

A Collection of Integral Inequalities and Applications

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Abstract

In this article, structurally new integral inequalities are introduced and rigorously proved. Integral inequalities play an essential role in Analysis and there is an intense area of research which is still active. The inequalities presented involve integrals, and then the inequalities are employed to obtain estimates involving real valued functions of a single variable. The article is rigorous, the proofs are detailed, applications are discussed, accompanied by appropriate graphical evidence.

1 Introduction

Inequalities involving integrals are at the heart of Analysis. These types of inequalities play an essential role in various fields where approximations are required and obtaining a closed formula for a given problem seems impossible. There are various works in the literature that contribute to the field, one of the most celebrated book by analysts all over the world is the work of Polya-Hardy-Littlewood on various types of analytic inequalities [1]. Another inspiration for this article is the monograph of Mitrinovic on analytic inequalities [2]. Additionally, other works in the literature [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14] serve a main motivation for this article where the authors derive various classes of integral inequalities. The field is contemporary and integral inequalities are at the heart of Modern Analysis. Concerning the current article, the work is organized as follows: First, the main inequalities involving integrals are presented in forms of theorems. Then, in the next section, rigorous proofs are included. The next section is dedicated to applications where weak estimates involving real valued functions of a single variable are obtained, using as test functions the unilateral Gaussian function $\exp(-t^2)$, the classical unilateral exponential function $\exp()$ and the unilateral inverse trigonometric function $\arctan()$. The next section includes graphical illustrations of the derived estimates. Ultimately, the last section the conclusion, explaining the core of the article and substantiating the novelty of the work. This article serves as motivation for further research in Analysis and as a pedagogical approach to demonstrate the elegance of integral inequalities. Concerning the work presented, the novelty lies in the following: 1. The integral inequalities presented are not the standard textbook inequalities, comparing with the classical works in analysis [1], [2], [3], [26], [27], [28], [29], [30], [31]. 2. The corollaries derived, involve weak inequalities of real valued functions of single variable, such as the Error function, the Gaussian function, the Sigmoid function, the exponential function and the

inverse tangent function. These inequalities are non-trivial and contribute to the literature of recorded mathematical inequalities. Using integral methods to derive inequalities is a fresh approach and one recent work of the French analyst Christophe Chesneau has employed integral methods to find estimates for trigonometric functions [15].

2 Presentation of the Main Results

In this section, the main results are presented in forms of theorems.

Theorem 2.1. Let $u : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$, $u \in W^{1,2}([a, b])$, $W^{1,2}([a, b]) = \{u \mid u \in L_2([a, b]), u' \in L_2([a, b])\}$. Then,

$$\left(\int_a^b u \, dt \right)^2 + \left(\int_a^b u' \, dt \right)^2 \leq |b - a| \|u\|_{W^{1,2}([a,b])}^2. \quad (1)$$

Theorem 2.2. Let $u : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$, $u \in C^\infty([a, b])$. Then

$$\begin{aligned} \left| \left(\int_a^b u \, dt \right)^3 + \left(\int_a^b u' \, dt \right)^3 \right| &\leq (b - a)^2 \|u\|_{L_3([a,b])}^3 + (b - a)^{\frac{3}{2}} \|u'\|_{L_2([a,b])}^3, \\ \|u\|_{L_3([a,b])}^3 &< \infty, \quad \|u'\|_{L_2([a,b])}^3 < \infty. \end{aligned} \quad (2)$$

Theorem 2.3. Let $u : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$, $u \in C^\infty([a, b])$, $u(a) \neq u(b) \neq 0$. Then

$$\begin{aligned} \left| \frac{1}{u(a)} - \frac{1}{u(b)} \right| &\leq \|u'\|_{L_2([a,b])} \left\| \frac{1}{u} \right\|_{L_4([a,b])}^2, \\ \|u'\|_{L_2([a,b])} &< \infty, \quad \left\| \frac{1}{u} \right\|_{L_4([a,b])}^2 < \infty. \end{aligned} \quad (3)$$

Theorem 2.4. Let $u : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$, $u \in C^\infty([a, b])$. Then

$$\begin{aligned} &\left| \left(\int_a^b u \, dt \right)^2 + \left(\int_a^b u \, dt \right)^3 + \left(\int_a^b u \, dt \right)^4 \right| \\ &\leq |b - a| \sqrt{b - a} \left(\|u\|_{L_4([a,b])}^2 + |b - a| \|u\|_{L_6([a,b])}^3 + |b - a|^2 \|u\|_{L_8([a,b])}^4 \right), \\ &\|u\|_{L_4([a,b])}^2, \|u\|_{L_6([a,b])}^3, \|u\|_{L_8([a,b])}^4 < \infty. \end{aligned} \quad (4)$$

Theorem 2.5. Let $u : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$, $u \in C^\infty([a, b])$. Then

$$\begin{aligned} &(b^4 - a^4) \left(\int_a^b u \, dt \right)^2 + 4(\exp(b) - \exp(a)) \int_a^b u \, dt \\ &\leq (b - a)^2 \frac{4\sqrt{7}}{7} \sqrt{b^7 - a^7} \|u\|_{L_4([a,b])}^2 + (b - a) 2\sqrt{2} \sqrt{\exp(2b) - \exp(2a)} \|u\|_{L_2([a,b])}, \\ &\|u\|_{L_4([a,b])}^2, \|u\|_{L_2([a,b])} < \infty, \text{ assuming } u > 0 \text{ and monotonically increasing.} \end{aligned} \quad (5)$$

Theorem 2.6. Let $u : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$, $u \in C^\infty([a, b])$. Then

$$\begin{aligned} & (\cosh(b) - \cosh(a)) \int_a^b u \, dt + \left(\frac{b^4 - a^4}{4} \right) \left(\int_a^b u \, dt \right)^2 \\ & \leq \frac{(b-a)}{2} \sqrt{\sinh(2b) - \sinh(2a) - 2(b-a)} \|u\|_{L_2([a,b])} + \\ & \quad + \frac{(b-a)^2 \sqrt{7}}{7} \sqrt{b^7 - a^7} \|u\|_{L_4([a,b])}^2, \end{aligned} \quad (6)$$

$\|u\|_{L_4([a,b])}^2, \|u\|_{L_2([a,b])} < \infty$, assuming $u > 0$ and monotonically increasing.

Theorem 2.7. Let $u : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$, $u \in C^\infty([a, b])$. Then

$$\begin{aligned} & \left| \left(\int_a^b u \, dt \right)^2 - \left(\int_a^b u' \, dt \right)^2 \right| \leq (b-a) \|u\|_{W^{1,2}([a,b])}^2 + 2(b-a) \|u\|_{L_2([a,b])} \|u'\|_{L_2([a,b])}, \\ & \|u\|_{W^{1,2}([a,b])}^2, \|u\|_{L_2([a,b])}, \|u'\|_{L_2([a,b])} < \infty. \end{aligned} \quad (7)$$

Theorem 2.8. Let $u : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$, $u \in C^\infty([a, b])$, $u(a) \neq u(b) \neq 0$. Then,

$$\begin{aligned} & \left| \frac{1}{u^2(a) + 1} - \frac{1}{u^2(b) + 1} \right| \leq 2 \left(\int_a^b \frac{1}{(u^2 + 1)^4} dt \right)^{\frac{1}{2}} \|u\|_{L_4([a,b])} \|u'\|_{L_4([a,b])}, \\ & \|u\|_{L_4([a,b])}, \|u'\|_{L_4([a,b])} < \infty. \end{aligned} \quad (8)$$

Theorem 2.9. Let $u : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$, $u \in C^\infty([a, b])$. Then

$$\begin{aligned} & \left| \frac{b^2}{\exp(-u(b)) + 1} - \frac{a^2}{\exp(-u(a)) + 1} \right| \\ & \leq \left(\int_a^b \frac{\exp(-2u)}{(\exp(-u) + 1)^4} dt \right)^{\frac{1}{2}} \left(\frac{2\sqrt{3}}{3} \sqrt{b^3 - a^3} + \left(\int_a^b t^4 u'^2 dt \right)^{\frac{1}{2}} \right) + \\ & \quad + \frac{2\sqrt{3}}{3} \sqrt{b^3 - a^3} \left(\int_a^b \frac{1}{(\exp(-u) + 1)^4} dt \right)^{\frac{1}{2}} \end{aligned} \quad (9)$$

assuming that the integrals converge.

Theorem 2.10. Let $u : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$, $u \in C^\infty([a, b])$, $u(a) \neq u(b) \neq 0$. Then

$$\begin{aligned} & \left| \frac{\exp(u(b))}{\exp(u(b)) + 1} - \frac{\exp(u(a))}{\exp(u(a)) + 1} \right| \leq \left(\int_a^b \exp(4u) dt \right)^{\frac{1}{4}} \left(\int_a^b \frac{1}{(\exp(u) + 1)^8} dt \right)^{\frac{1}{4}} \|u'\|_{L_2([a,b])}, \\ & \|u'\|_{L_2([a,b])} < \infty \end{aligned} \quad (10)$$

assuming that the integrals converge.

3 Rigorous Proofs of the Main Results

In this section, detailed proofs of the estimates in the previous section are carried out. All the rigorous proofs follow below.

Proof. Proof of **Theorem 2.1**:

$$\begin{aligned}
 \left(\int_a^b u \, dt \right)^2 + \left(\int_a^b u' \, dt \right)^2 &= \left(\int_a^b u \, dt \right) \left(\int_a^b u \, dt \right) + \left| \int_a^b u' \, dt \right|^2 \\
 &\leq |b-a| \int_a^b u^2 \, dt + \left| \int_a^b u' \, dt \right|^2 \quad (\text{Schwarz}) \\
 &= |b-a| \int_a^b u^2 \, dt + \left| \int_a^b u' \, dt \right| \left| \int_a^b u' \, dt \right| \\
 &\leq |b-a| \int_a^b u^2 \, dt + \\
 &\quad + \sqrt{b-a} \left(\int_a^b u'^2 \, dt \right)^{\frac{1}{2}} \sqrt{b-a} \left(\int_a^b u'^2 \, dt \right)^{\frac{1}{2}} \quad (\text{Schwarz}) \\
 &= |b-a| \int_a^b u^2 \, dt + |b-a| \int_a^b u'^2 \, dt \\
 &= |b-a| \left(\int_a^b u^2 \, dt + \int_a^b u'^2 \, dt \right) \\
 &= |b-a| \left(\|u\|_{L_2([a,b])}^2 + \|u'\|_{L_2([a,b])}^2 \right) \\
 &= |b-a| \|u\|_{W^{1,2}([a,b])}^2.
 \end{aligned}$$

□

Proof. Proof of **Theorem 2.2**:

$$\begin{aligned}
 \left| \left(\int_a^b u \, dt \right)^3 + \left(\int_a^b u' \, dt \right)^3 \right| &= \left| \left(\int_a^b u \, dt \right) \left(\int_a^b u \, dt \right) \left(\int_a^b u \, dt \right) + \left(\int_a^b u' \, dt \right)^3 \right| \\
 &= \left| \left(\int_a^b u \, dt \right) \left(\int_a^b u \, dt \right) \left(\int_a^b u \, dt \right) + \left(\int_a^b u' \, dt \right) \left(\int_a^b u' \, dt \right) \left(\int_a^b u' \, dt \right) \right|
 \end{aligned}$$

$$\begin{aligned}
&\leq \left| \left(\int_a^b u \, dt \right) \left(\int_a^b u \, dt \right) \left(\int_a^b u \, dt \right) \right| + \\
&+ \left| \left(\int_a^b u' \, dt \right) \left(\int_a^b u' \, dt \right) \left(\int_a^b u' \, dt \right) \right| \quad (Triangle) \\
&\leq (b-a)^2 \left| \int_a^b u^3 \, dt \right| + \left| \left(\int_a^b u' \, dt \right) \left(\int_a^b u' \, dt \right) \left(\int_a^b u' \, dt \right) \right| \quad (Hölder) \\
&\leq (b-a)^2 \int_a^b |u|^3 \, dt + \left(\int_a^b |u'| \, dt \right) \left(\int_a^b |u'| \, dt \right) \left(\int_a^b |u'| \, dt \right) \quad (Triangle) \\
&\leq (b-a)^2 \int_a^b |u|^3 \, dt + (b-a)^{\frac{3}{2}} \left(\left(\int_a^b |u'|^2 \, dt \right)^{\frac{1}{2}} \right)^3 \quad (Schwarz) \\
&= (b-a)^2 \left(\left(\int_a^b |u|^3 \, dt \right)^{\frac{1}{3}} \right)^3 + (b-a)^{\frac{3}{2}} \left(\left(\int_a^b |u'|^2 \, dt \right)^{\frac{1}{2}} \right)^3 \\
&= (b-a)^2 \|u\|_{L_3([a,b])}^3 + (b-a)^{\frac{3}{2}} \|u'\|_{L_2([a,b])}^3.
\end{aligned}$$

□

Proof. Proof of **Theorem 2.3**: Let $u \neq 0$. Then by the quotient rule we have

$$\left(\frac{1}{u} \right)' = \frac{(1)'u - 1u'}{u^2} = -\frac{u'}{u^2}, \quad ' := \frac{d}{dt}.$$

From the above, it follows

$$-\int_a^b \frac{u'}{u^2} \, dt = \int_a^b \left(\frac{1}{u} \right)' \, dt = \left(\frac{1}{u} \right)_{\partial[a,b]} = \frac{1}{u(b)} - \frac{1}{u(a)}.$$

The above can be recast in the form

$$\int_a^b \frac{u'}{u^2} \, dt = \frac{1}{u(a)} - \frac{1}{u(b)}.$$

Then, the above yields

$$\begin{aligned}
\left| \frac{1}{u(a)} - \frac{1}{u(b)} \right| &= \left| \int_a^b \frac{u'}{u^2} \, dt \right| \\
&\quad (Triangle \, type \, inequality) \leq \int_a^b \left| \frac{u'}{u^2} \right| \, dt
\end{aligned}$$

$$\begin{aligned}
(\text{Schwarz inequality}) &\leq \left(\int_a^b |u'|^2 dt \right)^{\frac{1}{2}} \left(\int_a^b \left(\frac{1}{u^2} \right)^2 dt \right)^{\frac{1}{2}} \\
&= \left(\int_a^b |u'|^2 dt \right)^{\frac{1}{2}} \left(\int_a^b \left(\frac{1}{u^4} \right) dt \right)^{\frac{1}{2}} \\
&= \left(\int_a^b |u'|^2 dt \right)^{\frac{1}{2}} \left(\int_a^b \left(\frac{1}{u} \right)^4 dt \right)^{\frac{1}{4}} \Big)^2 \\
&= \|u'\|_{L_2([a,b])} \left\| \frac{1}{u} \right\|_{L_4([a,b])}^2.
\end{aligned}$$

□

Proof. Proof of **Theorem 2.4**:

$$\begin{aligned}
&\left| \left(\int_a^b u dt \right)^2 + \left(\int_a^b u dt \right)^3 + \left(\int_a^b u dt \right)^4 \right| \leq \\
&\leq \left| \left(\int_a^b u dt \right)^2 \right| + \left| \left(\int_a^b u dt \right)^3 \right| + \left| \left(\int_a^b u dt \right)^4 \right| \quad (\text{Triangle}) \\
&\leq (b-a) \int_a^b |u|^2 dt + (b-a)^2 \int_a^b |u|^3 dt + \\
&+ (b-a)^3 \int_a^b |u|^4 dt \quad (\text{Schwarz and Hölder}) \\
&\leq |b-a| \sqrt{b-a} \left(\int_a^b |u|^4 dt \right)^{\frac{1}{2}} + |b-a|^2 \sqrt{b-a} \left(\int_a^b |u|^6 dt \right)^{\frac{1}{2}} + \\
&+ |b-a|^3 \sqrt{b-a} \left(\int_a^b |u|^8 dt \right)^{\frac{1}{2}} \quad (\text{Schwarz}) \\
&= |b-a| \sqrt{b-a} \|u\|_{L_4([a,b])}^2 + |b-a|^2 \sqrt{b-a} \|u\|_{L_6([a,b])}^3 + \\
&+ |b-a|^3 \sqrt{b-a} \|u\|_{L_8([a,b])}^4 \\
&= |b-a| \sqrt{b-a} \left(\|u\|_{L_4([a,b])}^2 + |b-a| \|u\|_{L_6([a,b])}^3 + |b-a|^2 \|u\|_{L_8([a,b])}^4 \right).
\end{aligned}$$

□

Proof. Proof of **Theorem 2.5**: Let u be monotonically increasing on $[a, b]$. Then

$$\begin{aligned}
 & \left(\frac{b^4 - a^4}{4} \right) \left(\int_a^b u \, dt \right)^2 + (\exp(b) - \exp(a)) \left(\int_a^b u \, dt \right) \\
 &= \left(\int_a^b t^3 \, dt \right) \left(\int_a^b u \, dt \right)^2 + \left(\int_a^b \exp(t) \, dt \right) \left(\int_a^b u \, dt \right) \\
 &= \left(\int_a^b t^3 \, dt \right) \left(\int_a^b u \, dt \right) \left(\int_a^b u \, dt \right) + \left(\int_a^b \exp(t) \, dt \right) \left(\int_a^b u \, dt \right) \\
 & \quad \text{Chebyshev integral inequality } (u_{[a,b]} > 0) \leq \\
 & \quad (b-a)^2 \int_a^b t^3 u^2 \, dt + (b-a) \int_a^b \exp(t) u \, dt \\
 & \quad \text{Schwarz inequality} \leq (b-a)^2 \left(\int_a^b (t^3)^2 \, dt \right)^{\frac{1}{2}} \left(\int_a^b (u^2)^2 \, dt \right)^{\frac{1}{2}} + \\
 & \quad + (b-a) \left(\int_a^b \exp(2t) \, dt \right)^{\frac{1}{2}} \left(\int_a^b u^2 \, dt \right)^{\frac{1}{2}} \\
 & \leq (b-a)^2 \frac{\sqrt{7}}{7} \sqrt{b^7 - a^7} \|u\|_{L_4([a,b])}^2 + (b-a) \frac{\sqrt{2}}{2} \sqrt{\exp(2b) - \exp(2a)} \|u\|_{L_2([a,b])}.
 \end{aligned}$$

By polishing the previous inequality and multiplying by factor 4 both sides of the inequality, the final result is obtained

$$\begin{aligned}
 & (b^4 - a^4) \left(\int_a^b u \, dt \right)^2 + 4(\exp(b) - \exp(a)) \int_a^b u \, dt \\
 & \leq (b-a)^2 \frac{4\sqrt{7}}{7} \sqrt{b^7 - a^7} \|u\|_{L_4([a,b])}^2 + (b-a) 2\sqrt{2} \sqrt{\exp(2b) - \exp(2a)} \|u\|_{L_2([a,b])}, \\
 & \quad \|u\|_{L_4([a,b])}^2, \quad \|u\|_{L_2([a,b])} < \infty.
 \end{aligned}$$

□

Proof. Proof of **Theorem 2.6**: Let u be a monotonically increasing function on domain $[a, b]$. Then:

$$\begin{aligned}
 & (\cosh(b) - \cosh(a)) \left(\int_a^b u \, dt \right) + \left(\frac{b^4 - a^4}{4} \right) \left(\int_a^b u \, dt \right)^2 \\
 &= \left(\int_a^b \sinh(t) \, dt \right) \left(\int_a^b u \, dt \right) + \left(\int_a^b t^3 \, dt \right) \left(\int_a^b u \, dt \right)^2
 \end{aligned}$$

$$\begin{aligned}
&= \left(\int_a^b \sinh(t) dt \right) \left(\int_a^b u dt \right) + \left(\int_a^b t^3 dt \right) \left(\int_a^b u dt \right) \left(\int_a^b u dt \right) \\
&\quad \text{Chebyshev integral inequality } (u_{[a,b]} > 0) \leq \\
&\quad (b-a) \left(\int_a^b u \sinh(t) dt \right) + (b-a)^2 \left(\int_a^b t^3 u^2 dt \right) \\
&\quad \text{Schwarz inequality} \leq \\
&\quad (b-a) \left(\int_a^b \sinh^2(t) dt \right)^{\frac{1}{2}} \left(\int_a^b u^2 dt \right)^{\frac{1}{2}} + (b-a)^2 \left(\int_a^b t^6 dt \right)^{\frac{1}{2}} \left(\int_a^b u^4 dt \right)^{\frac{1}{2}} \\
&= \frac{(b-a)}{2} \sqrt{\sinh(2b) - \sinh(2a) - 2(b-a)} \|u\|_{L_2([a,b])} + \\
&\quad + \frac{(b-a)^2 \sqrt{7}}{7} \sqrt{b^7 - a^7} \|u\|_{L_4([a,b])}^2, \\
&\quad \|u\|_{L_4([a,b])}^2, \|u\|_{L_2([a,b])} < \infty.
\end{aligned}$$

□

Proof. Proof of **Theorem 2.7**:

$$\begin{aligned}
&\left| \left(\int_a^b u dt \right)^2 - \left(\int_a^b u' dt \right)^2 \right| = \left| \left(\int_a^b u dt \right) - \left(\int_a^b u' dt \right) \right| \left| \left(\int_a^b u dt \right) + \left(\int_a^b u' dt \right) \right| \\
&\quad \text{Triangle inequality} \leq \\
&\quad \left(\left| \int_a^b u dt \right| + \left| \int_a^b u' dt \right| \right) \left(\left| \int_a^b u dt \right| + \left| \int_a^b u' dt \right| \right) = \left(\left| \int_a^b u dt \right| + \left| \int_a^b u' dt \right| \right)^2 \\
&\quad = \left| \int_a^b u dt \right|^2 + 2 \left| \int_a^b u dt \right| \left| \int_a^b u' dt \right| + \left| \int_a^b u' dt \right|^2 \\
&\quad = \left(\int_a^b u dt \right)^2 + 2 \left| \int_a^b u dt \right| \left| \int_a^b u' dt \right| + \left| \int_a^b u' dt \right|^2 \\
&\quad \text{Schwarz inequality} \leq \\
&\quad (b-a) \int_a^b u^2 dt + 2 \left| \int_a^b u dt \right| \left| \int_a^b u' dt \right| + \left| \int_a^b u' dt \right|^2 \\
&\quad = (b-a) \|u\|_{L_2([a,b])}^2 + 2 \left| \int_a^b u dt \right| \left| \int_a^b u' dt \right| + \left| \int_a^b u' dt \right|^2 \\
&\quad \text{Schwarz inequality} \leq
\end{aligned}$$

$$\begin{aligned}
& (b-a) \|u\|_{L_2([a,b])}^2 + 2(b-a) \|u\|_{L_2([a,b])} \|u'\|_{L_2([a,b])} + (b-a) \|u'\|_{L_2([a,b])}^2 \\
& = (b-a) \|u\|_{W^{1,2}([a,b])}^2 + 2(b-a) \|u\|_{L_2([a,b])} \|u'\|_{L_2([a,b])}, \\
& \|u\|_{W^{1,2}([a,b])}^2, \|u\|_{L_2([a,b])}, \|u'\|_{L_2([a,b])} < \infty.
\end{aligned}$$

□

Proof. Proof of **Theorem 2.8**: By a bit of calculus and employing the quotient rule, we have:

$$\left(\frac{1}{u^2+1}\right)' = -\frac{2uu'}{(u^2+1)^2}.$$

From the above, we have

$$\int_a^b \frac{2uu'}{(u^2+1)^2} dt = \frac{1}{u^2(a)+1} - \frac{1}{u^2(b)+1}.$$

Then the bound follows

$$\begin{aligned}
& \left| \frac{1}{u^2(a)+1} - \frac{1}{u^2(b)+1} \right| = \left| \int_a^b \frac{2uu'}{(u^2+1)^2} dt \right| \\
& = 2 \left| \int_a^b \frac{uu'}{(u^2+1)^2} dt \right| \\
& \text{Triangle type inequality} \leq 2 \int_a^b \left| \frac{uu'}{(u^2+1)^2} \right| dt \\
& \text{Hölder's inequality} \leq 2 \left(\int_a^b \left(\frac{1}{(u^2+1)^2} \right)^2 dt \right)^{\frac{1}{2}} \left(\int_a^b |u|^4 dt \right)^{\frac{1}{4}} \left(\int_a^b |u'|^4 dt \right)^{\frac{1}{4}} \\
& = 2 \left(\int_a^b \frac{1}{(u^2+1)^4} dt \right)^{\frac{1}{2}} \|u\|_{L_4([a,b])} \|u'\|_{L_4([a,b])}.
\end{aligned}$$

□

Proof. Proof of **Theorem 2.9**: Using a bit of calculus and applying the quotient rule, this gives

$$\left(\frac{t^2}{\exp(-u)+1}\right)' = \frac{\exp(-u)(2t+t^2u')}{(\exp(-u)+1)^2} + \frac{2t}{(\exp(-u)+1)^2}.$$

Then from the above, it follows

$$\begin{aligned}
& \left| \frac{b^2}{\exp(-u(b))+1} - \frac{a^2}{\exp(-u(a))+1} \right| = \\
& \left| \int_a^b \frac{\exp(-u)(2t+t^2u')}{(\exp(-u)+1)^2} dt + \int_a^b \frac{2t}{(\exp(-u)+1)^2} dt \right|
\end{aligned}$$

$$\begin{aligned}
\text{Triangle inequality} &\leq \left| \int_a^b \frac{\exp(-u)(2t+t^2u')}{(\exp(-u)+1)^2} dt \right| + \left| \int_a^b \frac{2t}{(\exp(-u)+1)^2} dt \right| \\
&\leq \int_a^b \left| \frac{\exp(-u)(2t+t^2u')}{(\exp(-u)+1)^2} \right| dt + \int_a^b \left| \frac{2t}{(\exp(-u)+1)^2} \right| dt \\
\text{Schwarz inequality} &\leq \left(\int_a^b \frac{\exp(-2u)}{(\exp(-u)+1)^4} dt \right)^{\frac{1}{2}} \|2t+t^2u'\|_{L_2([a,b])} + \\
&\quad + 2 \left(\int_a^b \frac{1}{(\exp(-u)+1)^4} dt \right)^{\frac{1}{2}} \left(\int_a^b t^2 dt \right)^{\frac{1}{2}} \\
\text{Minkowski inequality} &\leq \left(\int_a^b \frac{\exp(-2u)}{(\exp(-u)+1)^4} dt \right)^{\frac{1}{2}} (\|2t\|_{L_2([a,b])} + \|t^2u'\|_{L_2([a,b])}) + \\
&\quad + 2 \left(\int_a^b \frac{1}{(\exp(-u)+1)^4} dt \right)^{\frac{1}{2}} \left(\int_a^b t^2 dt \right)^{\frac{1}{2}} \\
&= \left(\int_a^b \frac{\exp(-2u)}{(\exp(-u)+1)^4} dt \right)^{\frac{1}{2}} \left(\frac{2\sqrt{3}}{3} \sqrt{b^3-a^3} + \left(\int_a^b t^4 u'^2 dt \right)^{\frac{1}{2}} \right) + \\
&\quad + \frac{2\sqrt{3}}{3} \sqrt{b^3-a^3} \left(\int_a^b \frac{1}{(\exp(-u)+1)^4} dt \right)^{\frac{1}{2}}.
\end{aligned}$$

□

Proof. Proof of **Theorem 2.10**: Using a bit of calculus and employing the quotient rule of differentiation, we have

$$\left(\frac{\exp(u)}{\exp(u)+1} \right)' = \frac{u' \exp(u)}{(\exp(u)+1)^2}.$$

Then, it follows

$$\begin{aligned}
\left| \frac{\exp(u(b))}{\exp(u(b))+1} - \frac{\exp(u(a))}{\exp(u(a))+1} \right| &= \left| \int_a^b \frac{u' \exp(u)}{(\exp(u)+1)^2} dt \right| \\
&\leq \int_a^b \left| \frac{u' \exp(u)}{(\exp(u)+1)^2} \right| dt \\
\text{Hölder's inequality} &\leq \|u'\|_{L_2([a,b])} \left(\int_a^b \exp(4u) dt \right)^{\frac{1}{4}} \left(\int_a^b \frac{1}{(\exp(u)+1)^8} dt \right)^{\frac{1}{4}}.
\end{aligned}$$

□

4 Applications-Inequalities Involving Error, Exponential, Inverse Trigonometric and Sigmoid Functions

In this section, inequalities are derived involving error and sigmoid type functions, as immediate corollaries of the integral inequalities derived.

Corollary 4.1.

$$erf(x) \leq \sqrt{\frac{4x}{\pi} \int_0^x \exp(-2t^2) dt + \frac{16x}{\pi} \int_0^x t^2 \exp(-2t^2) dt - \frac{4}{\pi} (1 - \exp(-x^2))^2}, \quad (11)$$

$$x \in]0, +\infty[, \quad erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

Proof. The result follows immediately from (1), letting $a = 0$, $b = x$ and $u(t) = \exp(-t^2)$, $t \in (0, x)$ and by polishing the inequality by doing some routine calculations. $\square$

Corollary 4.2.

$$erf(x) \leq \frac{2}{\sqrt{\pi}} \sqrt[3]{x^2 \int_0^x \exp(-3t^2) dt + 8x^{\frac{3}{2}} \left(\int_0^x t^2 \exp(-2t^2) dt \right)^{\frac{3}{2}} + (1 - \exp(-x^2))^3}, \quad (12)$$

$$x \in]0, +\infty[, \quad erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

Proof. This result stems from the inequality (2), letting $a = 0$, $b = x$, $u(t) = \exp(-t^2)$, $t \in (0, x)$ and by polishing the inequality by doing some routine calculations. $\square$

Corollary 4.3.

$$\left| \frac{1}{2} - S(x^2) \right| \leq 2 \left(\int_0^x t^2 \exp(-2t^2) dt \right)^{\frac{1}{2}} \left(\int_0^x (S(t^2))^4 dt \right)^{\frac{1}{2}}, \quad (13)$$

$$x \in]0, +\infty[, \quad S(x) = \frac{1}{1 + \exp(-x)}.$$

Proof. This is an immediate corollary of (3), letting $a = 0$, $b = x$, $u(t) = \exp(-t^2) + 1$, $t \in (0, x)$. $\square$

Corollary 4.4.

$$\begin{aligned}
& (\exp(x) - 1)^2 + (\exp(x) - 1)^3 + (\exp(x) - 1)^4 \\
& \leq x^{\frac{3}{2}} \left(\frac{1}{2} \sqrt{\exp(4x) - 1} + \frac{x\sqrt{6}}{6} \sqrt{\exp(6x) - 1} + \frac{x^2\sqrt{8}}{8} \sqrt{\exp(8x) - 1} \right), \quad (14) \\
& x \in]0, +\infty[.
\end{aligned}$$

Proof. The bound stems from the inequality (4), letting $a = 0$, $b = x$, $u(t) = \exp(t)$, $t \in (0, x)$ and performing essential calculations. $\square$

Corollary 4.5.

$$\begin{aligned}
& x^4 \left(x \arctan(x) - \frac{\ln(1+x^2)}{2} \right)^2 + 4(\exp(x) - 1) \left(x \arctan(x) - \frac{\ln(1+x^2)}{2} \right) \\
& \leq \frac{x^2 4\sqrt{7}\sqrt{x^7}}{7} \left(\int_0^x \arctan^4(t) dt \right)^{\frac{1}{2}} + 2\sqrt{2x} \sqrt{\exp(2x) - 1} \left(\int_0^x \arctan^2(t) dt \right)^{\frac{1}{2}}, \quad (15) \\
& x \in]0, +\infty[.
\end{aligned}$$

Proof. The bound is an immediate corollary of (5), letting $a = 0$, $b = x$, $u(t) = \arctan(t)$, $t \in (0, x)$. $\square$

Corollary 4.6.

$$\begin{aligned}
& (\cosh(x) - 1) \left(x \arctan(x) - \frac{\ln(1+x^2)}{2} \right) + \frac{x^4}{4} \left(x \arctan(x) - \frac{\ln(1+x^2)}{2} \right)^2 \\
& \leq \frac{x}{2} \sqrt{\sinh(2x) - 2x} \left(\int_0^x \arctan^2(t) dt \right)^{\frac{1}{2}} + \frac{x^2\sqrt{7}\sqrt{x^7}}{7} \left(\int_0^x \arctan^4(t) dt \right)^{\frac{1}{2}}, \quad (16) \\
& x \in]0, +\infty[.
\end{aligned}$$

Proof. The bound is an immediate result of applying (6), by setting $a = 0$, $b = x$, $u(t) = \arctan(t)$, $t \in (0, x)$. $\square$

Corollary 4.7.

$$\begin{aligned}
& \frac{1}{x} \left| \frac{\pi}{4} (\operatorname{erf}(x))^2 - (\exp(-x^2) - 1)^2 \right| \\
& \leq \int_0^x \exp(-2t^2) dt + 4 \int_0^x t^2 \exp(-2t^2) dt + 4 \left(\int_0^x \exp(-2t^2) dt \right)^{\frac{1}{2}} \left(\int_0^x t^2 \exp(-2t^2) dt \right)^{\frac{1}{2}}, \quad (17) \\
& x \in]0, +\infty[.
\end{aligned}$$

Proof. The result follows from the application of the estimate (7), setting $a = 0$, $b = x$, $u(t) = \exp(-t^2)$ and performing all the appropriate calculations. $\square$

Corollary 4.8.

$$\left| \frac{1}{2} - S(2x^2) \right| \leq 4 \left(\int_0^x (S(2t^2))^4 dt \right)^{\frac{1}{2}} \left(\int_0^x \exp(-4t^2) dt \right)^{\frac{1}{4}} \left(\int_0^x t^4 \exp(-4t^2) dt \right)^{\frac{1}{4}}, \quad (18)$$

$$x \in]0, +\infty[, \quad S(x) = \frac{1}{1 + \exp(-x)}.$$

Proof. This bound stems from the inequality (8), for $a = 0$, $b = x$ and $u(t) = \exp(-t^2)$, $t \in (0, x)$. $\square$

Corollary 4.9.

$$\left| x^2 S(x^2) \right| \leq \left(\int_0^x \exp(-2t^2) (S(t^2))^4 dt \right)^{\frac{1}{2}} \left(\frac{2\sqrt{3}}{3} x^{\frac{3}{2}} + \frac{2\sqrt{7}}{7} x^{\frac{7}{2}} \right) + \frac{2\sqrt{3}}{3} x^{\frac{3}{2}} \left(\int_0^x (S(t^2))^4 dt \right)^{\frac{1}{2}}, \quad (19)$$

$$x \in]0, +\infty[, \quad S(x) = \frac{1}{1 + \exp(-x)}.$$

Proof. The result follows by employing the estimate (9), letting $a = 0$, $b = x$, $u(t) = t^2$, $t \in (0, x)$. $\square$

Corollary 4.10.

$$x^{-\frac{3}{2}} \left| \exp(-x^2) S(x^2) - \frac{1}{2} \right| \leq \frac{2\sqrt{3}}{3} \left(\int_0^x \exp(-4t^2) dt \right)^{\frac{1}{4}} \left(\int_0^x (S(t^2))^8 dt \right)^{\frac{1}{4}}, \quad (20)$$

$$x \in]0, +\infty[, \quad S(x) = \frac{1}{1 + \exp(-x)}.$$

Proof. The result is derived by applying the integral inequality (10), letting $a = 0$, $b = x$, $u(t) = -t^2$, $t \in (0, x)$ and using some trivial calculations to recast the inequality to the final form. $\square$

5 Graphical Illustrations

In this section we include graphical illustrations of the inequalities (11), (12), (13), (14), (15), (16), (17), (18), (19), (20) to verify our results. The red solid curve corresponds to the function that consist of the left hand side of the inequality, whereas the blue dashed curve corresponds to the function that consists of the right hand side of the inequality. The corresponding graphs follow below in one place, with captions included. The inequalities are global and hold for all $x > 0$.

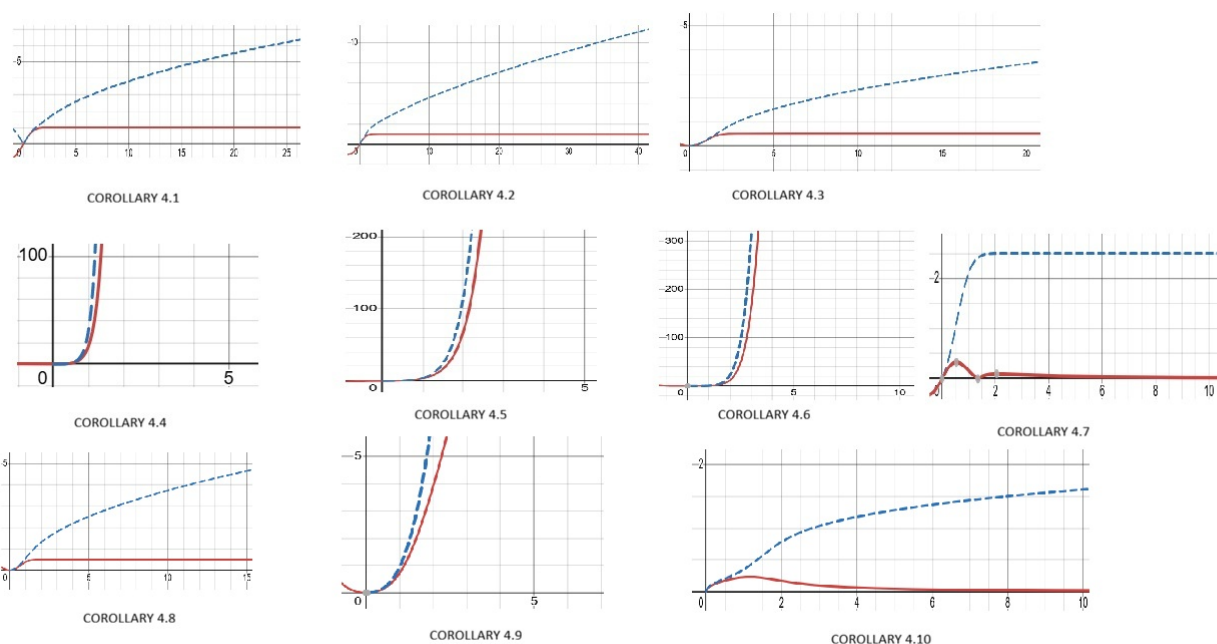


Figure 1: Graphical illustration of weak inequalities (11), (12), (13), (14), (15), (16), (17), (18), (19), (20).

6 Conclusion

In this article, integral inequalities have been derived and have been employed to obtain nontrivial weak inequalities involving real valued functions of a single variable. More precisely, weak type inequalities involving error, exponential, inverse trigonometric and sigmoid functions. All the inequalities yielded as global inequalities for $x > 0$. In terms of the weak global inequalities involving the error function obtained in this article, these do not overlap with existing works [16], [17], [18], [19], [20], [21], [22], [23]. The global bounds presented here involving the error function are new to the literature to the best of the author's knowledge. Also the analytical techniques employed are not the ones employed in the previously mentioned works. The analytical machinery in this article is pure use of integral inequalities. Concerning the weak global inequalities involving the sigmoid function, the one presented in the article are structurally different from the works of Nantomah [24] and Chesneau [25]. Both authors have employed classical methods such as convexity and monotonicity to obtain inequalities. In this work, only integral type inequalities are used. Last but not least, graphical illustrations have been presented to verify the theory and confirm the validity of the weak bounds.

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