

Fixed Point Theorem for Banach Contraction Mapping in Perturbed Multiplicative Metric Space

Clement Boateng Ampadu

¹ Independent Researcher, USA

e-mail: profampadu@gmail.com; drampadu@hotmail.com

Abstract

The notion of perturbed metric space appeared in [1], whilst the notion of multiplicative metric space appeared in [2]. In this paper, we combine both notions to introduce perturbed multiplicative metric space. We obtain the Banach fixed point theorem in this new space with an illustrative example.

1 Preliminaries

Definition 1.1. [1] Let $K, L : B \times B \mapsto [0, \infty)$ be two given mappings. We say that K is a perturbed metric on B with respect to L if

$$K - L : B \times B \mapsto \mathbb{R}$$

$$(a, b) \mapsto K(a, b) - L(a, b)$$

is a metric on B , that is, for all $a, b, c \in B$,

- (i) $(K - L)(a, b) \geq 0$.
- (ii) $(K - L)(a, b) = 0$ if and only if $a = b$.
- (iii) $(K - L)(a, b) = (K - L)(b, a)$.
- (iv) $(K - L)(a, b) \leq (K - L)(a, c) + (K - L)(c, b)$.

We call L a perturbed mapping, $j = K - L$ an exact metric, and (B, K, L) a perturbed metric space.

Example 1.2. [1] Let $K : \mathbb{R} \times \mathbb{R} \mapsto [0, \infty)$ be the mapping defined by

$$K(a, b) = |a - b| + a^2 b^4, \quad a, b \in \mathbb{R}.$$

Then K is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$L : \mathbb{R} \times \mathbb{R} \mapsto [0, \infty),$$

given by

$$L(a, b) = a^2 b^4, \quad a, b \in \mathbb{R}.$$

In this case, the exact metric is the mapping $j : \mathbb{R} \times \mathbb{R} \mapsto [0, \infty)$ defined by

$$j(a, b) = |a - b|, \quad a, b \in \mathbb{R}.$$

Example 1.3. [1] Let $K : \mathbb{N} \times \mathbb{N} \mapsto [0, \infty)$ be the mapping defined by

$$K(a, b) = (a - b)^2, \quad a, b \in \mathbb{N}.$$

Then K is a perturbed metric on $\mathbb{N}$, where the perturbed mapping

$$L : \mathbb{N} \times \mathbb{N} \mapsto [0, \infty)$$

is given by

$$L(a, b) = (a - b)^2 - |a - b|, \quad a, b \in \mathbb{N},$$

and the exact metric $j : \mathbb{N} \times \mathbb{N} \mapsto [0, \infty)$ is given by

$$j(a, b) = |a - b|, \quad a, b \in \mathbb{N}.$$

Theorem 1.4. [1] Let (B, K, L) be a complete perturbed metric space, and let $L : B \mapsto B$ be a given mapping. Assume that the following conditions hold:

(i) L is a perturbed continuous mapping.

(ii) There exists $\lambda \in (0, 1)$ such that

$$K(La, Lb) \leq \lambda K(a, b)$$

for all $a, b \in B$.

Then L admits a unique fixed point.

Definition 1.5. [2] A multiplicative metric is a mapping $j : B \times B \mapsto \mathbb{R}$ satisfying the following conditions:

(i) $j(a, b) \geq 1$ for all $a, b \in B$, and $j(a, b) = 1$ if and only if $a = b$.

(ii) $j(a, b) = j(b, a)$ for all $a, b \in B$.

(iii) $j(a, c) \leq j(a, b) \cdot j(b, c)$ for all $a, b, c \in B$ (multiplicative triangle inequality).

Definition 1.6. [3] Let (B, j) be a multiplicative metric space, $\{a_n\}$ be a sequence in B , and $a \in B$. If, for every multiplicative open ball $M_\epsilon(a)$, there exists a natural number N such that $n \geq N$ implies $a_n \in M_\epsilon(a)$, then the sequence $\{a_n\}$ is said to be multiplicative convergent to a , denoted by $a_n \rightarrow_* a$ ($n \rightarrow \infty$).

Lemma 1.7. [3] Let (B, j) be a multiplicative metric space, $\{a_n\}$ be a sequence in B , and $a \in B$. Then $a_n \rightarrow_* a$ ($n \rightarrow \infty$) if and only if $j(a_n, a) \rightarrow_* 1$ ($n \rightarrow \infty$).

Definition 1.8. [3] Let (B, j) be a multiplicative metric space, and $\{a_n\}$ be a sequence in B . The sequence is called a multiplicative Cauchy sequence if, for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $j(a_n, a_m) < \epsilon$ for $m, n \geq N$.

Lemma 1.9. [3] Let (B, j) be a multiplicative metric space, and $\{a_n\}$ be a sequence in B . Then $\{a_n\}$ is a multiplicative Cauchy sequence if and only if $j(a_n, a_m) \rightarrow_* 1$ ($m, n \rightarrow \infty$).

Definition 1.10. [3] We call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is multiplicative convergent to $a \in B$.

Definition 1.11. [3] Let (B, j) be a multiplicative metric space. A mapping $L : B \mapsto B$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0, 1)$ such that

$$j(L(a_1), L(a_2)) \leq j(a_1, a_2)^\lambda$$

for all $a_1, a_2 \in B$.

Theorem 1.12. [3] Let (B, j) be a multiplicative metric space, and let $L : B \mapsto B$ be a multiplicative contraction. If (B, j) is complete, then L has a unique fixed point.

2 Perturbed Multiplicative Metric Space

Definition 2.1. Let $K_m, L_m : B \times B \mapsto [1, \infty)$ be two given mappings. We say that K_m is a perturbed multiplicative metric on B with respect to L_m if

$$\frac{K_m}{L_m} : B \times B \mapsto [1, \infty)$$

$$(a, b) \mapsto \frac{K_m(a, b)}{L_m(a, b)}$$

is a multiplicative metric on B , that is, for all $a, b, c \in B$,

$$(i) \left(\frac{K_m}{L_m} \right)(a, b) \geq 1.$$

$$(ii) \left(\frac{K_m}{L_m} \right)(a, b) = 1 \text{ if and only if } a = b.$$

$$(iii) \left(\frac{K_m}{L_m} \right)(a, b) = \left(\frac{K_m}{L_m} \right)(b, a).$$

$$(iv) \left(\frac{K_m}{L_m} \right)(a, b) \leq \left(\frac{K_m}{L_m} \right)(a, c) \cdot \left(\frac{K_m}{L_m} \right)(c, b).$$

We call L_m a perturbed mapping, $m = \frac{K_m}{L_m}$ an exact multiplicative metric, and (B, K_m, L_m) a perturbed multiplicative metric space.

Example 2.2. Let $K_m : \mathbb{R} \times \mathbb{R} \mapsto [1, \infty)$ be the mapping defined by

$$K_m(a, b) = 2^{|a-b|+a^2b^4}, \quad a, b \in \mathbb{R}.$$

Then K_m is a perturbed multiplicative metric on $\mathbb{R}$ with respect to the perturbed mapping

$$L_m : \mathbb{R} \times \mathbb{R} \mapsto [1, \infty),$$

given by

$$L_m(a, b) = 2^{a^2b^4}, \quad a, b \in \mathbb{R}.$$

In this case, the exact multiplicative metric is the mapping $m : \mathbb{R} \times \mathbb{R} \mapsto [1, \infty)$ defined by

$$m(a, b) = 2^{|a-b|}, \quad a, b \in \mathbb{R}.$$

Example 2.3. Let $K_m : \mathbb{N} \times \mathbb{N} \mapsto [1, \infty)$ be the mapping defined by

$$K_m(n, q) = 2^{(n-q)^2}, \quad n, q \in \mathbb{N}.$$

Then K_m is a perturbed multiplicative metric on $\mathbb{N}$, where the perturbed mapping

$$L_m : \mathbb{N} \times \mathbb{N} \mapsto [1, \infty)$$

is given by

$$L_m(n, q) = 2^{(n-q)^2-|n-q|}, \quad n, q \in \mathbb{N},$$

and the exact multiplicative metric $m : \mathbb{N} \times \mathbb{N} \mapsto [1, \infty)$ is given by

$$m(n, q) = 2^{|n-q|}, \quad n, q \in \mathbb{N}.$$

Remark 2.4. If j is a multiplicative metric, then $\log j$ is an ordinary metric, and conversely. Applying this to $m = \frac{K_m}{L_m}$ being a multiplicative metric is equivalent to $\log K_m - \log L_m$ being an ordinary metric. This suggests that a perturbed multiplicative metric space (B, K_m, L_m) can be obtained directly from a perturbed metric space (B, K, L) in the sense of [1] via $K_m = \exp(K)$ and $L_m = \exp(L)$. In particular, Theorem 3.1 of this paper follows from Theorem 1.4 of [1]. Note that the multiplicative presentation in this paper offers natural example classes as in Example 2.2-2.3 and Example 3.3, or a more convenient formulation for certain applications.

Proposition 2.5. Let $K_m, L_m, R_m : B \times B \mapsto [1, \infty)$ be three given mappings, and let $\alpha > 0$.

- (i) If (B, K_m, L_m) and (B, K_m, R_m) are two perturbed multiplicative metric spaces, then $(B, K_m, (L_m \cdot R_m)^{\frac{1}{2}})$ is a perturbed multiplicative metric space.

(ii) If (B, K_m, L_m) is a perturbed multiplicative metric space, then $(B, K_m^\alpha, L_m^\alpha)$ is a perturbed multiplicative metric space.

Proof. (i) Since $\frac{K_m}{L_m}$ and $\frac{L_m}{R_m}$ are two multiplicative metrics on B , it follows that

$$\left(\frac{K_m}{L_m} \cdot \frac{K_m}{R_m} \right)^{\frac{1}{2}} = \frac{K_m}{(L_m \cdot R_m)^{\frac{1}{2}}}$$

is a multiplicative metric on B .

(ii) Since $\frac{K_m}{L_m}$ is a multiplicative metric on B , and $\alpha > 0$, it follows that

$$\left(\frac{K_m}{L_m} \right)^\alpha = \frac{K_m^\alpha}{L_m^\alpha}$$

is a multiplicative metric on B .

□

Definition 2.6. Let (B, K_m, L_m) be a perturbed multiplicative metric space, $\{a_n\}$ be a sequence in B , and $S : B \mapsto B$ be a mapping.

- (i) We say that $\{a_n\}$ is a perturbed convergent sequence in (B, K_m, L_m) , if $\{a_n\}$ is a multiplicative convergent sequence in the multiplicative metric space (B, m) , where m is the exact multiplicative metric, that is, $m = \frac{K_m}{L_m}$.
- (ii) We say that $\{a_n\}$ is a perturbed multiplicative Cauchy sequence in (B, K_m, L_m) , if $\{a_n\}$ is a multiplicative Cauchy sequence in the multiplicative metric space (B, m) .
- (iii) We say that (B, K_m, L_m) is a complete perturbed multiplicative metric space, if (B, m) is a complete multiplicative metric space, or, equivalently, if every perturbed multiplicative Cauchy sequence in (B, K_m, L_m) is a perturbed multiplicative convergent sequence in (B, K_m, L_m) .
- (iv) We say that S is a perturbed continuous mapping, if S is continuous with respect to the exact multiplicative metric m .

3 Generalization of Banach Fixed Point Theorem

Theorem 3.1. Let (B, K_m, L_m) be a complete perturbed multiplicative metric space and $S : B \mapsto B$ be a given mapping. Assume that the following conditions hold:

- (i) S is a perturbed continuous mapping.

(ii) There exists $\lambda \in (0, 1)$ such that

$$K_m(Sa, Sb) \leq K_m(a, b)^\lambda$$

for all $a, b \in B$.

Then S admits a unique fixed point.

Proof. Let $a_0 \in B$ be fixed. Consider the Picard sequence $\{a_n\} \subset B$ defined by $a_{n+1} = Sa_n$, $n \in \mathbb{N}$. Taking $(a, b) = (a_0, a_1)$ in the contraction definition of the theorem, we obtain, $K_m(Sa_0, Sa_1) \leq K_m(a_0, a_1)^\lambda$, that is, $K_m(a_1, a_2) \leq K_m(a_0, a_1)^\lambda$. Similarly, taking $(a, b) = (a_1, a_2)$ in the contraction definition of the theorem, we obtain, $K_m(a_2, a_3) \leq K_m(a_1, a_2)^\lambda$, that is, $K_m(a_2, a_3) \leq K_m(a_0, a_1)^{\lambda^2}$. Continuing in the same way, by induction, we obtain, $K_m(a_n, a_{n+1}) \leq \tau^{\lambda^n}$, where $\tau = K_m(a_0, a_1)$. Let $m = \frac{K_m}{L_m}$ be the exact multiplicative metric. Since $K_m(a_n, a_{n+1}) \leq \tau^{\lambda^n}$, we deduce that $m(a_n, a_{n+1}) \cdot L_m(a_n, a_{n+1}) \leq \tau^{\lambda^n}$, $n \in \mathbb{N}$. Since $m(a_n, a_{n+1}) \leq m(a_n, a_{n+1}) \cdot L_m(a_n, a_{n+1})$, it follows that $m(a_n, a_{n+1}) \leq \tau^{\lambda^n}$, $n \in \mathbb{N}$. Thus making use of this inequality and the multiplicative triangle inequality, we obtain that for all $n \geq 0$ and $m \geq 1$, that

$$\begin{aligned} m(a_n, a_{n+m}) &\leq m(a_n, a_{n+1}) \cdots m(a_{n+m-1}, a_{n+m}) \\ &= \tau^{\lambda^n + \cdots + \lambda^{n+m-1}} \\ &\leq \tau^{\frac{\lambda^n}{1-\lambda}} \rightarrow 1 \quad \text{as } n, m \rightarrow \infty. \end{aligned}$$

It follows that $\{a_n\}$ is a multiplicative Cauchy sequence in the multiplicative metric space (B, m) , that is, $\{a_n\}$ is a perturbed multiplicative Cauchy sequence in the perturbed multiplicative metric space (B, K_m, L_m) . By the completeness of the perturbed multiplicative metric space (B, K_m, L_m) , we deduce there exists $a^* \in B$ such that $\lim_{n \rightarrow \infty} m(a_n, a^*) = 1$. We now show that a^* is a fixed point of S . Since S is a perturbed continuous mapping, $\lim_{n \rightarrow \infty} m(a_n, a^*) = 1$ yields $\lim_{n \rightarrow \infty} m(Sa_n, Sa^*) = 1$, that is, $\lim_{n \rightarrow \infty} m(a_{n+1}, Sa^*) = 1$. Since $m = \frac{K_m}{L_m}$ is a multiplicative metric on B , by the uniqueness of the limit, we get $a^* = Sa^*$, that is, a^* is a fixed point of S . We now show that S admits a unique fixed point. We argue by contradiction supposing that $a, b \in B$ are two distinct fixed points of S . By the contraction definition of the theorem, we have $K_m(a, b) = K_m(Sa, Sb) \leq K_m(a, b)^\lambda$, which yields $m(a, b) \cdot L_m(a, b) \leq (m(a, b) \cdot L_m(a, b))^\lambda$, that is, $(m(a, b) \cdot L_m(a, b))^{1-\lambda} \leq 1$. Since $a \neq b$, $m(a, b) \cdot L_m(a, b) \neq 1$, and the last inequality yields $\lambda \geq 1$, which contradicts the condition $\lambda \in (0, 1)$. Consequently, a^* is the unique fixed point of S . This completes the proof. $\square$

Corollary 3.2. (Banach's fixed point theorem) Let (B, m) be a complete multiplicative metric space, and let $S : B \rightarrow B$ be a given mapping. Assume that there exists $\lambda \in (0, 1)$ such that $m(Sa, Sb) \leq m(a, b)^\lambda$ for all $a, b \in B$. Then S admits a unique fixed point.

Proof. Let $K_m = m$ and $L_m \equiv 1$ ($L_m(a, b) = 1$ for all $a, b \in B$). Then (B, K_m, L_m) is a perturbed multiplicative metric space. Furthermore, by the contraction definition of the theorem, S is continuous with respect to the exact multiplicative metric m , and (ii) of Theorem 3.1 holds. Then Theorem 3.1 applies, this completes the proof. $\square$

Example 3.3. Let $B = \{A_1, A_2, A_3, A_4\} \subset \mathbb{R}^3$, where A_i , $i = 1, 2, 3, 4$, are the vertices of a regular tetrahedron with $\|A_i - A_j\| = 1$, $i \neq j$. Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^3$. We consider the mapping $S : B \mapsto B$ defined by

$$SA_1 = A_1, SA_2 = A_3, SA_3 = A_4, SA_4 = A_1.$$

We introduce the mapping $L_m : B \times B \mapsto [1, \infty)$ defined by

$$\begin{aligned} L_m(A_1, A_2) &= L_m(A_2, A_1) = 2^4, \\ L_m(A_1, A_3) &= L_m(A_3, A_1) = 2^3, \\ L_m(A_1, A_4) &= L_m(A_4, A_1) = 2^2, \\ L_m(A_2, A_3) &= L_m(A_3, A_2) = 2^4, \\ L_m(A_2, A_4) &= L_m(A_4, A_2) = 2^9, \\ L_m(A_3, A_4) &= L_m(A_4, A_3) = 2^3, \\ L_m(A_i, A_i) &= 1, \quad i = 1, 2, 3, 4. \end{aligned}$$

We also consider the mapping $K_m : B \times B \mapsto [1, \infty)$ defined by $K_m(A_i, A_j) = 2^{\|A_i - A_j\|} \cdot L_m(A_i, A_j)$, $i, j \in \{1, 2, 3, 4\}$. Observe that, (B, K_m, L_m) is a perturbed multiplicative metric space. In this case, the exact multiplicative metric is the discrete multiplicative metric $m : B \times B \mapsto [1, \infty)$ defined by $m(A_i, A_j) = 2^{\|A_i - A_j\|}$, $i, j \in \{1, 2, 3, 4\}$, that is, $m(A_i, A_j) = 2$, if $i \neq j$, and $m(A_i, A_j) = 1$, if $i = j$. We now show that the mapping S satisfies condition (i) and (ii) of Theorem 3.1. It is clear that S is a perturbed continuous mapping, that is, (i) is satisfied. On the other hand, (ii) of Theorem 3.1 is equivalent to

$$2^{\|SA_i - SA_j\|} \cdot L_m(SA_i, SA_j) \leq (2^{\|A_i - A_j\|} \cdot L_m(A_i, A_j))^\lambda$$

for every $i, j \in \{1, 2, 3, 4\}$. Observe that if $i = j$, by the definition of the perturbed mapping L_m , we have $2^{\|SA_i - SA_i\|} \cdot L_m(SA_i, SA_i) = 1$, which shows the contraction definition of the theorem is satisfied for all $\lambda > 0$. Assume that $i \neq j$. Table 1 provides the values of $\log(2^{\|SA_i - SA_j\|} \cdot L_m(SA_i, SA_j))$ and $\log(2^{\|A_i - A_j\|} \cdot L_m(A_i, A_j))$. From Table 1, we deduce that

$$\max_{1 \leq i \leq j \leq 4} \frac{\log(2^{\|SA_i - SA_j\|} \cdot L_m(SA_i, SA_j))}{\log(2^{\|A_i - A_j\|} \cdot L_m(A_i, A_j))} \leq \frac{4}{5}.$$

Table 1: The values of $\log(2^{\|SA_i - SA_j\|} \cdot L_m(SA_i, SA_j))$ and $\log(2^{\|A_i - A_j\|} \cdot L_m(A_i, A_j))$

(i, j)	$\log(2^{\ SA_i - SA_j\ } \cdot L_m(SA_i, SA_j))$	$\log(2^{\ A_i - A_j\ } \cdot L_m(A_i, A_j))$
(1,2)	$4\log(2)$	$5\log(2)$
(1,3)	$3\log(2)$	$4\log(2)$
(1,4)	0	$3\log(2)$
(2,3)	$4\log(2)$	$5\log(2)$
(2,4)	$4\log(2)$	$10\log(2)$
(3,4)	$3\log(2)$	$4\log(2)$

Then, by symmetry (notice that L_m is a symmetric mapping), we deduce that the contraction definition of the example holds for all $\lambda \in [\frac{4}{5}, 1)$, which shows that condition (ii) of Theorem 3.1 is satisfied. Observe that the only fixed point of S is the point A_1 , which confirms the result provided by Theorem 3.1. We point out that the Banach fixed point theorem is not applicable in the multiplicative metric space (B, m) , where m is the multiplicative metric defined by $m(A_i, A_j) = 2$ if $i \neq j$, and $m(A_i, A_j) = 1$, if $i = j$. This can easily be seen by observing for instance that

$$\frac{\log m(SA_1, SA_2)}{\log m(A_1, A_2)} = \frac{\log m(A_1, A_3)}{\log m(A_1, A_2)} = 1.$$

Acknowledgement

The author sincerely thanks the anonymous reviewer(s) for their constructive comments and suggestions, which greatly improved this manuscript.

References

- [1] Jleli, M., & Samet, B. (2025). On Banach's fixed point theorem in perturbed metric space. *Journal of Applied Analysis and Computation*, 15(2), 993–1001. <https://doi.org/10.11948/20240242>
- [2] Bashirov, A. E., Kurpinar, E. M., & Ozyapici, A. (2008). Multiplicative calculus and its applications. *Journal of Mathematical Analysis and Applications*, 337(1), 36–48. <https://doi.org/10.1016/j.jmaa.2007.03.081>
- [3] Ozavsar, M., & Cevikel, A. C. (2012). *Fixed points of multiplicative contraction mappings on multiplicative metric spaces* [Preprint]. arXiv:1205.5131. <https://arxiv.org/abs/1205.5131>

This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted, use, distribution and reproduction in any medium, or format for any purpose, even commercially provided the work is properly cited.
