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Bounded Factorial Diophantine Equations via Permutation Resonances

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Abstract

Let $P \in \mathbb{Z}[x]$. We study when nonzero integers $x_1, x_2, \ldots$, uniformly bounded in absolute value, can satisfy

$$\prod_{k=1}^{n} \left(k + \frac{P(k)}{x_k} \right) = n!$$

for infinitely many n. A degree obstruction forces $\deg P \leq 1$, and the two homogeneous affine cases are solved explicitly. For a genuinely mixed affine polynomial $P(x) = ax + b$, we develop a periodic block-rescaling construction governed by a permutation $\sigma \in S_L$, leading to the resonance polynomial

$$R_\sigma(t) = \prod_{r=1}^{L} (1 - td_r) - 1, \qquad d_r = \sigma(r) - r.$$

We isolate the additional integrality and nonvanishing requirements by introducing the notion of an *admissible resonance*. We show that 0 is always a double root, establish a reflection symmetry, and reformulate R_σ in terms of elementary symmetric functions of the displacement vector. An exhaustive exact search through block length 9 identifies the first rational resonances and shows that $L = 2, 4, 7$ are barren, while a partial search at length 10 finds further examples. We prove an exact direct-sum law for resonance polynomials and use coprime witness lengths to obtain *cofinite realization* theorems: a slope realized at two coprime admissible block lengths is realized at every sufficiently large length, with one fixed admissible pair (a, b). We give a closed-form proof that block length 4 is barren, and show that the master construction cannot produce a mixed affine example when $b = \pm 1$. The paper concludes with conjectures and open questions concerning the full bounded sequence problem beyond the periodic block-rescaling framework.

1 The problem

Let $P \in \mathbb{Z}[x]$. We ask for which polynomials P there exist a constant $M \geq 1$ and, for infinitely many positive integers n, nonzero integers $x_1, \ldots, x_n$ satisfying

$$|x_k| \leq M \qquad (1 \leq k \leq n)$$

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and

$$\prod_{k=1}^{n}\left(k+\frac{P(k)}{x_k}\right)=n!. \tag{1.1}$$

The equation (1.1) is a multiplicative variant of the classical *polynomial-factorial* Diophantine equation $n!=Q(x)$, in which a fixed polynomial Q is asked to take the value $n!$ for some integer x. That equation has a long history, going back to the Brocard–Ramanujan problem $n!+1=x^2$, and has been studied systematically by Berend and Harmse [4], who established unconditional finiteness results for broad classes of Q; see Novaković [5, 6] for recent developments on Diophantine equations built from factorials and products of factorials. Here the roles are reversed: instead of asking when a fixed polynomial can equal $n!$ exactly, we allow n independent, but uniformly bounded, rational perturbations of the individual factors $1,2,\dots,n$, and ask when their product can still reproduce $n!$ for infinitely many n. Boundedness of the perturbations x_k, rather than a fixed target value, is thus the essential constraint, and the mechanism developed below is combinatorial and multiplicative – governed by permutations of a finite block – rather than analytic.

The essential point is that the polynomial P is fixed, whereas the integers x_k may depend on n, subject only to a uniform bound.

The question is naturally multiplicative. In particular, the individual factors need not equal the corresponding integers $1,2,\dots,n$; rather, the factors may be rearranged or rescaled in such a way that their product remains $n!$.

The first result shows that the degree of P is severely restricted.

The block-rescaling mechanism of Section 6 was inspired by the elementary period-2 telescoping identities of Bagdasar, Baker, and Conniss [3], given there for two specific Diophantine equations proposed by Bencze [1, 2]. Propositions 4.1 and 4.2 below are the natural analogues of their constructions for the present affine family; the master periodic block construction of Section 6 generalizes the underlying mechanism to blocks of arbitrary length and to an arbitrary permutation, and Sections 11 and 12 develop it into cofinite-realization and structural-obstruction theorems with no counterpart in that earlier note.

2 Main results

The paper has two complementary parts. The first is valid for arbitrary bounded solutions of (1.1): boundedness forces P to be affine. The second develops a concrete construction for the genuinely mixed affine case. Its main ingredients are the four results below, stated here informally; precise statements, hypotheses, and proofs are given at the labelled locations.

> **Theorem A** (degree obstruction, Theorem 3.1). *If a uniform bound on the x_k in (1.1) holds for infinitely many n, then* $\deg P\le 1$.

Theorem B (master permutation construction, Proposition 6.4). *For an affine $P(x) = ax+b$, a derangement $\sigma \in S_L$ with displacement vector $d_r = \sigma(r) - r$ produces a bounded periodic solution of (1.1) whenever (a, b) satisfies the explicit divisibility, nonvanishing, and resonance conditions of Proposition 6.4 – equivalently, whenever a/b is an admissible resonance for σ in the sense of Definition 6.2.*

Theorem C (cofinite realization, Theorem 11.2). *If a slope t_0 is an admissible resonance for derangements at two coprime block lengths L_1, L_2, then it is an admissible resonance at every block length $L \geq (L_1 - 1)(L_2 - 1)$, with a single pair (a, b) working uniformly across this entire cofinite set of lengths.*

Theorem D ($L = 4$ is barren, Theorem 12.3). *No derangement of $\{1, 2, 3, 4\}$ admits a nonzero rational resonance; this follows in closed form from the discriminant of Q_σ, with no exhaustive search needed.*

First, Theorem A treats (1.1) for arbitrary bounded sequences, prior to and independent of any specific construction: it shows that the mixed affine case $P(x) = ax + b$ ($ab \neq 0$) is the only case that can possibly require the construction developed in Theorem B, since the homogeneous cases $ab = 0$ are already settled directly (Section 4).

Second, Theorem B reduces the search for bounded solutions in the mixed affine case to a search for admissible resonances of the polynomial

$$R_\sigma(t) = \prod_{r=1}^{L}(1 - td_r) - 1, \qquad d_r = \sigma(r) - r.$$

A nonzero rational root $t = a/b$ of R_σ yields an actual integer block only when the corresponding divisibility and nonvanishing conditions are also satisfied; this motivates Definition 6.2.

Third, Theorem C shows that resonance is compatible with direct sums of permutations: a slope realized at two coprime block lengths propagates to every sufficiently large block length. This cofinite realization theorem is the principal structural consequence of the construction, and turns the sporadic examples found by computation into infinite families.

Fourth, exact computation through block length 9 shows that $L = 2, 4, 7$ are barren, while several other lengths support rational resonances (Section 10). Theorem D upgrades the case $L = 4$ from a computational fact to a closed-form structural theorem; $L = 7$ remains barren only by exhaustive verification, and is identified as an open problem (Section 12).

The novelty of the present work is therefore not the elementary period-2 telescoping mechanism itself, which goes back to the constructions of Bagdasar, Baker, and Conniss [3], but its extension to arbitrary permutation blocks and the resulting resonance-polynomial, direct-sum, and cofinite-realization theory (Theorems B, C, and D).

3 A degree obstruction

Theorem 3.1. *Let $P \in \mathbb{Z}[x]$ be nonzero. Suppose that there exists a constant $M \geq 1$ such that (1.1) has a solution for infinitely many positive integers n, with $1 \leq |x_k| \leq M$. Then* $\deg P \leq 1$.

Proof. Suppose, for contradiction, that $d := \deg P \geq 2$. Write

$$P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0, \qquad a_d \neq 0.$$

There exist constants $c > 0$ and $K \geq 1$ such that

$$|P(k)| \geq ck^d \qquad (k \geq K).$$

Since $|x_k| \leq M$, we have

$$\left|k + \frac{P(k)}{x_k}\right| \geq \frac{|P(k)|}{|x_k|} - k \geq \frac{c}{M}k^d - k.$$

Because $d \geq 2$, after increasing K if necessary, there is a constant $C > 0$ such that

$$\left|k + \frac{P(k)}{x_k}\right| \geq Ck^d \qquad (k \geq K).$$

For the finitely many indices $1 \leq k < K$, the factors are nonzero: indeed, if

$$k + \frac{P(k)}{x_k} = 0,$$

then the product in (1.1) would vanish. Hence, since there are only finitely many possibilities for $x_k \in \{\pm 1, \ldots, \pm M\}$, there is a constant $A > 0$ such that

$$\prod_{k=1}^{K-1} \left|k + \frac{P(k)}{x_k}\right| \geq A.$$

Therefore

$$n! = \left|\prod_{k=1}^{n} \left(k + \frac{P(k)}{x_k}\right)\right| \geq AC^{n-K+1} \prod_{k=K}^{n} k^d.$$

Thus

$$n! \geq AC^{n-K+1} \left(\frac{n!}{(K-1)!}\right)^d,$$

and hence

$$1 \geq \frac{A}{(K-1)!^d} C^{n-K+1} (n!)^{d-1}.$$

But the right-hand side tends to $+\infty$ as $n \to \infty$, a contradiction. □

Remark 3.2. *The theorem is sharp with respect to degree: all affine polynomials with one of the two coefficients equal to zero admit very simple bounded constructions, as we now show.*

4 The homogeneous affine cases

4.1 The case $P(x) = ax$

Proposition 4.1. *Let $P(x) = ax$, where $a \in \mathbb{Z} \setminus \{0\}$. For every even integer $n = 2m$, define*

$$x_{2j-1} = a, \qquad x_{2j} = -2a \qquad (1 \le j \le m).$$

Then

$$\prod_{k=1}^{2m} \left(k + \frac{ak}{x_k} \right) = (2m)!.$$

In particular, the sequence (x_k) is bounded by $2|a|$.

Proof. For $k = 2j - 1$,

$$(2j-1) + \frac{a(2j-1)}{a} = 2(2j-1).$$

For $k = 2j$,

$$2j + \frac{2aj}{-2a} = j.$$

Thus each pair contributes

$$2(2j-1) \cdot j = (2j-1)(2j),$$

and therefore

$$\prod_{k=1}^{2m} \left(k + \frac{ak}{x_k} \right) = \prod_{j=1}^{m} (2j-1)(2j) = (2m)!.$$

□

4.2 The case $P(x) = b$

Proposition 4.2. *Let $P(x) = b$, where $b \in \mathbb{Z} \setminus \{0\}$. For every even integer $n = 2m$, define*

$$x_{2j-1} = b, \qquad x_{2j} = -b \qquad (1 \le j \le m).$$

Then

$$\prod_{k=1}^{2m} \left(k + \frac{b}{x_k} \right) = (2m)!.$$

In particular, the sequence (x_k) is bounded by $|b|$.

Proof. We have

$$(2j-1) + \frac{b}{b} = 2j$$

and

$$2j + \frac{b}{-b} = 2j - 1.$$

Hence each pair contributes $2j(2j-1)$, and the result follows. □

Corollary 4.3. *Every affine polynomial $P(x) = ax + b$ with $ab = 0$ admits a uniformly bounded solution for infinitely many n. The case $P = 0$ is immediate; the two nonzero cases are given by Propositions 4.1 and 4.2.*

5 A first mixed question: period two

The preceding constructions suggest trying a fixed periodic sequence for the mixed affine polynomial

$$P(x) = ax + b, \qquad ab \neq 0.$$

The most immediate possibility is to use two fixed nonzero integers u, v:

$$x_{2j-1} = u, \qquad x_{2j} = v.$$

If one insists, as in the homogeneous constructions, that every pair reproduce the corresponding pair $(2j-1)(2j)$, then this simple period-2 mechanism is impossible in the genuinely mixed case.

Proposition 5.1 (No mixed constant period-2 pattern)**.** *Suppose $a, b \neq 0$ and $u, v \neq 0$. If*

$$\left(2j - 1 + \frac{a(2j-1)+b}{u}\right)\left(2j + \frac{2aj+b}{v}\right) = (2j-1)(2j)$$

for every $j \geq 1$, then $a = 0$, a contradiction.

Proof. Put $t = 2j - 1$. Then the assumed identity becomes

$$(At + C)(B(t+1) + D) = t(t+1),$$

where

$$A = 1 + \frac{a}{u}, \qquad C = \frac{b}{u}, \qquad B = 1 + \frac{a}{v}, \qquad D = \frac{b}{v}.$$

Comparing the coefficients of t^2, t, and 1, respectively, gives

$$AB = 1 \qquad A(B+D) + BC = 1 \qquad \text{and} \qquad C(B+D) = 0.$$

Since $b \neq 0$ and $u \neq 0$, we have $C \neq 0$, so $B + D = 0$. Consequently $1 + \dfrac{a+b}{v} = 0$, and hence $v = -(a+b)$. Therefore

$$B = 1 + \frac{a}{-(a+b)} = \frac{b}{a+b}.$$

Since $AB = 1$, $A = \dfrac{a+b}{b}$. But $A = 1 + \dfrac{a}{u}$, so, because $a \neq 0$,

$$1 + \frac{a}{u} = 1 + \frac{a}{b},$$

which gives $u = b$. Thus $C = 1$. The coefficient of t now gives $BC = 1$, hence $B = 1$. But $B = \dfrac{b}{a+b} = 1$ forces $a = 0$, contradicting the hypothesis. □

This failure motivates a more flexible construction in which a whole block of factors is rescaled and permuted.

6 The master periodic block construction

Let $P(x) = ax + b,\ \ a, b \in \mathbb{Z}$, and let $L \geq 2$. Let $\sigma \in S_L$ be a permutation. Write

$$d_r = \sigma(r) - r \qquad (1 \leq r \leq L)$$

for its displacement vector.

We seek constants $c_1, \ldots, c_L \in \mathbb{Z}^*$ and nonzero constants $\lambda_1, \ldots, \lambda_L$ such that, for every integer $k \geq 1$ and every $1 \leq r \leq L$,

$$k + r - 1 + \frac{a(k + r - 1) + b}{c_r} = \lambda_r\big(k + \sigma(r) - 1\big).$$

Comparing the coefficients of k gives $\lambda_r = 1 + \dfrac{a}{c_r}$. Comparing constant terms then gives

$$r - 1 + \frac{a(r-1) + b}{c_r} = \left(1 + \frac{a}{c_r}\right)(\sigma(r) - 1),$$

and therefore

$$\frac{b}{c_r} = \sigma(r) - r\left(1 + \frac{a}{c_r}\right).$$

Equivalently, $b = d_r c_r + a d_r$, or

$$c_r = \frac{b}{d_r} - a. \tag{6.1}$$

Thus, within this block-rescaling framework, the values c_r are forced by the permutation.

For the factors to be permuted and rescaled without changing the product of the block, we need

$$\prod_{r=1}^{L} \lambda_r = 1.$$

Using (6.1),

$$1 + \frac{a}{c_r} = 1 + \frac{a d_r}{b - a d_r} = \frac{b}{b - a d_r}.$$

Hence

$$\prod_{r=1}^{L} \lambda_r = \prod_{r=1}^{L} \frac{b}{b - a d_r} = \frac{1}{\prod_{r=1}^{L}(1 - (a/b) d_r)}.$$

Consequently the required condition is

$$\prod_{r=1}^{L} \left(1 - \frac{a}{b} d_r\right) = 1. \tag{6.2}$$

This leads to the following polynomial.

Definition 6.1. *For a permutation $\sigma \in S_L$ with displacement vector*

$$d_r = \sigma(r) - r,$$

define its resonance polynomial *by*

$$R_\sigma(t) = \prod_{r=1}^{L}(1 - td_r) - 1. \tag{6.3}$$

Definition 6.2. *Let $\sigma \in S_L$ be a derangement with displacement vector $d = (d_1, \dots, d_L)$, and let $t = p/q \in \mathbb{Q}^*$ be written in lowest terms ($\gcd(p, q) = 1$). We call t an* admissible resonance *for σ if*

$$R_\sigma(t) = 0 \qquad \text{and} \qquad t\,d_r \neq 1 \text{ for every } r.$$

Remark 6.3 (Realizing an admissible resonance)**.** *Admissibility in the sense of Definition 6.2 is an intrinsic property of the rational number t and the displacement vector d; it does not by itself specify the integer pair (a, b) used in the master construction (Proposition 6.4 below), and in practice (a, b) is* not *the reduced pair (p, q): the reduced denominator q is typically too small to be divisible by every $|d_r|$. Concretely, if $t = p/q$ is an admissible resonance for σ, set*

$$N = \operatorname{lcm}_r\left(\frac{|d_r|}{\gcd(|d_r|, q)}\right), \qquad (a, b) = (pN,\, qN).$$

Then $d_r \mid b$ for every r (since $|d_r|/\gcd(|d_r|, q)$ divides N, and $q/\gcd(|d_r|, q)$ is an integer, so $|d_r| \mid qN$), and

$$\frac{b}{d_r} - a = \frac{N(q - pd_r)}{d_r} \neq 0$$

for every r, precisely because $q - pd_r = 0$ would mean $td_r = 1$, excluded by Definition 6.2. Thus $(a, b) = (pN, qN)$ always satisfies conditions (i) and (ii) of Proposition 6.4; any common multiple of this (a, b) works equally well, so the pair is not unique, only its existence matters here.

Proposition 6.4 (Master periodic construction)**.** *Let*

$$P(x) = ax + b, \qquad a, b \in \mathbb{Z},\ b \neq 0,$$

and let $\sigma \in S_L$ be a derangement. Suppose that

(i) *$d_r \mid b$ for every r;*

(ii) *$b/d_r - a \neq 0$ for every r;*

(iii) *$R_\sigma(a/b) = 0$.*

(Here a, b are arbitrary integers with $a/b = t$ for some admissible resonance t of σ in the sense of Definition 6.2 – not necessarily $\gcd(a, b) = 1$; Remark 6.3 shows such a pair always exists once t is admissible.) Define $c_r = \dfrac{b}{d_r} - a$ and extend periodically by

$$x_{qL+r} = c_r \qquad (q \geq 0,\ 1 \leq r \leq L).$$

Then, for every positive integer m,

$$\prod_{k=1}^{mL}\left(k+\frac{ak+b}{x_k}\right)=(mL)!.$$

In particular, the sequence (x_k) is bounded.

Proof. Let $k=qL+r$ with $q\geq 0$ and $1\leq r\leq L$, so that $k=(qL+1)+(r-1)$.

By construction, $c_r=\dfrac{b}{d_r}-a$, so $1+\dfrac{a}{c_r}=\dfrac{b}{b-ad_r}$. Moreover,

$$r-1+\frac{a(r-1)+b}{c_r}=\left(1+\frac{a}{c_r}\right)(\sigma(r)-1).$$

After adding the common block offset, this gives

$$k+\frac{ak+b}{x_k}=\lambda_r\big(qL+\sigma(r)\big),$$

where $\lambda_r=1+\dfrac{a}{c_r}$. Therefore the product over one block is

$$\left(\prod_{r=1}^{L}\lambda_r\right)\prod_{r=1}^{L}(qL+\sigma(r)).$$

Condition (iii) is precisely

$$\prod_{r=1}^{L}\lambda_r=1.$$

Since σ is a permutation,

$$\prod_{r=1}^{L}(qL+\sigma(r))=\prod_{r=1}^{L}(qL+r).$$

Multiplying over $q=0,\dots,m-1$ gives

$$\prod_{k=1}^{mL}\left(k+\frac{ak+b}{x_k}\right)=\prod_{k=1}^{mL}k=(mL)!.$$

□

Remark 6.5. *The proposition gives a complete characterization only within this specific block-rescaling construction. It does not imply that every bounded solution must be periodic, nor does it rule out solutions arising from other permutations, other block lengths, or genuinely non-periodic sequences. Theorem 11.2 and Remark 11.3 below show, however, that non-periodic constructions are already available within the present framework, by concatenating blocks of different lengths.*

7 Structural properties of the resonance polynomial

The resonance polynomial has an important universal feature. We give two proofs of the next lemma: a direct calculation, and a second proof via elementary symmetric functions which also clarifies the higher coefficients of R_σ and situates the polynomial within a standard algebraic framework.

Lemma 7.1. *For every permutation $\sigma \in S_L$, $R_\sigma(0) = R'_\sigma(0) = 0$, and, if σ is nontrivial,*

$$R''_\sigma(0) = -\sum_{r=1}^{L} d_r^2 < 0.$$

Consequently, for every nontrivial permutation, 0 is a root of R_σ of multiplicity exactly 2. More precisely, $R_\sigma(t) = t^2 Q_\sigma(t)$, where

$$Q_\sigma(0) = -\frac{1}{2}\sum_{r=1}^{L} d_r^2 < 0$$

and $\deg Q_\sigma \leq L-2$, with equality if and only if σ is a derangement (the leading coefficient of R_σ being $(-1)^L \prod_r d_r$).

Proof. Clearly $R_\sigma(0) = 0$. Differentiating (6.3),

$$R'_\sigma(0) = -\sum_{r=1}^{L} d_r.$$

But

$$\sum_{r=1}^{L} d_r = \sum_{r=1}^{L} \sigma(r) - \sum_{r=1}^{L} r = 0.$$

Hence $R'_\sigma(0) = 0$. A second differentiation gives

$$R''_\sigma(0) = 2 \sum_{1 \leq r < s \leq L} d_r d_s.$$

Since

$$\left(\sum_{r=1}^{L} d_r\right)^2 = \sum_{r=1}^{L} d_r^2 + 2\sum_{r<s} d_r d_s = 0,$$

we obtain

$$R''_\sigma(0) = -\sum_{r=1}^{L} d_r^2.$$

For a nontrivial permutation, at least one d_r is nonzero, so the right-hand side is strictly negative. Thus the multiplicity of the root 0 is exactly 2. □

Remark 7.2 (Symmetric-function reformulation)**.** *Expanding the defining product (6.3) in terms of the elementary symmetric functions*

$$e_k(d) = e_k(d_1, \ldots, d_L) = \sum_{1 \leq r_1 < \cdots < r_k \leq L} d_{r_1} \cdots d_{r_k}, \qquad e_0(d) = 1,$$

gives

$$\prod_{r=1}^{L}(1-td_r)=\sum_{k=0}^{L}(-1)^k e_k(d)\,t^k,$$

and hence

$$R_\sigma(t)=\sum_{k=1}^{L}(-1)^k e_k(d)\,t^k. \tag{7.1}$$

This identity depends on σ only through the multiset $\{d_1,\dots,d_L\}$*, since every elementary symmetric function does; in particular R_σ is unchanged under any relabelling of the positions r that leaves the multiset of displacements fixed. Since $e_1(d)=\sum_r d_r=0$ always (the displacement vector of a permutation sums to zero), the linear term of (7.1) vanishes identically, which already gives $R_\sigma(0)=R'_\sigma(0)=0$ without differentiating. The quadratic coefficient is*

$$e_2(d)=\sum_{r<s}d_r d_s=-\frac{1}{2}\sum_r d_r^2$$

by the same identity $(\sum_r d_r)^2=\sum_r d_r^2+2e_2(d)=0$ used above, recovering $R''_\sigma(0)/2=e_2(d)=-\frac{1}{2}\sum_r d_r^2$, i.e. the second half of Lemma 7.1. More generally, the coefficients of Q_σ in Lemma 7.1 are, up to sign, the elementary symmetric functions $e_3(d),\dots,e_L(d)$ of the displacement vector, so that the search for rational resonances is exactly a search for rational roots of a polynomial whose coefficients are elementary symmetric functions of $\{d_r\}$ (see e.g. [9] for the general theory, including the Newton identities relating e_k to the power sums $p_k(d)=\sum_r d_r^k$, which could in principle be used to bound or locate rational roots). This connects the resonance polynomial to the broader combinatorics of permutation displacement statistics, such as Spearman's footrule $D(\sigma)=\sum_r|d_r|$ [7].

The next symmetry is also useful.

Lemma 7.3 (Reflection symmetry)**.** *Let $\sigma\in S_L$, and define*

$$\sigma'(r)=L+1-\sigma(L+1-r).$$

Then $d'_r=-d_{L+1-r}$, and consequently $R_{\sigma'}(t)=R_\sigma(-t)$. In particular, nonzero rational resonances occur in opposite pairs.

Proof. We have

$$d'_r=\sigma'(r)-r=L+1-\sigma(L+1-r)-r.$$

Putting $s=L+1-r$ gives $d'_r=s-\sigma(s)=-d_s=-d_{L+1-r}$. Therefore

$$R_{\sigma'}(t)=\prod_{r=1}^{L}(1-td'_r)-1=\prod_{r=1}^{L}(1+td_r)-1=R_\sigma(-t).$$

□

8 The period-2 master case

For $L = 2$, the only derangement is the transposition $\sigma = (1\,2)$, whose displacement vector is $(d_1, d_2) = (1, -1)$. Hence

$$R_\sigma(t) = (1-t)(1+t) - 1 = -t^2.$$

Corollary 8.1. *There is no nonzero rational resonance at block length $L = 2$. Consequently, the master periodic block construction produces no genuinely mixed affine example with $L = 2$.*

9 The first mixed example

The first nonzero resonance occurs for $L = 3$. Take $\sigma = (1\,2\,3)$, so that

$$(d_1, d_2, d_3) = (1, 1, -2).$$

Then $R_\sigma(t) = (1-t)^2(1+2t) - 1$. Expanding, $R_\sigma(t) = t^2(2t-3)$. Thus the nonzero resonance is $t = \dfrac{3}{2}$. Take $a = 3,\ \ b = 2$, so that $\dfrac{a}{b} = \dfrac{3}{2}$. Formula (6.1) gives $c_1 = c_2 = -1,\ \ c_3 = -4$. Hence we may take

$$x = (-1, -1, -4, -1, -1, -4, \ldots).$$

The bound is $M = 4$.

Example 9.1. *For $P(x) = 3x + 2$, the periodic sequence*

$$x_{3q+1} = x_{3q+2} = -1, \qquad x_{3q+3} = -4$$

satisfies

$$\prod_{k=1}^{3m} \left(k + \frac{3k+2}{x_k} \right) = (3m)!$$

for every $m \geq 1$.

By the reflection lemma, the opposite slope $-\dfrac{3}{2}$ is obtained from the reflected permutation.

10 Computational survey of resonances

The master construction reduces the search for mixed affine examples to the following finite problem for each block length L: enumerate derangements $\sigma \in S_L$, construct

$$R_\sigma(t) = \prod_{r=1}^{L} (1 - t d_r) - 1,$$

remove its universal factor t^2, and determine the nonzero rational roots of the remaining polynomial.

The number of derangements is

$$!L = L! \sum_{j=0}^{L} \frac{(-1)^j}{j!}.$$

The exhaustive search was carried out for $2 \leq L \leq 9$. A partial search was also carried out for $L = 10$, restricting to the subcase $\sigma(1) = 2$. The computational results are summarized below.

Table 1: First representative resonances for small block lengths.

L	one representative resonance	one admissible |b|
3	$\frac{3}{2}$	2
5	$-\frac{2}{3}$	6
6	$\frac{3}{4}$	4
6	$\frac{7}{6}$	12
6	$\frac{13}{12}$	12
8	$\frac{1}{4}$	12
8	$\frac{5}{12}$	12
8	$\frac{1}{2}$	6
8	$\frac{5}{6}$	12
8	$\frac{2}{3}$	6
9	$\frac{3}{10}$	20
9	$\frac{7}{12}$	12
9	$\frac{21}{20}$	60

Here the displayed value of $|b|$ is one admissible value for a realizing permutation, obtained from the divisibility condition $d_r \mid b$. No claim of global minimality over all possible realizing permutations is intended.

For example, at $L = 5$, one of the displacement vectors producing the resonance $-2/3$ is

$$(3, 3, -2, -1, -3),$$

for which

$$R_\sigma(t) = t^2(3t + 2)(18t^2 + 9t - 8).$$

Thus

$$t = -\frac{2}{3}$$

is a nonzero rational resonance.

The same displacement vector also gives an explicit mixed example. Taking

$$b = 6, \qquad a = -4,$$

we have

$$\frac{a}{b} = -\frac{2}{3}.$$

Formula (6.1) gives

$$(c_1, c_2, c_3, c_4, c_5) = (6, 6, 1, -2, 2).$$

Thus

$$x = (6, 6, 1, -2, 2, 6, 6, 1, -2, 2, \ldots)$$

has bound $M = 6$ and satisfies

$$\prod_{k=1}^{5m} \left(k + \frac{-4k + 6}{x_k} \right) = (5m)!$$

for every $m \geq 1$.

10.1 The exhaustive search through $L = 9$

The exhaustive search gives the following qualitative picture.

L	number of derangements	nonzero rational resonances found
2	1	none
3	2	$\frac{3}{2}$ and its negative
4	9	none
5	44	$\frac{2}{3}$ and its negative
6	265	several
7	1854	none
8	14833	several
9	133496	several

In particular, the lengths

$$2, \qquad 4, \qquad 7$$

are barren up to $L = 9$, meaning that no derangement at those lengths produces a nonzero rational root of its resonance polynomial.

At $L = 6$, for example, the search produces the slopes

$$\frac{3}{4}, \qquad \frac{7}{6}, \qquad \frac{13}{12},$$

in addition to resonances already occurring at smaller lengths.

At $L = 8$, the search produces, among others,

$$\frac{1}{4}, \quad \frac{5}{12}, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{5}{6}.$$

At $L = 9$, it produces

$$\frac{3}{10}, \quad \frac{7}{12}, \quad \frac{21}{20},$$

together with several resonances already appearing at smaller block lengths.

10.2 The partial search at $L = 10$

There are $!10 = 1334961$ derangements of $\{1, \ldots, 10\}$, making a complete search somewhat larger. As a partial computation, we restricted to $\sigma(1) = 2$. There are 148329 derangements satisfying this condition.

This partial search already produces the new rational resonance $\frac{4}{3}$. Thus the computational evidence suggests that new resonances continue to appear beyond the first few block lengths.

10.3 Computational implementation

The computational results reported in this section were obtained using Python. The symbolic and exact-arithmetic components were implemented with the `SymPy` library, in particular for constructing and manipulating the resonance polynomials

$$R_\sigma(t) = \prod_{r=1}^{L}(1 - td_r) - 1,$$

and for performing exact checks of identities such as

$$\prod_{k=1}^{n}\left(k + \frac{ak + b}{x_k}\right) = n!.$$

For the larger searches, especially those for $L = 8, 9, 10$, exact rational arithmetic was handled with Python's built-in `fractions.Fraction` class. This allowed the rational-root tests to be carried out exactly, with no floating-point approximations.

The computational procedure was as follows. For each derangement σ, we formed its displacement vector

$$(d_1, \ldots, d_L), \qquad d_r = \sigma(r) - r,$$

constructed R_σ, and wrote

$$R_\sigma(t) = t^2 Q_\sigma(t).$$

We then tested the rational roots of Q_σ exactly. A rational root $t_0 = p/q$ (in lowest terms) was retained only when it was an admissible resonance in the sense of Definition 6.2, i.e. when

$$t_0\, d_r \neq 1 \quad (1 \leq r \leq L).$$

For each admissible t_0, an explicit integer pair (a, b) with $a/b = t_0$ was then constructed as in Remark 6.3 – *not* simply $(a, b) = (p, q)$, since q is generally too small to be divisible by every $|d_r|$ – and the resulting values were independently checked against the original factorial identity for the corresponding periodic sequence.

All computations in this section use exact integer or rational arithmetic; no floating-point root detection is involved.

11 Additivity and cofinite realization of resonances under direct sums

The block construction of Section 6 treats a single block of length L, repeated periodically. We now show that distinct blocks realizing the *same* resonance can be freely combined, in any order, and that this turns the sporadic resonances of Section 10 into infinite families covering all sufficiently large block lengths.

11.1 The direct-sum lemma

For permutations $\sigma_1 \in S_{L_1}$ and $\sigma_2 \in S_{L_2}$, define their *direct sum* $\sigma_1 \oplus \sigma_2 \in S_{L_1+L_2}$ by

$$(\sigma_1 \oplus \sigma_2)(r) = \begin{cases} \sigma_1(r), & 1 \leq r \leq L_1, \\ L_1 + \sigma_2(r - L_1), & L_1 < r \leq L_1 + L_2. \end{cases}$$

If σ_1, σ_2 are both derangements, so is $\sigma_1 \oplus \sigma_2$, since neither block contributes a fixed point.

Lemma 11.1 (Additivity of resonances)**.** *Let $\sigma_1 \in S_{L_1}$, $\sigma_2 \in S_{L_2}$ be derangements, with displacement vectors $d^{(1)}, d^{(2)}$. The displacement vector of $\sigma_1 \oplus \sigma_2$ is the concatenation $(d^{(1)}, d^{(2)})$, and consequently*

$$R_{\sigma_1\oplus\sigma_2}(t) = \Big(\prod_{r=1}^{L_1}(1 - td_r^{(1)})\Big)\Big(\prod_{s=1}^{L_2}(1 - td_s^{(2)})\Big) - 1 = \big(R_{\sigma_1}(t) + 1\big)\big(R_{\sigma_2}(t) + 1\big) - 1.$$

In particular, if t_0 is a common root of R_{σ_1} and R_{σ_2}, then t_0 is a root of $R_{\sigma_1\oplus\sigma_2}$.

Proof. For $1 \leq r \leq L_1$, $(\sigma_1 \oplus \sigma_2)(r) - r = \sigma_1(r) - r = d_r^{(1)}$. For $r = L_1 + s$ with $1 \leq s \leq L_2$,

$$(\sigma_1 \oplus \sigma_2)(r) - r = L_1 + \sigma_2(s) - (L_1 + s) = \sigma_2(s) - s = d_s^{(2)}.$$

This proves the displacement-vector claim. The formula for $R_{\sigma_1\oplus\sigma_2}$ follows from the definition (6.3) applied to the concatenated vector, which splits the defining product into the product over the first L_1 indices times the product over the last L_2 indices. If $R_{\sigma_1}(t_0) = R_{\sigma_2}(t_0) = 0$, i.e. both factors equal 1 at t_0, then $R_{\sigma_1\oplus\sigma_2}(t_0) = 1 \cdot 1 - 1 = 0$. □

For a fixed nonzero rational t_0, write

$$\mathcal{S}(t_0) = \{L \geq 2 : \text{some derangement } \sigma \in S_L \text{ satisfies } R_\sigma(t_0) = 0\},$$

and define the admissible-length set

$$\mathcal{A}(t_0) = \{L \geq 2 : \text{some derangement } \sigma \in S_L \text{ has } t_0 \text{ as an admissible resonance, in the sense of Definition 6.2}\}.$$

The distinction between these two sets is important: a rational root of R_σ need not satisfy the nonvanishing condition $t_0 d_r \neq 1$ required for the master construction to apply (Remark 6.3). The direct-sum lemma implies that both sets are closed under addition, while the following theorem concerns the admissible set.

Theorem 11.2 (Cofinite realization of admissible resonances)**.** *Fix a nonzero rational $t_0 = p/q$ in lowest terms, and suppose that*

$$L_1, L_2 \in \mathcal{A}(t_0), \qquad \gcd(L_1, L_2) = 1,$$

witnessed by derangements $\sigma_1 \in S_{L_1}$, $\sigma_2 \in S_{L_2}$ with displacement vectors $d^{(1)}, d^{(2)}$. Then every integer

$$L \geq (L_1 - 1)(L_2 - 1)$$

belongs to $\mathcal{A}(t_0)$. Moreover, a single pair of integers (a, b) with $a/b = t_0$ – not, in general, the reduced pair (p, q) – can be used uniformly for every such L in the master construction of Proposition 6.4.

Proof. By Sylvester's theorem on the two-coin Frobenius problem (see [8]), every integer $L \geq (L_1-1)(L_2-1)$ can be written as $L = mL_1 + nL_2$ with $m, n \geq 0$. Taking the direct sum of m copies of σ_1 and n copies of σ_2 (in any order) gives, by repeated application of Lemma 11.1, a derangement $\sigma \in S_L$ with $R_\sigma(t_0) = 0$; its displacement vector is the concatenation of m copies of $d^{(1)}$ and n copies of $d^{(2)}$, so every entry of this vector already occurs in $d^{(1)}$ or $d^{(2)}$. Since $L_1, L_2 \in \mathcal{A}(t_0)$, every entry d_r of $d^{(1)}$ and of $d^{(2)}$ satisfies $t_0 d_r \neq 1$ (Definition 6.2); hence the same holds for every entry of the concatenated vector, and t_0 is an admissible resonance for σ. This proves $L \in \mathcal{A}(t_0)$.

For the uniform pair (a, b), apply Remark 6.3 to the *combined* finite set of displacement values occurring in $d^{(1)}$ and $d^{(2)}$: set

$$N = \operatorname{lcm}_r \left(\frac{|d_r|}{\gcd(|d_r|, q)} \right), \qquad (a, b) = (pN, qN),$$

the least common multiple ranging over every entry d_r of $d^{(1)}$ and of $d^{(2)}$ (a fixed finite set, independent of L). By Remark 6.3 applied entrywise, $d_r \mid b$ and $b/d_r - a \neq 0$ for every such entry, hence for every entry of any concatenation of copies of $d^{(1)}$ and $d^{(2)}$. Thus (a, b) satisfies conditions (i) and (ii) of Proposition 6.4 for every $L = mL_1 + nL_2$ in the semigroup generated by L_1, L_2, and condition (iii) holds by the first part of the proof. □

Remark 11.3 (Aperiodic concatenations)**.** *The construction in the proof of Theorem 11.2 need not be periodic. The admissible blocks of lengths L_1 and L_2 may be concatenated in any order, and each completed*

block contributes exactly its corresponding factorial interval to the product. Thus an arbitrary infinite sequence of such blocks gives a bounded sequence $(x_k)_{k\geq 1}$ *for which*

$$\prod_{k=1}^{n}\left(k+\frac{ak+b}{x_k}\right)=n!$$

holds whenever n *is a cumulative block endpoint. If the order of the blocks is not eventually periodic, the resulting sequence is aperiodic. This is a concrete family of non-periodic realizations within the present block-rescaling framework; it does not address the stronger question of whether bounded solutions exist outside that framework.*

11.2 An explicit example: $L_1 = 5$, $L_2 = 8$

The witnessing displacement vectors found in the search of Section 10 include

$$d^{(1)} = (2,3,1,-3,-3) \text{ is the displacement vector of } \sigma_1 \in S_5,$$
$$d^{(2)} = (1,1,3,-3,3,1,-3,-3) \text{ is the displacement vector of } \sigma_2 \in S_8,$$

both with $t_0 = 2/3$ (in lowest terms, $p = 2$, $q = 3$) as an admissible resonance: $R_{\sigma_i}(2/3) = 0$, and one checks directly that $\frac{2}{3}d_r \neq 1$ for every entry of $d^{(1)}$ and $d^{(2)}$ (indeed $\frac{2}{3}d_r \in \mathbb{Z}$ only for $d_r = 3$, where it equals $2 \neq 1$). Since $\gcd(5,8) = 1$, both lie in $\mathcal{A}(2/3)$. Following Remark 6.3, with the entries' absolute values $\{1,2,3\}$ and $q = 3$,

$$N = \operatorname{lcm}\left(\frac{1}{\gcd(1,3)}, \frac{2}{\gcd(2,3)}, \frac{3}{\gcd(3,3)}\right) = \operatorname{lcm}(1,2,1) = 2,$$
$$(a,b) = (pN, qN) = (4,6).$$

Every entry of $d^{(1)}$ and $d^{(2)}$ divides $b = 6$, and $b/d_r - a \neq 0$ for every entry, so this pair satisfies conditions (i) and (ii) of Proposition 6.4 for both witnesses at once.

By Sylvester's theorem [8], the Frobenius number of $\langle 5,8\rangle$ is $5\cdot 8 - 5 - 8 = 27$, so every $L \geq 28 = (L_1-1)(L_2-1)$ lies in $\mathcal{A}(2/3)$, along with every smaller L representable as $5m+8n$ $(m,n \geq 0)$: explicitly, every positive integer except

$$1,2,3,4,6,7,9,11,12,14,17,19,22,27$$

lies in the numerical semigroup $\langle 5,8\rangle$.

As a single instance, concatenating $d^{(1)}$ and $d^{(2)}$ once each gives $L = 13$ and the periodic block

$$c = (-1,-2,2,-6,-6,\ 2,2,-2,-6,-2,2,-6,-6),$$

and one verifies by exact rational arithmetic that

$$\prod_{k=1}^{13m}\left(k+\frac{4k+6}{x_k}\right) = (13m)! \qquad (m = 1,2,3,\ldots),$$

where (x_k) is the periodic extension of c with period 13. This example is not obtained from any single block of length ≤ 9 reported in Section 10.

A second coprime pair was located in the same search: $L = 8$ and $L = 9$ both realize the resonances $\pm 1/4$ and $\pm 1/2$, with $(L_1 - 1)(L_2 - 1) = 7 \cdot 8 = 56$, giving a second infinite family of resonant lengths for these slopes.

11.3 Densifying the founding example: a second witness for $3/2$

The slope $3/2$ of Example 9.1 – the first mixed example of the paper – occurs only at multiples of 3 among $L \leq 9$ (Section 10), and the same is true at $L = 10, 11$: an exhaustive check (using the same rational-root method as Section 10) finds no derangement of length 10 or 11 with resonance $\pm 3/2$. To find a coprime witness, we exploit the following consequence of unique factorization: if $R_\sigma(3/2) = 0$, i.e. $\prod_{r=1}^{L}(2 - 3d_r) = 2^L$, then, since the left-hand side is an integer product equal to a power of 2, *no* factor $2 - 3d_r$ may carry an odd prime divisor – otherwise that prime would divide the product but not 2^L. Hence every displacement value d_r occurring in a resonance-$3/2$ derangement must satisfy $2 - 3d_r = \pm 2^a$ for some integer $a \geq 0$, i.e.

$$d_r \in \left\{ \tfrac{2-2^a}{3} : a \text{ odd} \right\} \cup \left\{ \tfrac{2+2^a}{3} : a \text{ even} \right\} = \{\ldots, -42, -10, -2\} \cup \{1, 2, 6, 22, 86, \ldots\},$$

keeping only the integer values (e.g. $a = 1$ gives $d_r = 0$, excluded). This reduces the search for resonance-$3/2$ witnesses at a given length to a small integer-programming problem over this restricted alphabet, which is solved far more easily than an exhaustive permutation search. Solving it for $L \not\equiv 0 \pmod 3$ produces the multiset

$$\{\underbrace{1, \ldots, 1}_{12}, 2, 6, -10, -10\}, \qquad L = 16,$$

which is realizable as an actual displacement vector: the derangement $\sigma \in S_{16}$ given by

$$\sigma = (1\,2\,3\,4\,5\,7\,8\,9\,10\,11)(6\,12\,13\,14\,15\,16)$$

has $d = (1, 1, 1, 1, 2, 6, 1, 1, 1, 1, -10, 1, 1, 1, 1, -10)$ and satisfies $R_\sigma(3/2) = 0$ by direct computation. One checks $\frac{3}{2}d_r \neq 1$ for every entry of this vector and of $(1, 1, -2)$ (the $L = 3$ witness of Example 9.1), so both lie in $\mathcal{A}(3/2)$, and since $\gcd(3, 16) = 1$, Theorem 11.2 applies. Following Remark 6.3, with $t_0 = 3/2$ ($p = 3$, $q = 2$) and combined entry set $\{1, 2, 6, 10\}$,

$$N = \operatorname{lcm}\left(\frac{1}{\gcd(1,2)}, \frac{2}{\gcd(2,2)}, \frac{6}{\gcd(6,2)}, \frac{10}{\gcd(10,2)} \right) = \operatorname{lcm}(1, 1, 3, 5) = 15,$$
$$(a, b) = (3N, 2N) = (45, 30).$$

Every displacement value occurring in any concatenation of the $L = 3$ and $L = 16$ blocks divides $b = 30$, and the corresponding values $b/d_r - a$ are all nonzero. Explicitly, the $L = 3$ block gives

$c_r \in \{-15, -15, -60\}$, while the $L = 16$ block gives c_r among $-15, -30, -40, -48$. Hence the common bound furnished by these two witnesses is

$$M = 60.$$

Corollary 11.4. *For $P(x) = 45x + 30$, every L in the numerical semigroup $\langle 3, 16\rangle$ admits a bounded sequence (x_k), obtained by concatenating admissible blocks of lengths 3 and 16, such that*

$$\prod_{k=1}^{L} \left(k + \frac{45k + 30}{x_k}\right) = L!, \qquad |x_k| \leq 60.$$

Repeating the resulting block of total length L periodically gives the same identity at every positive multiple mL. In particular, every $L \geq (3-1)(16-1) = 30$ is covered, together with

$$L = 3, 6, 9, 12, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28.$$

The only lengths not covered are

$$1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 17, 20, 23, 26, 29.$$

This extends the paper's founding example (Example 9.1, the slope 3/2 realized only periodically at multiples of 3) into a genuine cofinite-realization theorem, in the same way that Theorem 11.2 did for the slope 2/3 above.

Remark 11.5. *Theorem 11.2 does not by itself establish Conjecture 14.1 below, since it only shows that a length L is non-barren once* some *slope t_0 has two coprime witnesses at lengths $\leq L$; it says nothing about lengths for which every slope's witnesses share a common factor. It does, however, replace an open-ended computational conjecture by a concrete, checkable sufficient condition – the existence of one coprime pair of witnesses per slope – and every instance of that condition found in Section 10 converts immediately into a genuine cofinite-realization theorem, as in the examples above.*

12 Structural obstructions for the barren lengths

Exhaustive search (Section 10) shows that $L = 2, 4, 7$ are barren among $2 \leq L \leq 9$: no derangement at these lengths yields a nonzero rational resonance. Corollary 8.1 already settles $L = 2$, from the unique derangement's resonance polynomial $R_\sigma(t) = -t^2$. We now give a closed-form proof for $L = 4$, and explain precisely why the same style of argument does not immediately extend to $L = 7$, sharpening the latter into a concrete open problem.

12.1 $L = 4$ is barren: a closed-form proof

For $L = 4$, Q_σ has degree 2; by (7.1),

$$Q_\sigma(t) = e_4(d)t^2 - e_3(d)t + e_2(d).$$

This quadratic has a rational root if and only if its discriminant

$$D_\sigma = e_3(d)^2 - 4e_2(d)e_4(d)$$

is the square of a rational number; since $D_\sigma \in \mathbb{Z}$, this means D_σ is a perfect square (or $D_\sigma = 0$, which would give a root at $t = e_3/(2e_4)$, a special case of the same test). Every derangement of $\{1,2,3,4\}$ has cycle type $(2,2)$ (three permutations) or (4) (six permutations).

Proposition 12.1 (Double transpositions)**.** *Let $\sigma \in S_4$ be a product of two disjoint transpositions, moving points at distances (spans) $m_1, m_2 \geq 1$. Then*

$$D_\sigma = \begin{cases} 8m_1^6, & m_1 = m_2, \\ 4m_1^2 m_2^2 (m_1^2 + m_2^2), & m_1 \neq m_2. \end{cases}$$

In the first case $D_\sigma = 2(2m_1^3)^2$ has squarefree part 2 and is never a perfect square. In the second case $D_\sigma = (2m_1 m_2)^2(m_1^2 + m_2^2)$ is a perfect square if and only if $m_1^2 + m_2^2$ is. The three perfect matchings of $\{1,2,3,4\}$ realize spans $(m_1, m_2) \in \{(1,1),(2,2),(1,3)\}$; the first two are equal-span, and $1^2 + 3^2 = 10$ is not a perfect square. Hence D_σ is never a perfect square for σ of type $(2,2)$ in S_4.

Proof. Write the four displacement values as the multiset $\{m_1, -m_1, m_2, -m_2\}$; by Remark 7.2, Q_σ depends only on this multiset. Direct expansion of the elementary symmetric functions gives $e_2 = -(m_1^2 + m_2^2)$, $e_3 = 0$, $e_4 = m_1^2 m_2^2$, whence $D_\sigma = -4e_2e_4 = 4m_1^2m_2^2(m_1^2 + m_2^2)$; setting $m_1 = m_2 = m$ gives $D_\sigma = 4m^4 \cdot 2m^2 = 8m^6$. The perfect-square claims follow at once from these closed forms, and the three matchings of $\{1,2,3,4\}$ into two pairs – $\{12,34\}$, $\{13,24\}$, $\{14,23\}$ – have spans $(1,1)$, $(2,2)$, $(1,3)$ respectively. □

Proposition 12.2 (Four-cycles)**.** *For each of the six 4-cycles in S_4, $D_\sigma \in \{-8, 80, 448\}$, each value attained twice (by a reflection pair, in the sense of Lemma 7.3). None of -8, $80 = 16 \cdot 5$, $448 = 64 \cdot 7$ is a perfect square: -8 is negative, and $5, 7$ are squarefree and greater than 1.*

Proof. Direct computation. The six 4-cycles and their displacement vectors are

$$(1,1,1,-3),\ (1,2,-2,-1),\ (2,-1,1,-2),\ (2,2,-1,-3),\ (3,-1,-1,-1),\ (3,1,-2,-2),$$

with respective discriminants $-8, 80, 80, 448, -8, 448$; the pairing into three repeated values matches the reflection symmetry of Lemma 7.3, which leaves D_σ invariant (it acts on Q_σ by $t \mapsto -t$, which changes the sign of e_3 alone and hence fixes $e_3^2 - 4e_2e_4$). □

Theorem 12.3. *$L = 4$ is barren: for every derangement $\sigma \in S_4$, the resonance polynomial R_σ has no nonzero rational root.*

Proof. Immediate from Propositions 12.1 and 12.2, which cover both cycle types occurring in S_4. □

12.2 Why $L = 7$ resists the same approach

For $L = 7$, $\deg Q_\sigma = L - 2 = 5$, and no single discriminant governs the rational roots of a quintic, so Theorem 12.3's argument does not directly generalize. The 1854 derangements of $\{1, \dots, 7\}$ split into four cycle types, with counts

$$(7) : 720, \qquad (5, 2) : 504, \qquad (4, 3) : 420, \qquad (3, 2, 2) : 210,$$

which sum to 1854 as required. It is tempting to hope that, as in the $L = 4$ case, a cycle type built from smaller pieces – such as $(3, 2, 2)$, which contains two disjoint transpositions much as the $(2, 2)$ type of S_4 does – might again yield a clean closed form. This hope is only partially justified: the cycles of a $(3, 2, 2)$-type derangement of $\{1, \dots, 7\}$ act on three disjoint subsets of positions, but these subsets need not be contiguous, so σ is generally *not* a direct sum $\sigma_1 \oplus \sigma_2 \oplus \sigma_3$ in the sense of Lemma 11.1, and R_σ does not factor as a product over the three cycles. Such a factorization applies only in the special *contiguous* sub-case, in which the three orbits occupy three consecutive intervals (for instance $\{1, 2, 3\}$, $\{4, 5\}$, $\{6, 7\}$). This sub-case accounts for only a handful of the 210 derangements of type $(3, 2, 2)$, so the following computation does *not* prove that this cycle type is barren. In the contiguous sub-case, a transposition of adjacent points contributes $R_2(t) + 1 = 1 - t^2$. A 3-cycle on consecutive points contributes $(1 - t)^2(1 + 2t)$ (Section 9) or its reflection $(1 + t)^2(1 - 2t)$. Up to $t \mapsto -t$, one therefore obtains the equation

$$(1 - t)^2(1 + 2t) \cdot (1 - t^2)^2 = 1 \qquad \text{(contiguous sub-case only)},$$

which has no nonzero rational solution. This is consistent with, but does not imply, the general fact.

We record the search for a genuinely structural (as opposed to purely computational) obstruction at $L = 7$, and more generally a classification of which cycle types can carry rational resonances, as Question 2 in the list of open questions of Section 14. The exhaustive verification of Section 10, via the rational root theorem applied to the exact integer coefficients $e_2(d), \dots, e_7(d)$ of Remark 7.2, remains a complete and rigorous proof that $L = 7$ is barren; what is missing is a conceptual explanation of the kind Theorem 12.3 provides for $L = 4$.

Remark 12.4 (A partial pattern)**.** *Factoring Q_σ over $\mathbb{Q}$ for all* 1854 *derangements of* $\{1, \dots, 7\}$ *shows that Q_σ is irreducible of degree* 5 *in* 1826 *cases, and factors as an irreducible quadratic times an irreducible cubic in the remaining* 28 *cases; no linear factor ever occurs, consistent with Section 10. The* 28 *exceptional cases occur only for cycle types* $(2, 5)$ *and* $(3, 4)$*, never for* (7) *or* $(3, 2, 2)$*, and in* 27 *of the* 28 *cases the quadratic factor is one of the two fixed polynomials $2t^2 \mp t - 2$ (a reflection pair in the sense of Lemma 7.3), independent of which particular derangement is used. This is consistent with, but not implied by, the elementary observation that for a $(2, 5)$-type derangement the transposition contributes a factor $1 - s^2t^2$ (where s is its span) to $R_\sigma(t) + 1 = \prod_r (1 - td_r)$, since $R_\sigma + 1$ and R_σ itself need not share factors. We record this recurring quadratic as a further, still unexplained, regularity for future work, rather than as a step toward a proof.*

13 A useful obstruction when $b = 1$ or $b = -1$

The master construction has an especially strong restriction when the constant term of the affine polynomial is ± 1.

Proposition 13.1. *Suppose that $b = \pm 1$. Then the master periodic block construction cannot produce a genuinely mixed affine polynomial*

$$P(x) = ax + b, \qquad a \neq 0.$$

Proof. Condition (i) of Proposition 6.4 requires $d_r \mid b$. Since $b = \pm 1$, we must have $d_r \in \{-1, 1\}$ for every r.

We first determine the permutations having this property. Since $d_r = \sigma(r) - r$, every point must move exactly one step to the left or one step to the right. The point 1 cannot move left, so

$$\sigma(1) = 2.$$

Then 2 must move to 1; otherwise it would have to move to 3, but then no value could map to 1. Hence $(1\,2)$ is a cycle. We now argue by induction. Suppose that, for some $j \geq 1$, we have shown $\sigma(2i-1) = 2i$ and $\sigma(2i) = 2i - 1$ for $1 \leq i \leq j$, so that σ maps $\{1, \ldots, 2j\}$ onto itself. If $2j = L$ we stop. Otherwise σ maps $\{2j+1, \ldots, L\}$ onto itself. Hence $2j+1$ cannot move left, and $\sigma(2j+1) = 2j+2$. The point $2j+1$ must be the image of $2j$ or of $2j+2$. Since $\sigma(2j) = 2j-1$, it follows that $\sigma(2j+2) = 2j+1$. This completes the induction and gives

$$\sigma = (1\,2)(3\,4)\cdots(L-1\,L).$$

In particular, L must be even and the displacement vector consists of $L/2$ copies of 1 and $L/2$ copies of -1.

Therefore

$$R_\sigma(t) = (1 - t^2)^{L/2} - 1.$$

If t is rational and $R_\sigma(t) = 0$, then

$$(1 - t^2)^{L/2} = 1.$$

Hence $1 - t^2 = \pm 1$. The first possibility gives $t = 0$, while the second gives $t^2 = 2$, which has no rational solution. Thus there is no nonzero rational resonance. □

Remark 13.2. *Proposition 13.1 is a restriction only on the master periodic block-rescaling mechanism. It does not settle the original problem for $P(x) = ax \pm 1$, because the original problem allows arbitrary bounded sequences (x_k) and does not require the block-rescaling form.*

14 The resulting conjectural picture

The computations suggest that the resonance mechanism is considerably richer than the first example at $L = 3$, and Theorem 11.2 shows that, wherever two coprime witnesses occur for the same slope, this richness persists to all sufficiently large block lengths.

Among the rational slopes a/b known to arise from the master construction are representatives of the pairs

$$\pm\frac{3}{2}, \quad \pm\frac{2}{3}, \quad \pm\frac{3}{4}, \quad \pm\frac{7}{6}, \quad \pm\frac{13}{12},$$

as well as

$$\pm\frac{1}{4}, \quad \pm\frac{5}{12}, \quad \pm\frac{1}{2}, \quad \pm\frac{5}{6},$$

and

$$\pm\frac{3}{10}, \quad \pm\frac{7}{12}, \quad \pm\frac{21}{20}, \quad \pm\frac{4}{3}.$$

The last value is obtained in the partial $L = 10$ search described above. By Theorem 11.2, the slopes $\pm 2/3$ and $\pm 1/4, \pm 1/2$ are in fact realized at *every* sufficiently large block length, via the coprime pairs $(5, 8)$ and $(8, 9)$ respectively.

These values should be regarded as a list of presently known resonant slopes, not as a classification.

The exhaustive computations through $L = 9$ suggest the following.

Conjecture 14.1. *The only barren block lengths are*

$$L = 2, \qquad L = 4, \qquad L = 7.$$

Equivalently, every integer

$$L \geq 3, \qquad L \notin \{4, 7\},$$

admits a derangement $\sigma \in S_L$ for which R_σ has a nonzero rational root.

Remark 14.2. *Theorem 11.2 provides a concrete route toward Conjecture 14.1. It suffices to find enough coprime witness pairs, with a common admissible slope for each pair, to cover all sufficiently large lengths by numerical semigroups, and then to settle the finitely many remaining small lengths. For example, the pair $(5, 8)$ already gives all lengths $L \geq 28$ for the slope $2/3$, while the pair $(8, 9)$ gives all $L \geq 56$ for the slopes $1/4$ and $1/2$. These examples do not by themselves prove the conjecture, but they illustrate a concrete strategy for reducing it to further finite computations and structural arguments.*

A second natural question concerns the full original problem.

Conjecture 14.3. *For a genuinely mixed affine polynomial*

$$P(x) = ax + b, \qquad ab \neq 0,$$

the existence of a uniformly bounded sequence (x_k) satisfying (1.1) for infinitely many n is governed by a finite algebraic resonance phenomenon of the type exhibited by the periodic block construction.

At present, however, this stronger statement is open. In particular, the degree obstruction in Theorem 3.1 applies to arbitrary bounded sequences, whereas the resonance polynomial analysis applies only to the particular periodic block-rescaling mechanism.

This distinction leaves several questions open.

1. Are there genuinely mixed affine examples arising from bounded non-periodic sequences but from no periodic block construction? (Theorem 11.2 produces non-periodic realizations that are nonetheless built from periodic blocks; the question is whether examples exist outside the block-rescaling framework entirely.)
2. Theorem 12.3 explains the barren length 4 by a closed-form discriminant argument, reducing to cycle type. Does an analogous, genuinely structural obstruction (as opposed to the exhaustive rational-root check of Section 10) explain the barren length 7? More generally, is there a criterion, in terms of the cycle type or other invariants of σ, for when R_σ can have a nonzero rational root?
3. Is Conjecture 14.1 true for all L? By Remark 14.2, it would suffice to find, for the finitely many currently uncovered lengths, either a direct resonance or a further coprime pair of witnesses among already-known resonant slopes.
4. Can one classify all rational roots of the resonance polynomials R_σ as σ ranges over all finite permutations? By Remark 7.2, this is equivalent to describing the rational roots of $\sum_k (-1)^k e_k(d) t^k$ as d ranges over displacement multisets of permutations, a question that connects the problem to the combinatorics of permutation displacement statistics such as Spearman's footrule [7].
5. Can the full original bounded-sequence problem be characterized purely in terms of the rational slope a/b?

15 Conclusion

The original product identity

$$\prod_{k=1}^{n} \left(k + \frac{P(k)}{x_k} \right) = n!$$

looks at first like a flexible Diophantine equation with infinitely many possible choices of the integers x_k. The boundedness hypothesis, however, immediately forces a strong algebraic restriction: the polynomial P must be affine.

The two homogeneous affine cases admit elementary period-2 constructions. For mixed affine polynomials, the situation is more subtle. A natural periodic block construction transforms the problem into the search for rational roots of the resonance polynomials

$$R_\sigma(t) = \prod_{r=1}^{L} (1 - t d_r) - 1.$$

The universal double root at 0, the reflection symmetry, and the computationally observed rational resonances provide a coherent algebraic framework for constructing further examples, and Section 7 shows that this framework sits naturally inside the theory of elementary symmetric functions of the displacement vector.

The first genuinely mixed construction occurs already at block length 3, while the exhaustive search through length 9 reveals a much richer collection of resonant slopes. Crucially, Theorem 11.2 shows that this richness is not confined to small, sporadically discovered block lengths: whenever a slope is realized at two coprime lengths, it is realized at every sufficiently large length, by freely concatenating (not necessarily periodically) the two witnessing blocks. This converts part of the computational survey of Section 10 into genuine cofinite-realization theorems, and reduces Conjecture 14.1 to a concrete, finitely checkable search for further coprime witness pairs. Among the barren lengths, we give a fully structural, closed-form proof that $L = 4$ is barren (Theorem 12.3), via the discriminant of the resulting quadratic Q_σ; the same style of argument does not yet close the case $L = 7$, whose barrenness rests, for now, on exhaustive verification rather than a conceptual reason (Section 12). At the same time, the failure of the mechanism for $b = \pm 1$ and the existence of unexplored non-periodic possibilities show that the original bounded problem remains substantially broader than the periodic theory developed here.

Acknowledgment

The block-rescaling mechanism studied here generalizes the period-2 telescoping constructions of Bagdasar, Baker, and Conniss [3], given there for the Diophantine equations OQ.4836 and OQ.4837 proposed by Bencze [1, 2]. The author is grateful to the authors of that note, whose elementary constructions were the starting point for the periodic block construction of Section 6.

A Exact-search methodology

For reproducibility, we record the exact search procedure used for the computational survey. No floating-point arithmetic is needed. For each L, the program enumerates all derangements $\sigma \in S_L$, forms the integer displacement vector $d_r = \sigma(r) - r$, constructs

$$R_\sigma(t) = \prod_{r=1}^{L}(1 - td_r) - 1 = t^2 Q_\sigma(t),$$

and factors Q_σ over $\mathbb{Q}$. Every rational root $t_0 = p/q$ of Q_σ, in lowest terms, is then tested for admissibility in the sense of Definition 6.2: $t_0 d_r \neq 1$ for every r. This condition depends only on p, q and the displacement values, not on any further choice – unlike the divisibility condition $d_r \mid b$, which does depend on the integer pair (a, b) ultimately used to build the block, and which is generally *false* for the reduced pair (p, q) itself (the reduced denominator q is typically too small to be divisible by every $|d_r|$; see Remark 6.3 and the

worked examples of Section 11). Consequently the search does not test $d_r \mid q$ directly; instead, once t_0 passes the admissibility test, an explicit realizing pair is constructed as

$$N = \operatorname{lcm}_r \left(\frac{|d_r|}{\gcd(|d_r|, q)} \right), \qquad (a, b) = (pN, qN),$$

which is guaranteed by Remark 6.3 to satisfy $d_r \mid b$ and $b/d_r - a \neq 0$ for every r. Each retained resonance is then checked directly in the original factorial identity using the corresponding block $c_r = b/d_r - a$.

A compact pseudocode description is:

```
for L in target_lengths:
  for sigma in derangements(L):
    d = [sigma[r] - r for r in 1..L]
    R = product(1 - t*d[r] for r in 1..L) - 1
    Q = cancel(R / t**2)
    for t0 = p/q in rational_roots(Q):     # lowest terms
      if all(t0*d[r] != 1 for d[r] in d):  # admissible? (Def. 6.2)
        terms = [abs(d[r]) // gcd(abs(d[r]), q) for d[r] in d]
        N = lcm(terms)
        a, b = p*N, q*N                    # realizing pair (Rem. 6.3)
        record(L, sigma, t0, a, b)
        verify_original_identity(L, sigma, a, b)
```

The exhaustive computations reported in Section 10 cover $2 \leq L \leq 9$. The computation at $L = 10$ is partial and is restricted to $\sigma(1) = 2$. The implementation uses exact integer and rational arithmetic (in particular Python's `Fraction` and SymPy's exact polynomial arithmetic), so the reported rational-root tests do not depend on numerical tolerances.

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