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Fixed Point Result of Weak Paired Contractions

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Abstract

In this paper, we introduce a new class of mappings called *paired weak contractions*, which unify and extend two recent concepts in fixed point theory: weak contractions in Banach spaces (Berinde) and paired contractive mappings in metric spaces (Chand & Rohen). For a mapping $T : X \to X$ on a metric space X, we define the condition:

$$d(Tx, Ty) + d(Ty, Tz) \leq \delta(d(x, y) + d(y, z)) + L(d(y, Tx) + d(z, Ty)),$$

for all $x, y, z \in X$ which are pairwise distinct, where $\delta \in (0, 1)$ and $L \geq 0$. We establish existence and uniqueness of fixed points. We also apply our result to the Fredholm integral equation. Our results recover the main theorems of Chand & Rohen as special cases.

1 Preliminaries

Definition 1.1 ([1]). In the context of a metric space (X, d) with a cardinality of at least three, denoted as $|X| \geq 3$, we will say that the mapping $T : X \to X$ will have PC on X if there exists a real number $\delta \in [0, 1)$ such that the following inequality holds:

$$d(Tx, Ty) + d(Ty, Tz) \leq \delta(d(x, y) + d(y, z)),$$

for all $x, y, z \in X$ such that x, y, and z are pairwise distinct.

Definition 1.2. [2] Let (X, d) be a metric space. A map $T : X \to X$ is called a *weak contraction* if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$.

Definition 1.3. [1] A point $x \in X$ is termed a periodic point of period n if $T^n(x) = x$. The smallest positive integer m for which $T^m(x) = x$ is referred to as the prime period of x.

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2 Main Result

Inspired by the definitions provided in the previous section, we introduce a new mapping.

Definition 2.1. Let (X, d) be a metric space with a cardinality of at least three, denoted as $|X| \geq 3$. We say that a mapping $T : X \to X$ is a *paired weak contraction* (PWC) on X if there exists a real number $\delta \in (0, 1)$ and some $L \geq 0$ such that the following inequality holds:

$$d(Tx, Ty) + d(Ty, Tz) \leq \delta(d(x, y) + d(y, z)) + L(d(y, Tx) + d(z, Ty)), \tag{2.1}$$

for all $x, y, z \in X$ such that x, y, and z are pairwise distinct.

We now establish a fixed-point theorem for this newly defined mapping.

Theorem 2.1. *Let (X, d) be a complete metric space with a cardinality of at least three. If the mapping $T : X \to X$ is PWC on X, then it can be concluded that T has a fixed point.*

Proof. Let $\{x_n\}$ be a sequence with no two consecutive terms equal, defined by $x_n = Tx_{n-1}$ for all $n = 1, 2, \cdots$, and observe that

$$\begin{aligned}
d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) &= d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) \\
&\leq \delta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + L[d(x_n, x_n) + d(x_{n+1}, x_{n+1})] \\
&\leq \delta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&\vdots \\
&\leq \delta^n[d(x_0, x_1) + d(x_1, x_2)].
\end{aligned}$$

Let $J_i = d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})$. Then we get

$$J_n \leq \delta J_{n-1} \leq \delta^2 J_{n-2} \leq \cdots \leq \delta^n J_0. \tag{2.2}$$

Now, suppose that for a $j \in \mathbb{N}$ such that $j \geq 3$ and some $0 \leq i \leq j$, $x_j = x_i$, then

$$J_i = d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) = d(x_j, x_{j+1}) + d(x_{j+1}, x_{j+2}) = J_j.$$

This contradicts Equation (2.2). Therefore, such j and i do not exist.

Now, observe that

$$d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) = J_n \leq \delta^n J_0.$$

Thus, making use of the above inequality and the triangle inequality together, we obtain that for all $n \geq 0$

and $m \geq 1$, we have

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &\leq (\delta^n + \delta^{n+1} + \cdots + \delta^{n+m-1}) J_0 \\ &= \delta^n \frac{1-\delta^m}{1-\delta} J_0 \\ &\leq \frac{\delta^n}{1-\delta} J_0. \end{aligned}$$

Letting n and m tend to infinity in the above inequality, we get $d(x_n, x_{n+m}) \to 0$. This shows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), ensuring its convergence to a point $x^* \in X$, i.e. $d(x_n, x^*) \to 0$. Now we show that x^* is a fixed point of T. Observe that

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_n) + d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_n) + \delta \left[d(x_{n-1}, x_n) + d(x_n, x^*)\right] + L\left[d(x_n, x_n) + d(x^*, x_{n+1})\right]. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we have

$$d(x^*, Tx^*) \to 0,$$

and hence $d(x^*, Tx^*) = 0$, thus, $x^* = Tx^*$. □

Theorem 2.2 (Continuation of Theorem 2.1)**.** *If X and T are as defined in Theorem 2.1 with $1-\delta-2L \geq 0$, then T has a fixed point if and only if T does not have periodic points of prime period two.*

Proof. (Necessary condition) Let T have a fixed point (say ℓ). Suppose that T has a point of prime period two. Then there exists $y \in X$ such that $Ty = T(Tx) = x$. Using the definition of T, we get:

$$\begin{aligned} d(y, \ell) + d(\ell, x) &= d(Tx, T\ell) + d(T\ell, Ty) \\ &\leq \delta\Big(d(x, \ell) + d(\ell, y)\Big) + L\Big[d(\ell, Tx) + d(y, T\ell)\Big] \\ &\leq \delta d(x, \ell) + (\delta + 2L) d(\ell, y) \\ (1-\delta) d(x, \ell) + (1-\delta-2L) d(\ell, y) &\leq 0. \end{aligned}$$

It follows that $x = \ell$ and $\ell = y$, implying $x = y$, which is a contradiction. Therefore, T does not have periodic points of prime period two.

(Sufficient Condition) Suppose that T has no points of prime period two. Let $\{x_n\}$ be a sequence starting from some $x_0 \in X$ such that no term of the sequence x_n is fixed, then for each $i = 1, 2, 3, \cdots$ we can prove that every x_i is different from the others. The absence of fixed points in the sequence x_n implies that $x_{i+1} \neq x_i$, and the absence of a point of prime period two implies that $x_{i+2} \neq x_i$. Following this argument, all elements are different from one another. Now,

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) = d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}).$$

From here, by mimicking the proof of Theorem 2.1, we get that T has a fixed point. □

Theorem 2.3 (Continuation of Theorem 2.1)**.** *If X and T are as defined in Theorem 2.1 with $1-\delta-L>0$, then T has at most two distinct fixed points.*

Proof. Assume that T has three pairwise distinct fixed points a, b, and c. Using Equation 2.1 for these three points, we get

$$\begin{aligned} d(a,b)+d(b,c) &= d(Ta,Tb)+d(Tb,Tc) \\ &\leq \delta[d(a,b)+d(b,c)]+L[d(b,Ta)+d(c,Tb)] \\ &\leq (\delta+L)\Big(d(a,b)+d(b,c)\Big), \end{aligned}$$

which is a contradiction. Hence, T can possess at most two fixed points. □

Now, we propose a variant of the PWC mapping and prove that this variant, under the designed conditions, has a unique fixed point.

Theorem 2.4. *Let (X,d) be a complete metric space with $|X|\geq 3$. Consider the mapping $T: X \to X$ to be a PWC on X such that*

$$d(Tx,Ty)+d(Ty,Tz)\leq \delta(d(x,y)+d(y,z))+L(d(x,Tx)+d(y,Ty)),$$

for some $\delta\in(0,1)$ and $L\geq 0$ such that $\kappa=\delta+L<1$. Then T has a unique fixed point.

Proof. Let $\{x_n\}$ be a sequence with no two consecutive terms equal, defined by $x_n = Tx_{n-1}$ for all $n=1,2,\cdots$, and observe that

$$\begin{aligned} d(x_n,x_{n+1})+d(x_{n+1},x_{n+2}) &= d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1}) \\ &\leq \delta[d(x_{n-1},x_n)+d(x_n,x_{n+1})]+L[d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\ &\leq (\delta+L)[d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\ &\leq \kappa[d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\ &\ \vdots \\ &\leq \kappa^n[d(x_0,x_1)+d(x_1,x_2)]. \end{aligned}$$

Let $J_i = d(x_i,x_{i+1})+d(x_{i+1},x_{i+2})$, then we get

$$J_n \leq \kappa J_{n-1}\leq \kappa^2 J_{n-2}\leq\cdots\leq \kappa^n J_0. \tag{2.3}$$

Now, following the steps of Theorem 2.1, we get the existence of a fixed point.

For uniqueness, suppose that a, b and c are three pairwise distinct fixed points of T. Now, from the contractive condition of the theorem, we have

$$\begin{aligned} d(a,b)+d(b,c) &= d(Ta,Tb)+d(Tb,Tc) \\ &\leq \delta[d(a,b)+d(b,c)]+L[d(a,Ta)+d(b,Tb)]. \end{aligned}$$

From the above inequality, we have $d(a,b)+d(b,c) \leq \kappa(d(a,b)+d(b,c))$. Since $\kappa < 1$, it follows that $d(a,b)+d(b,c) = 0$, implying $a = b$ and $b = c$. Thus, T possesses a unique fixed point. This completes the proof. $\square$

Remark 2.5. In Theorems 2.1, 2.2, 2.3, and 2.4 presented above, taking $L = 0$ immediately return theorems of [1].

Example 2.6. Let $X = \{1,2,3\}$ and define the distance function d as follows: $d(1,1) = d(2,2) = d(3,3) = 0$, $d(1,2) = d(2,1) = \frac{1}{4}$, $d(2,3) = d(3,2) = \frac{1}{3}$, and $d(1,3) = d(3,1) = \frac{1}{2}$. Define the mapping $T : X \mapsto X$ by $T(1) = 1$, $T(2) = 2$ and $T(3) = 1$. Then T is a PWC on X with $\delta = \frac{1}{2}$ and $L = 1$. Furthermore, T does not possess a periodic point of prime period two.

Proof. It is clear that T does not possess a periodic point of prime period two. Now, we show that T is a PWC mapping on X as follows. We need to consider six cases:

<u>Case 1:</u> $x = 1, y = 2, z = 3$. In this case, we have

$$d(T1,T2)+d(T2,T3) < \frac{1}{2}[d(1,2)+d(2,3)]+[d(2,T1)+d(3,T2)].$$

Evaluating both sides of the inequality, we get

$$0.5 < 0.875.$$

<u>Case 2:</u> $x = 2, y = 1, z = 3$. In this case, we have

$$d(T2,T1)+d(T1,T3) < \frac{1}{2}[d(2,1)+d(1,3)]+[d(1,T2)+d(3,T1)].$$

Evaluating both sides of the inequality, we get

$$0.25 < 1.125.$$

<u>Case 3:</u> $x = 2, y = 3, z = 1$. In this case, we have

$$d(T2,T3)+d(T3,T1) < \frac{1}{2}[d(2,3)+d(3,1)]+[d(3,T2)+d(1,T3)]$$

Evaluating both sides of the inequality, we get

$$0.25 < 0.75.$$

<u>Case 4:</u> $x = 3, y = 2, z = 1$. In this case, we have

$$d(T3,T2)+d(T2,T1) < \frac{1}{2}[d(3,2)+d(2,1)]+[d(2,T3)+d(1,T2)].$$

Evaluating both sides of the inequality, we get

$$0.5 < 0.791667.$$

Case 5: $x = 1, y = 3, z = 2$. In this case, we have

$$d(T1, T3) + d(T3, T2) < \frac{1}{2}[d(1, 3) + d(3, 2)] + [d(3, T1) + d(2, T3)].$$

Evaluating both sides of the inequality, we get

$$0.25 < 1.16667.$$

Case 6: $x = 3, y = 1, z = 2$. In this case, we have

$$d(T3, T1) + d(T1, T2) < \frac{1}{2}[d(3, 1) + d(1, 2)] + [d(1, T3) + d(2, T1)].$$

Evaluating both sides of the inequality, we get

$$0.25 < 0.625.$$

Thus, combining cases 1 through 6, we see that T is a PWC on X. This completes the proof. Note that this example illustrates Theorem 2.2 and is not intended to show that $L > 0$ is necessary for the contraction to hold. □

3 Application to Fredholm Integral Equation

We apply our result to establish an existence theorem for a non-linear Fredholm integral equation. Let $Y = C[0, 1]$ be the set of all real-valued continuous functions on $[0, 1]$ equipped with the metric $p(b, c) = \max_{t\in[0,1]} |b(t) - c(t)|$, for all $b, c \in Y$. Then (Y, p) is a complete metric space. Now we consider the non-linear Fredholm integral equation

$$b(t) = c(t) + \int_0^1 K(t, s, b(s))ds, \tag{3.1}$$

where $s, t \in [0, 1]$. Assume that $K : [0, 1] \times [0, 1] \times Y \mapsto \mathbb{R}$ and $c : [0, 1] \mapsto \mathbb{R}$ are continuous, where $c(t)$ is a given function in Y.

Theorem 3.1. *Suppose (Y, p) is a metric space equipped with the metric $p(b, c) = \max_{t\in[0,1]} |b(t) - c(t)|$ for all $b, c \in Y$, and let $F : Y \mapsto Y$ be an operator on Y defined by*

$$Fb(t) = c(t) + \int_0^1 K(t, s, b(s))ds. \tag{3.2}$$

If there exist $\delta \in (0,1)$ and $L \geq 0$ such that, for all pairwise distinct points $b, c, d \in Y$ and $s, t \in [0,1]$, the following inequality is satisfied:

$$\begin{aligned}|K(t,s,b(s)) - K(t,s,c(s))| &+ |K(t,s,c(s)) - K(t,s,d(s))| \\ &\leq \delta[|b(s)-c(s)| + |c(s)-d(s)|] + L[|c(s) - Fb(s)| + |d(s) - Fc(s)|].\end{aligned}$$

Then the integral equation (3.2) has a solution in Y.

Proof. From (3.1) and (3.2), we obtain

$$\begin{aligned}|Fb(t) - Fc(t)| + |Fc(t) - Fd(t)| &= \left| \int_0^1 K(t,s,b(s))ds - \int_0^1 K(t,s,c(s))ds \right| \\ &\quad + \left| \int_0^1 K(t,s,c(s))ds - \int_0^1 K(t,s,d(s))ds \right| \\ &\leq \int_0^1 (|K(t,s,b(s)) - K(t,s,c(s))| \\ &\quad + |K(t,s,c(s)) - K(t,s,d(s))|)ds \\ &\leq \int_0^1 (\delta[|b(s)-c(s)| + |c(s)-d(s)|] \\ &\quad + L[|c(s) - Fb(s)| + |d(s) - Fc(s)|])ds.\end{aligned}$$

Taking the maximum on both sides for all $t \in [0,1]$, we conclude that

$$p(Fb, Fc) + p(Fc, Fd) \leq \delta[p(b,c) + p(c,d)] + L[p(c, Fb) + p(d, Fc)]$$

which shows that F satisfies the condition of a PWC. Since $Y = C[0,1]$ is a complete metric space, we find that all the conditions of Theorem 2.1 are satisfied, and hence the integral equation (3.2) has a solution in Y. □

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