x^{-1}, q) = \mu(e_G x^{-1}, q) \leq C(\mu(e_G, q), \mu(x, q)) = C(0, \mu(x, q)) = \mu(x, q)$$

and so $\mu(x^{-1}, q) \leq \mu(x, q)$. Now

$$\mu(xy, q) = \mu(x((y)^{-1})^{-1}, q) \leq C(\mu(x, q), \mu(y^{-1}, q)) \leq C(\mu(x, q), \mu(y, q))$$

and so $\mu(xy, q) \leq C(\mu(x, q), \mu(y, q))$. Thus $\mu \in AQFSC(G)$. □

Proposition 3.7. *If* $\mu \in AQFSC(G)$, *then*

$$H = \{x \in G \mid f_{G\times Q}(x, q) = 0\}$$

is a subgroup of G.

Proof. Let $x, y \in H$ and $q \in Q$. As $\mu \in AQFSC(G)$, then

$$\mu(xy^{-1}, q) \leq C(\mu(x, q), \mu(y, q)) = C(0, 0) = 0.$$

This implies that $\mu(xy^{-1}, q) = 0$ and then $xy^{-1} \in H$ and from Proposition 2.3 we get H is a subgroup of G. □

Proposition 3.8. *Let $\mu \in AQFSC(G)$ and C be idempotent t-conorm. Then*

$$H = \{x \in G \mid \mu(x, q) = \mu(e_G, q)\}$$

is a subgroup of G.

Proof. Let $x, y \in H$, $q \in Q$ and $\mu \in AQFSC(G)$. Then

$$\mu(xy^{-1}, q) \le C(\mu(x, q), \mu(y, q)) = C(\mu(e_G, q), \mu(e_G, q)) = \mu(e_G, q) \le \mu(xy^{-1}, q)$$

and so $\mu(xy^{-1}, q) = \mu(e_G, q)$. Then $xy^{-1} \in H$ and Proposition 2.3 gives us H is a subgroup of G. □

Proposition 3.9. *Let $\mu \in AQFSC(G)$ and $\mu(xy^{-1}, q) = 0$. Then*

$$\mu(x, q) = \mu(y, q)$$

for all $x, y \in G$ and $q \in Q$.

Proof. Let $\mu \in AQFSC(G)$ and $x, y \in G$, $q \in Q$. Then

$$\mu(x, q) = \mu(xy^{-1}y, q) \le C(\mu(xy^{-1}, q), \mu(y, q)) = C(0, \mu(y, q)) = \mu(y, q)$$

$$= \mu(y^{-1}, q) = \mu(x^{-1}xy^{-1}, q) \le C(\mu(x^{-1}, q), \mu(xy^{-1}, q))$$

$$= C(\mu(x^{-1}, q), 0) = \mu(x^{-1}, q) = \mu(x, q).$$

Hence $\mu(x, q) = \mu(y, q)$. □

Proposition 3.10. *Let $\mu \in AQFSC(G)$. Then*

$$\mu(xy, q) = \mu(yx, q)$$

if and only if

$$\mu(x, q) = \mu(y^{-1}xy, q)$$

for all $x, y \in G$ and $q \in Q$.

Proof. Let $x, y \in G$, $q \in Q$ and $\mu(xy, q) = \mu(yx, q)$. Then

$$\mu(y^{-1}xy, q) = \mu(y^{-1}(xy), q) = \mu(xyy^{-1}, q) = \mu(xe_G, q) = \mu(x, q).$$

Conversely, let $\mu(x, q) = \mu(y^{-1}xy, q)$, then we obtain $\mu(xy, q) = \mu(x(yx)x^{-1}, q)$ $= \mu(yx, q)$. □

Proposition 3.11. *Let* $\mu \in AQFSC(G)$. *If* $\mu(xy^{-1}, q) = 1$, *then either* $\mu(x, q) = 1$ *or* $\mu(y, q) = 1$ *for all* $x, y \in G$ *and* $q \in Q$.

Proof. As $\mu \in AQFSC(G)$ so for all $x, y \in G$ and $q \in Q$ we have that

$$1 = \mu(xy^{-1}, q) \leq C(\mu(x, q), \mu(y, q))$$

and then either $\mu(x, q) = 1$ or $\mu(y, q) = 1$.

Definition 3.12. The *union* of $\mu, \nu \in [0, 1]^{G\times Q}$ is defined by

$$(\mu \cup \nu)(x, q) = C(\mu(x, q), \nu(x, q))$$

for all $x \in G$ and $q \in Q$.

Proposition 3.13. *Let* $\mu, \nu \in AQFSC(G)$. *Then* $\mu \cup \nu \in AQFSC(G)$.

Proof. Let $x, y \in G$, $q \in Q$. Then

$$\begin{aligned}(\mu \cup \nu)(xy, q) &= C(\mu(xy, q), \nu(xy, q)) \\ &\leq C(C(\mu(x, q), \mu(y, q)), C(\nu(x, q), \nu(y, q))) \\ &= C(C(\mu(x, q), \nu(x, q)), C(\mu(y, q), \nu(y, q))) \quad \text{(Lemma 2.9)} \\ &= C((\mu \cup \nu)(x, q), (\mu \cup \nu)(y, q)),\end{aligned}$$

also

$$\begin{aligned}(\mu \cup \nu)(x^{-1}, q) &= C(\mu(x^{-1}, q), \nu(x^{-1}, q)) \\ &\leq C(\mu(x, q), \nu(x, q)) = (\mu \cup \nu)(x, q).\end{aligned}$$

Hence $\mu \cup \nu \in AQFSC(G)$. □

Proposition 3.14. *Let* $\mu \in AQFSC(G)$ *and* $x, y \in G$, $q \in Q$. *If* C *is idempotent t-conorm and* $\mu(x, q) \neq \mu(y, q)$, *then*

$$\mu(xy, q) = C(\mu(x, q), \mu(y, q)).$$

Proof. Let $\mu(x, q) < \mu(y, q)$ for all $x, y \in G$ and $q \in Q$ and we get that $\mu(x, q) < \mu(xy, q)$ and so

$$\mu(y, q) = C(\mu(x, q), \mu(y, q))$$

and

$$\mu(xy, q) = C(\mu(x, q), \mu(xy, q)).$$

Now

$$\begin{aligned} \mu(xy, q) &\leq C(\mu(x, q), \mu(y, q)) = \mu(y, q) \\ &= \mu(x^{-1}xy, q) \leq C(\mu(x^{-1}, q), \mu(xy, q)) \\ &= C(\mu(x, q), \mu(xy, q)) = \mu(xy, q) \end{aligned}$$

and so

$$\mu(xy, q) = \mu(y, q) = C(\mu(x, q), \mu(y, q)). \qquad \square$$

Definition 3.15. We say that $\mu \in AQFSC(G)$ is a *normal* if $\mu(xyx^{-1}, q) = \mu(y, q)$ for all $x, y \in G$ and $q \in Q$. We denote by $NAQFSC(G)$ the set of all normal anti Q-fuzzy subgroups of G with respect to t-conorm C.

Proposition 3.16. *Let* $\mu_1, \mu_2 \in NAQFSC(G)$. *Then* $\mu_1 \cup \mu_2 \in NAQFSC(G)$.

Proof. If $x, y \in G$ and $q \in Q$, then

$$\begin{aligned} (\mu_1 \cup \mu_2)(xyx^{-1}, q) &= C(\mu_1(xyx^{-1}, q), \mu_2(xyx^{-1}, q)) \\ &= C(\mu_1(y, q), \mu_2(y, q)) \\ &= (\mu_1 \cup \mu_2)(y, q). \end{aligned} \qquad \square$$

Corollary 3.17. *Let* $I_n = \{1, 2, ..., n\}$. *If* $\{\mu_i \mid i \in I_n\} \subseteq NAQFSC(G)$, *then*

$$\mu = \cup_{i \in I_n} \mu_i \in NAQFSC(G).$$

Definition 3.18. Let $(G, \cdot), (H, \cdot)$ be any two groups such that $\mu \in AQFSC(G)$ and $\nu \in AQFSC(H)$. The *product* of μ and ν, denoted by $\mu \times \nu \in [0, 1]^{(G \times H) \times Q}$, is defined as

$$(\mu\times\nu)((x,\ y),\ q) = C(\mu(x,\ q),\ \nu(y,\ q))$$

for all $x \in G,\ y \in H,\ q \in Q.$

Throughout this paper, H denotes an arbitrary group with identity element e_H.

Proposition 3.19. *If* $\mu \in AQFSC(G)$ *and* $\nu \in AQFSC(H)$, *then* $\mu\times\nu \in AQFSC(G\times H)$.

Proof. Let $(x_1,\ y_1),\ (x_2,\ y_2) \in G\times H$ and $q \in Q.$ Then

$$\begin{aligned}
&(\mu\times\nu)((x_1,\ y_1)(x_2,\ y_2),\ q)\\
&= (\mu\times\nu)((x_1x_2,\ y_1y_2),\ q)\\
&= C(\mu(x_1x_2,\ q),\ \nu(y_1y_2,\ q))\\
&\le C(C(\mu(x_1,\ q),\ \mu(x_2,\ q)),\ C(\nu(y_1,\ q),\ \nu(y_2,\ q)))\\
&= C(C(\mu(x_1,\ q),\ \nu(y_1,\ q)),\ C(\mu(x_2,\ q),\ \nu(y_2,\ q)))\ \text{(Lemma 2.9)}\\
&= C((\mu\times\nu)((x_1,\ y_1),\ q),\ (\mu\times\nu)((x_2,\ y_2),\ q)).
\end{aligned}$$

Also

$$\begin{aligned}
(\mu\times\nu)((x_1,\ y_1)^{-1},\ q) &= (\mu\times\nu)((x_1^{-1},\ y_1^{-1}),\ q)\\
&= (\mu(x_1^{-1},\ q),\ \nu(y_1^{-1},\ q))\\
&\le C(\mu(x_1,\ q),\ \nu(y_1,\ q)) = (\mu\times\nu)((x_1,\ y_1),\ q).
\end{aligned}$$

Hence $\mu\times\nu \in AQFSC(G\times H).$ □

Proposition 3.20. *Let* $\mu \in [0,\ 1]^{G\times Q}$ *and* $\nu \in [0,\ 1]^{H\times Q}$. *If C be idempotent t-conorm and* $\mu\times\nu \in AQFSC(G\times H)$, *then at least one of the following two statements must hold.*

(1) $\nu(e_H,\ q) \le \mu(x,\ q)$, *for all* $x \in G$ *and* $q \in Q$,

(2) $\mu(e_G,\ q) \le \nu(y,\ q)$, *for all* $y \in H$ *and* $q \in Q$.

Proof. Let none of the statements (1) and (2) holds, then we can find $g \in G$ and

$h \in H$ such that $\mu(g, q) < \nu(e_H, q)$ and $\nu(h, q) < \mu(e_G, q)$. Now

$$\begin{aligned}(\mu \times \nu)((g, h), q) &= C(\mu(g, q), \nu(h, q)) \\ &< C(\mu(e_G, q), \nu(e_H, q)) \\ &= (\mu \times \nu)((e_H, e_H), q)\end{aligned}$$

and it is contradiction with $\mu \times \nu \in QFST(G \times H)$ (Lemma 3.3). This completes the proof. □

Proposition 3.21. *Let* $\mu \in [0, 1]^{G \times Q}$ *and* $\nu \in [0, 1]^{H \times Q}$ *such that* $\mu \times \nu \in AQFSC(G \times H)$ *and* C *is idempotent t-conorm. Then we obtain the following statements*:

(1) *If* $\mu(x, q) \geq \nu(e_H, q)$, *then* $\mu \in AQFSC(G)$ *for all* $x \in G$ *and* $q \in Q$.

(2) *If* $\nu(x, q) \geq \nu(e_G, q)$, *then* $\nu \in AQFSC(H)$ *for all* $x \in H$ *and* $q \in Q$.

(3) *Either* $\mu \in QFST(G)$ *or* $\nu \in QFST(H)$.

Proof. (1) Let $\mu(x, q) \geq \nu(e_H, q)$ for all $x \in G$, $q \in Q$ and then $\mu(xy^{-1}, q) \geq \nu(e_H, q)$ and $\mu(y, q) \geq \nu(e_H, q)$ for all $x, y \in G$, $q \in Q$. Then

$$\begin{aligned}\mu(xy^{-1}, q) &= C(\mu(xy^{-1}, q), \nu(e_H, q)) = C(\mu(xy^{-1}, q), \nu(e_H e_H, q)) \\ &= (\mu \times \nu)((xy^{-1}, e_H e_H), q) = (\mu \times \nu)((x, e_H)(y^{-1}, e_H), q) \\ &\leq C((\mu \times \nu)((x, e_H), q), (\mu \times \nu)((y^{-1}, e_H), q)) \\ &\leq C((\mu \times \nu)((x, e_H), q), (\mu \times \nu)((y, e_H), q)) \\ &= C(C(\mu(x, q), \nu(e_H, q)), C(\mu(y, q), \nu(e_H, q))) \\ &\leq C(C(\mu(x, q), \mu(x, q)), C(\mu(y, q), \mu(y, q))) \\ &= C(\mu(x, q), \mu(y, q)).\end{aligned}$$

Now from Proposition 3.5 we obtain that $\mu \in AQFSC(G)$.

(2) Let $\nu(x, q) \geq \mu(e_G, q)$ for all $x \in H$, $q \in Q$. Then $\nu(xy^{-1}, q) \geq \mu(e_G, q)$ and

$\nu(y, q) \geq \mu(e_G, q)$ for all $x, y \in H, q \in Q$. Then

$$\begin{aligned}
\nu(xy^{-1}, q) &= C(\nu(xy^{-1}, q), \mu(e_G, q)) \\
&= C(\nu(xy^{-1}, q), \mu(e_G e_G, q)) = C(\mu(e_G e_G, q), \nu(xy^{-1}, q)) \\
&= (\mu \times \nu)((e_G e_G, xy^{-1}), q) = (\mu \times \nu)((e_G, x)(e_G, y^{-1}), q) \\
&\leq C((\mu \times \nu)((e_G, x), q), (\mu \times \nu)((e_G, y^{-1}), q)) \\
&= C((\mu \times \nu)((e_G, x), q), (\mu \times \nu)((e_G, y), q)) \\
&= C(C(\mu(e_G, q), \nu(x, q)), C(\mu(e_G, q), \nu(y, q))) \\
&\leq C(C(\nu(x, q), \nu(x, q)), C(\nu(y, q), \nu(y, q))) \\
&= C(\nu(x, q), \nu(y, q)).
\end{aligned}$$

Now from Proposition 3.5 we obtain that $\nu \in AQFSC(H)$.

(3) Straightforward. □

4. Homomorphisms and Anti-homomorphisms of $AQFSC(G)$ and $NAQFSC(G)$

Proposition 4.1. *Let φ be an epimorphism from group G into group H. If $\mu \in AQFSC(G)$, then $\varphi(\mu) \in AQFSC(H)$.*

Proof. Let $h_1, h_2 \in H$ and $q \in Q$. Then

$$\begin{aligned}
\varphi(\mu)(h_1 h_2, q) &= \inf\{\mu(g_1 g_2, q) \mid g_1, g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\} \\
&\leq \inf\{C(\mu(g_1, q), \mu(g_2, q)) \mid g_1, g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\} \\
&= C((\inf\{\mu(g_1, q) \mid g_1 \in G, \varphi(g_1) = h_1\}), (\inf\{\mu(g_2, q) \mid g_2 \in G, \varphi(g_2) = h_2\})) \\
&= C(\varphi(\mu)(h_1, q), \varphi(\mu)(h_2, q)).
\end{aligned}$$

Also

$$\begin{aligned}
\varphi(\mu)(h_1^{-1}, q) &= \inf\{\mu(g_1^{-1}, q) \mid g_1 \in G, \varphi(g_1^{-1}) = h_1^{-1}\} \\
&\leq \inf\{\mu(g_1, q) \mid g_1 \in G, \varphi(g_1, q) = h_1\} = \varphi(\mu)(h_1, q).
\end{aligned}$$

Therefore $\varphi(\mu) \in AQFSC(H)$. □

Proposition 4.2. *Let* φ *be a homomorphism from group G into group H. If* $\nu \in AQFSC(H)$, *then* $\varphi^{-1}(\nu) \in AQFSC(G)$.

Proof. Let $g_1, g_2 \in G$ and $q \in Q$. Then

$$\begin{aligned}\varphi^{-1}(\nu)(g_1g_2, q) &= \nu(\varphi(g_1g_2), q) = \nu(\varphi(g_1)\varphi(g_2), q)\\ &\leq C(\nu(\varphi(g_1), q), \nu(\varphi(g_2), q))\\ &= C(\varphi^{-1}(\nu)(g_1, q), \varphi^{-1}(\nu)(g_2, q)).\end{aligned}$$

Moreover

$$\varphi^{-1}(\nu)(g_1^{-1}, q) = \nu(\varphi(g_1^{-1}), q) = \nu(\varphi^{-1}(g_1), q) \leq \nu(\varphi(g_1), q) = \varphi^{-1}(\nu)(g_1, q).$$

Then $\varphi^{-1}(\nu) \in AQFSC(G)$. □

Proposition 4.3. *Let* φ *be an anti-homomorphism from group G into group H. If* $\nu \in AQFSC(H)$, *then* $\varphi^{-1}(\nu) \in AQFSC(G)$.

Proof. Let $g_1g_2 \in G$ and $q \in Q$. Then

$$\begin{aligned}\varphi^{-1}(\nu)(g_1g_2, q) &= \nu(\varphi(g_1g_2), q) = \nu(\varphi(g_2)\varphi(g_1), q)\\ &\leq C(\nu(\varphi(g_2), q), \nu(\varphi(g_1), q))\\ &= C(\varphi^{-1}(\nu)(g_2, q), \varphi^{-1}(\nu)(g_1, q))\\ &= C(\varphi^{-1}(\nu)(g_1, q), \varphi^{-1}(\nu)(g_2, q)).\end{aligned}$$

Also

$$\varphi^{-1}(\nu)(g_1^{-1}, q) = \nu(\varphi(g_1^{-1}), q) = \nu(\varphi^{-1}(g_1), q) \leq \nu(\varphi(g_1), q) = \varphi^{-1}(\nu)(g_1, q).$$

Thus $\varphi^{-1}(\nu) \in AQFSC(G)$. □

Proposition 4.4. *Let* $\mu \in NAQFSC(G)$ *and H be a group. Suppose that* φ *is an epimorphism of G onto H. Then* $f(\mu) \in NAQFSC(H)$.

Proof. By Proposition 4.1 we have $\varphi(\mu) \in AQFSC(H)$. Let $x, y \in H$ and $q \in Q$.

Since φ is a surjection, $\varphi(u) = x$ for some $u \in G$. Then

$$\varphi(\mu)(xyx^{-1}, q) = \inf\{\mu(w, q) \mid w \in G, \varphi(w) = xyx^{-1}\}$$

$$= \inf\{\mu(u^{-1}wu, q) \mid w \in G, \varphi(u^{-1}wu) = y\}$$

$$= \inf\{\mu(w, q) \mid w \in G, \varphi(w) = y\} = \varphi(\mu)(y, q). \qquad \square$$

Proposition 4.5. *Let H be a group and* $\nu \in NAQFSC(H)$. *Suppose that* φ *is a homomorphism of G into H. Then* $\varphi^{-1}(\nu) \in NAQFSC(G)$.

Proof. By Proposition 4.2 we obtain that $\varphi^{-1}(\nu) \in AQFSC(G)$. Now for any $x, y \in G$ and $q \in Q$ we obtain

$$\varphi^{-1}(\nu)(xyx^{-1}, q) = \nu(\varphi(xyx^{-1}), q) = \nu(\varphi(x)\varphi(y)\varphi(x^{-1}), q)$$

$$= \nu(\varphi(x)\varphi(y)\varphi^{-1}(x), q) = \nu(\varphi(y), q) = \varphi^{-1}(\nu)(y, q).$$

Therefore $\varphi^{-1}(\nu) \in NAQFSC(G)$. $\square$

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