Earthline Journal of Mathematical Sciences
ISSN (Online): 2581-8147
Volume 4, Number 1, 2020, Pages 1-11
https://doi.org/10.34198/ejms.4120.111

Coefficient Estimates of Some Classes of Univalent Functions using Subordination Principle

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Abstract

In this work, two classes $T(b, \lambda)$ and $V(b, \lambda)$ were defined. Coefficient bounds, Fekete-Szegö functional and Hankel determinants for the classes were obtained. The results obtained generalized some earlier ones.

1. Introduction and Preliminaries

Let A be the class of analytic functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Furthermore, let S represent the family of all functions in A which are univalent in $\mathbb{U}$.

Received: February 27, 2020; Accepted: March 28, 2020

2010 Mathematics Subject Classification: Primary 30C45; Secondary 30C80.

Keywords and phrases: subordination principle, univalent functions, modified sigmoid function, Fekete-Szegö functional, Hankel determinant.

Similarly, the class P of all analytic functions ϕ with positive real part in $\mathbb{D}$ such that $\phi(0) = 1$ had been used by many authors. For functions in the class P expressed in the form

$$\phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n;$$

sharp bound $|c_n| \leq 2,\ n = 1, 2, \ldots$ exist.

Using subordination principle defined in literature such as Miller and Mocanu [6], some results have been established by many authors.

Kuroki and Owa [4] obtained the conditions necessary and sufficient for any function $f(z) \in A$ to satisfy the subordination

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{(\beta - \alpha)}{\pi} i \log\left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}\right),$$

and remarked as follows:

Let $S_{\alpha,\beta}(z) : \mathbb{U} \to \mathbb{C}$

$$S_{\alpha,\beta}(z) = 1 + \frac{(\beta - \alpha)}{\pi} i \log\left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}\right) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{2(\beta - \alpha)}{n\pi} \sin\left(\frac{n\pi(1 - \alpha)}{\beta - \alpha}\right) \qquad \alpha < 1, \quad \beta > 1.$$

(For details, see Remark 1.3 in [4]).

Fadipe-Joseph et al. [3] defined the modified sigmoid function

$$G(z) = \frac{2}{1 + e^{-z}} = 1 + \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right]^m\right) \in P.$$

Altinkaya and Yalçin [1] gave the coefficient estimate of a class of bi-univalent functions using the Schwarz function

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right)z^2 + \dots.$$

Malik et al. [5] also used Schwarz function in obtaining the coefficient inequality of functions in the class they defined using subordination principle.

Hence, subordination principle was used to establish the coefficient bounds of some classes of univalent functions in this paper.

2. Main Results

Definition 2.1. A function $f \in A$ is said to be in the class $T(b, \lambda)$; $0 \neq b \in \mathbb{C}$, $\lambda \geq 1$ if the following subordination holds

$$1 + \frac{1}{b}\left(\frac{z(f'(z))^{\lambda}}{f(z)} - 1\right) \prec S_{\alpha,\beta}(z).$$

Theorem 2.1. *Let* $f \in T(b, \lambda)$. *Then*

$$|a_2| \leq \frac{2|b|(\beta-\alpha)}{(2\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)$$

$$|a_3| \leq \frac{2|b|(\beta-\alpha)}{(3\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\left|\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right) + \frac{2|b|(\beta-\alpha)|4\lambda-2\lambda^2-1|}{\pi(2\lambda-1)^2}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right|$$

$$\begin{aligned} |a_4| \leq \frac{2|b|(\beta-\alpha)}{(4\lambda-1)\pi}&\left(4 + 4\left|\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right| + \left|\cos^2\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right) - \frac{1}{3}\sin^2\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right|\right. \\ &+ \frac{|11\lambda-6\lambda^2-2||b|(\beta-\alpha)}{(2\lambda-1)(3\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right) \\ &2\left(1 + \left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha} - 1\right| + \frac{2|b|(\beta-\alpha)|4\lambda-2\lambda^2-1|}{\pi(2\lambda-1)^2}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right) \\ &\left. + \frac{4\left|6\lambda^2 - \frac{4}{3}\lambda^3 - \frac{20}{3}\lambda + 1\right| \cdot |b|^2(\beta-\alpha)^2}{(2\lambda-1)^3\pi^2}\sin^2\frac{\pi(1-\alpha)}{\beta-\alpha}\right). \end{aligned}$$

Proof. Suppose $f \in T(b, \lambda)$, then by definition

$$1 + \frac{1}{b}\left(\frac{z(f'(z))^{\lambda}}{f(z)} - 1\right) \prec S_{\alpha, \beta}(z).$$

Consider

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \in P.$$

Let

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{p_1}{2} z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \frac{1}{2}\left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right)z^3 + \ldots.$$

Therefore,

$$1 + \frac{1}{b}\left(\frac{z(f'(z))^{\lambda}}{f(z)} - 1\right) = S_{\alpha, \beta}(w(z))$$

$$1 + \frac{2\lambda - 1}{b} a_2 z + \frac{((3\lambda - 1)a_3 + (2\lambda^2 - 4\lambda + 1)a_2^2)z^2}{b}$$

$$\frac{\left((4\lambda - 1)a_4 + (6\lambda^2 - 11\lambda + 2)a_2 a_3 + \left(\frac{4}{3}\lambda^3 - 6\lambda^2 + \frac{20}{3}\lambda - 1\right)a_2^3\right)z^3}{b} + \ldots$$

$$= 1 + \frac{B_1 p_1}{2} z + \left(\frac{B_1 p_2}{2} + \frac{(B_2 - B_1) p_1^2}{4}\right)z^2$$

$$+ \left(\frac{B_1 p_3}{2} + \frac{(B_2 - B_1) p_1 p_2}{2} + \frac{(B_1 - 2B_2 + B_3) p_1^3}{8}\right)z^3 + \ldots.$$

Comparing coefficients of z, z^2 and z^3;

$$a_2 = \frac{b p_1}{2(2\lambda - 1)}\left(\frac{2(\beta - \alpha)}{\pi}\sin\left(\frac{\pi(1 - \alpha)}{\beta - \alpha}\right)\right)$$

$$|a_2| \leq \frac{2|b|(\beta - \alpha)}{(2\lambda - 1)\pi}\sin\left(\frac{\pi(1 - \alpha)}{\beta - \alpha}\right). \tag{2.1}$$

Similarly,

$$a_3 = \frac{b(\beta-\alpha)}{(3\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\left(p_2 + \frac{p_1^2}{2}\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right) - \frac{p_1^2}{2}\right.$$

$$\left. + \frac{bp_1^2(\beta-\alpha)(4\lambda-2\lambda^2-1)}{\pi(2\lambda-1)^2}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right)$$

$$|a_3| \le \frac{2|b|(\beta-\alpha)}{(3\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)$$

$$\left|\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right) + \frac{2|b|(\beta-\alpha)|4\lambda-2\lambda^2-1]}{\pi(2\lambda-1)^2}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right|, \tag{2.2}$$

$$a_4 = \frac{b(\beta-\alpha)}{(4\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\left(\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) + \left(p_1p_2 - \frac{p_1^3}{2}\right)\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right.$$

$$\left. + \left(\frac{p_1^3}{8}\right)\left(\frac{2}{3}\left(3\cos^2\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right) - \sin^2\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right)\right)\right)$$

$$+ \frac{11\lambda-6\lambda^2-2}{(2\lambda-1)(3\lambda-1)}\left(\frac{bp_1(\beta-\alpha)}{\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right)$$

$$\left(p_2 + \frac{p_1^2}{2}\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right) - \frac{p_1^2}{2} + \frac{bp_1^2(\beta-\alpha)(4\lambda-2\lambda^2-1)}{\pi(2\lambda-1)^2}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right)$$

$$\left. + \frac{\left(\frac{4}{3}\lambda^3 - 6\lambda^2 + \frac{20}{3}\lambda - 1\right)b^2p_1^3(\beta-\alpha)^2}{3\pi^2(2\lambda-1)^3}\sin^2\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right)$$

$$|a_4| \le \frac{2|b|(\beta-\alpha)}{(4\lambda-1)\pi}\left(4 + 4\left|\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right|\left|\cos^2\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right) - \frac{1}{3}\sin^2\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right|\right.$$

$$+ \frac{|11\lambda-6\lambda^2-2||b|(\beta-\alpha)}{(2\lambda-1)(3\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)$$

$$2\left(1+\left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha}-1\right|+\frac{2|b|(\beta-\alpha)|4\lambda-2\lambda^2-1|}{\pi(2\lambda-1)^2}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right)$$

$$+\frac{4\left|6\lambda^2-\frac{4}{3}\lambda^3-\frac{20}{3}\lambda+1\right|\cdot|b|^2(\beta-\alpha)^2}{(2\lambda-1)^3\pi^2}\sin^2\frac{\pi(1-\alpha)}{\beta-\alpha}\Bigg). \tag{2.3}$$

Theorem 2.2. *Let* $f\in T(b,\lambda)$. *Then*

$$|a_3-\mu a_2^2|\le\frac{2|b|(\beta-\alpha)}{(3\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\left(1+\left|\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)-1\right|\right.$$

$$\left.+\frac{2|(4-3\mu)\lambda-2\lambda^2-(1-\mu)||b|(\beta-\alpha)}{\pi(2\lambda-1)^2}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right).$$

Proof.

$$a_3-\mu a_2^2=\frac{b}{3\lambda-1}\left(\frac{B_1p_2}{2}+\frac{(B_2-B_1)p_1^2}{4}+\frac{(4\lambda-2\lambda^2-1)}{b}a_2^2\right)-\mu a_2^2$$

$$=\frac{b(\beta-\alpha)}{(3\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\left(\left(p_2-\frac{p_1^2}{2}\right)+\frac{p_1^2}{2}\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right.$$

$$\left.+\frac{(4-3\mu)\lambda-2\lambda^2-(1-\mu)}{\pi^2(2\lambda-1)^2}bp_1^2(\beta-\alpha)\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right)$$

$$|a_3-\mu a_2^2|\le\left|\frac{b(\beta-\alpha)}{(3\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right|\left(\left|p_2-\frac{p_1^2}{2}+\frac{p_1^2}{2}\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right|\right.$$

$$\left.+\left|\frac{(4-3\mu)\lambda-2\lambda^2-(1-\mu)}{\pi^2(2\lambda-1)^2}bp_1^2(\beta-\alpha)\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right|\right)$$

$$|a_3-\mu a_2^2|\le\frac{2|b|(\beta-\alpha)}{(3\lambda-1)\pi}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\left(1+\left|\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)-1\right|\right.$$

$$\left.+\frac{2\left|(4-3\mu)\lambda-2\lambda^2-(1-\mu)\right||b|(\beta-\alpha)}{\pi(2\lambda-1)^2}\sin\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)\right).$$

Theorem 2.3. *Let* $f \in T(b, \lambda)$. *Then*

$$|a_4 - a_2 a_3| \leq \frac{2(\beta-\alpha)|b|}{(4\lambda-1)\pi} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}\left(1 + 2\left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha} - 1\right|\right.$$

$$+ \left|1 - 2\cos\frac{\pi(1-\alpha)}{\beta-\alpha} + \cos^2\frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{1}{3}\sin^2\frac{\pi(1-\alpha)}{\beta-\alpha}\right|$$

$$+ \frac{|7\lambda - 6\lambda^2 - 1| \cdot |b|(\beta-\alpha)}{(2\lambda-1)(3\lambda-1)\pi}\left(1 + \left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha} - 1\right|\right)\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$

$$\left.\frac{4\left|\frac{34}{3}\lambda^3 - 4\lambda^4 - 8\lambda^2 + \frac{5}{3}\lambda\right||b|^2(\beta-\alpha)^2}{(2\lambda-1)^3(3\lambda-1)\pi^2}\sin^2\frac{\pi(1-\alpha)}{\beta-\alpha}\right).$$

Proof.

$$a_4 - a_2 a_3 = \frac{bB_1 p_3}{2(4\lambda-1)} + \frac{b(B_2 - B_1)p_1 p_2}{2(4\lambda-1)}$$

$$+ \frac{b(B_1 - 2B_2 + B_3)p_1^3}{8(4\lambda-1)} - \frac{\frac{4}{3}\lambda^3 - 6\lambda^2 + \frac{20}{3}\lambda - 1}{4\lambda-1} a_2^3$$

$$- \left(\frac{6\lambda^2 - 11\lambda + 2}{4\lambda-1} + 1\right) a_2 a_3$$

$$= \frac{bB_1 p_3}{2(4\lambda-1)} + \frac{b(B_2 - B_1)p_1 p_2}{2(4\lambda-1)} + \frac{b(B_1 - 2B_2 + B_3)p_1^3}{8(4\lambda-1)}$$

$$- \frac{6\lambda^2 - 7\lambda + 1}{4\lambda-1}\left(\frac{(2p_2 - p_1^2)B_1 + p_1^2 B_2}{4(3\lambda-1)}\right) a_2 b$$

$$- \frac{\frac{4}{3}\lambda^3 - 6\lambda^2 + \frac{20}{3}\lambda - 1}{4\lambda-1} a_2^3 - \frac{2\lambda^2 - 4\lambda + 1}{3\lambda-1} a_2^3$$

$$= \frac{\beta-\alpha}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\left(\frac{bp_3}{(4\lambda-1)} + \frac{bp_1 p_2}{4\lambda-1}\left(\cos\frac{\pi(1-\alpha)}{\beta-\alpha} - 1\right)\right.$$

$$+\frac{bp_1^3}{4(4\lambda-1)}\left(1-2\cos\frac{\pi(1-\alpha)}{\beta-\alpha}+\cos^2\frac{\pi(1-\alpha)}{\beta-\alpha}+\frac{1}{3}\sin^2\frac{\pi(1-\alpha)}{\beta-\alpha}\right)$$

$$+\frac{2(7\lambda-6\lambda^2-1)}{4\lambda-1}\frac{b^2p_1(\beta-\alpha)}{(2\lambda-1)\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\left(\frac{2p_2+p_1^2\left(\cos\frac{\pi(1-\alpha)}{\beta-\alpha}-1\right)}{4(3\lambda-1)}\right)$$

$$-\frac{\frac{34}{3}\lambda^3-4\lambda^4-8\lambda^2+\frac{5}{3}\lambda}{(4\lambda-1)(3\lambda-1)}\frac{b^3p_1^3(\beta-\alpha)^2}{(2\lambda-1)^3\pi^2}\sin^2\frac{\pi(1-\alpha)}{\beta-\alpha}\Bigg)$$

$$|a_4-a_2a_3|\le\frac{2(\beta-\alpha)|b|}{(4\lambda-1)\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\left(1+2\left|\cos\left(\frac{\pi(1-\alpha)}{\beta-\alpha}\right)-1\right|\right.$$

$$+\left|1-2\cos\frac{\pi(1-\alpha)}{\beta-\alpha}+\cos^2\frac{\pi(1-\alpha)}{\beta-\alpha}+\frac{1}{3}\sin^2\frac{\pi(1-\alpha)}{\beta-\alpha}\right|$$

$$+\frac{|7\lambda-6\lambda^2-1|\cdot|b|(\beta-\alpha)}{(2\lambda-1)(3\lambda-1)\pi}\left(1+\left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha}-1\right|\right)\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$

$$\left.\frac{4\left|\frac{34}{3}\lambda^3-4\lambda^4-8\lambda^2+\frac{5}{3}\lambda\right||b|^2(\beta-\alpha)^2}{(2\lambda-1)^3(3\lambda-1)\pi^2}\sin^2\frac{\pi(1-\alpha)}{\beta-\alpha}\right).$$

Remark 2.1. Let $f(z)\in T(1,1)$. Then $|a_2|$ and $|a_3|$ agree with the bounds in Kuroki and Owa [4].

Definition 2.2. A function $f\in A$ is said to be in the class $V(b,\lambda)$; $0\ne b\in\mathbb{C}$, $\lambda\ge 1$ if the following subordination holds.

$$1+\frac{1}{b}\left(\frac{z(f'(z))^\lambda}{f(z)}-1\right)\prec G(z)=1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\cdots.$$

Theorem 2.4. *Let* $f\in V(b,\lambda)$. *Then*

$$|a_2|\le\frac{|b|}{2(2\lambda-1)}$$

$$|a_3| \le \frac{|b|}{4(3\lambda-1)}\left(2+\frac{|b|\cdot|4\lambda-2\lambda^2-1|}{(2\lambda-1)^2}\right)$$

$$|a_4| \le \frac{|b|}{8(4\lambda-1)}\left(31\frac{2}{3}+\frac{|b|\cdot|11\lambda-6\lambda^2-2|}{(2\lambda-1)(3\lambda-1)}\left(2+\frac{|b|\cdot|4\lambda-2\lambda^2-1|}{(2\lambda-1)^2}\right)\right.$$

$$\left.+\frac{|b|^2\left|6\lambda^2-\frac{4}{3}\lambda^3-\frac{20}{3}\lambda+1\right|}{(2\lambda-1)^3}\right).$$

Proof.

$$1+\frac{1}{b}\left(\frac{z(f'(z))^{\lambda}}{f(z)}-1\right)=G(w(z))$$

$$1+\frac{2\lambda-1}{b}a_2z+\frac{((3\lambda-1)a_3+(2\lambda^2-4\lambda+1)a_2^2)z^2}{b}$$

$$+\frac{\left((4\lambda-1)a_4+(6\lambda^2-11\lambda+2)a_2a_3+\left(\frac{4}{3}\lambda^3-6\lambda^2+\frac{20}{3}\lambda-1\right)a_2^3\right)z^3}{b}+\dots$$

$$=1+\frac{p_1}{4}z+\frac{1}{8}(2p_2-p_1^2)z^2+\frac{1}{8}\left(4p_3-4p_1p_2+\frac{23}{24}p_1^3\right)z^3+\dots.$$

Comparing coefficient of z, z^2 and z^3;

$$a_2=\frac{bp_1}{4(2\lambda-1)}$$

$$|a_2| \le \frac{|b|}{2(2\lambda-1)}$$

$$a_3=\frac{b}{8(3\lambda-1)}\left((2p_2-p_1^2)+\frac{bp_1^2(4\lambda-2\lambda^2-1)}{2(2\lambda-1)^2}\right)$$

$$|a_3| \le \frac{|b|}{4(3\lambda-1)}\left(2+\frac{|b|\cdot|4\lambda-2\lambda^2-1|}{(2\lambda-1)^2}\right)$$

$$a_4 = \frac{1}{(4\lambda - 1)}\left(\frac{b}{8}\left(4p_3 - 4p_1p_2 + \frac{23}{24}p_1^3\right)\right.$$

$$+ (11\lambda - 6\lambda^2 - 2)\left(\frac{bp_1}{4(2\lambda - 1)}\right)\left(\frac{1}{8(3\lambda - 1)}\left(b(2p_2 - p_1^2) + \frac{b^2p_1^2(4\lambda - 2\lambda^2 - 1)}{2(2\lambda - 1)^2}\right)\right)$$

$$\left. + \left(6\lambda^2 - \frac{4}{3}\lambda^3 - \frac{20}{3}\lambda + 1\right)\left(\frac{b^3p_1^3}{64(2\lambda - 1)^3}\right)\right)$$

$$|a_4| \le \frac{|b|}{8(4\lambda - 1)}\left(31\frac{2}{3} + \frac{|b|\cdot|11\lambda - 6\lambda^2 - 2|}{(2\lambda - 1)(3\lambda - 1)}\left(2 + \frac{|b|\cdot|4\lambda - 2\lambda^2 - 1|}{(2\lambda - 1)^2}\right)\right.$$

$$\left. + \frac{|b|^2\left|6\lambda^2 - \frac{4}{3}\lambda^3 - \frac{20}{3}\lambda + 1\right|}{8(2\lambda - 1)^3}\right).$$

Theorem 2.5. *Let* $f \in V(b, \lambda)$. *Then*

$$|a_3 - \mu a_2^2| \le \frac{|b|}{2(3\lambda - 1)}\left(1 + \frac{|(4 - 3\mu)\lambda - 2\lambda^2 - (1 - \mu)|\cdot|b|}{2(2\lambda - 1)^2}\right)$$

Proof.

$$a_3 - \mu a_2^2 = \frac{b(2p_2 - p_1^2)}{8(3\lambda - 1)} - \frac{(2\lambda^2 - 4\lambda + 1)a_2^2}{(3\lambda - 1)} - \mu a_2^2$$

$$= \frac{b}{(3\lambda - 1)}\left(\frac{2p_2 - p_1^2}{8} + \frac{((4 - 3\mu)\lambda - 2\lambda^2 - (1 - \mu))bp_1^2}{16(2\lambda - 1)^2}\right)$$

$$|a_3 - \mu a_2^2| \le \frac{|b|}{2(3\lambda - 1)}\left(1 + \frac{|(4 - 3\mu)\lambda - 2\lambda^2 - (1 - \mu)|\cdot|b|}{2(2\lambda - 1)^2}\right).$$

Theorem 2.6. *Let* $f \in V(b, \lambda)$. *Then*

$$|a_4 - a_2a_3| \le \frac{|b|}{8(4\lambda - 1)}\left(31\frac{2}{3} + \frac{4|7\lambda - 6\lambda^2 - 1|}{3\lambda - 1} + \frac{|b|^2\cdot\left|8\lambda^4 - \frac{56}{3}\lambda^3 + 10\lambda^2 - \frac{44}{3}\lambda + 2\right|}{4(3\lambda - 1)(2\lambda - 1)^3}\right).$$

Proof.

$$a_4 - a_2a_3 = \frac{1}{4\lambda - 1}\left(\frac{b}{8}\left(4p_3 - 4p_1p_2 + \frac{23}{24}p_1^3\right) - \left(\frac{4}{3}\lambda^3 - 6\lambda^2 + \frac{20}{3}\lambda - 1\right)a_2^3\right)$$

$$- \frac{(6\lambda - 11\lambda + 2)a_2a_3}{4\lambda - 1} - a_2a_3$$

$$= \frac{b}{8(4\lambda - 1)}\left(4p_3 - 4p_1p_2 + \frac{23}{24}p_1^3 - \frac{(6\lambda^2 - 7\lambda + 1)(2p_2 - p_1^2)}{3\lambda - 1}\right.$$

$$\left. + \frac{\left(8\lambda^4 - \frac{56}{3}\lambda^3 + 10\lambda^2 - \frac{44}{3}\lambda + 2\right)b^2p_1^3}{32(3\lambda - 1)(2\lambda - 1)^3}\right)$$

$$|a_4 - a_2a_3| \leq \frac{|b|}{8(4\lambda - 1)}\left(31\frac{2}{3} + \frac{4|7\lambda - 6\lambda^2 - 1|}{3\lambda - 1} + \frac{|b|^2 \cdot \left|8\lambda^4 - \frac{56}{3}\lambda^3 + 10\lambda^2 - \frac{44}{3}\lambda + 2\right|}{4(3\lambda - 1)(2\lambda - 1)^3}\right).$$

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