

Radius Constants for the Cesàro Operator of a Class of Harmonic Mappings Defined by a Third-Order Differential Inequality

Umut Can Kılıç^{1,*}, Sibel Yalçın² and Hasan Bayram³

¹ Bursa Uludağ University, Faculty of Arts and Sciences, Department of Mathematics, 16059 Görükle, Bursa, Türkiye
 e-mail: kilicumutcan@uludag.edu.tr

² Bursa Uludağ University, Faculty of Arts and Sciences, Department of Mathematics, 16059 Görükle, Bursa, Türkiye
 e-mail: syalcin@uludag.edu.tr

³ Bursa Uludağ University, Faculty of Arts and Sciences, Department of Mathematics, 16059 Görükle, Bursa, Türkiye
 e-mail: hbayram@uludag.edu.tr

Abstract

For $\lambda \geq \delta \geq 0$, let $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ denote the class of normalized harmonic mappings $f = h + \bar{g}$ in the open unit disk $\mathbb{D}$ satisfying the third-order differential inequality $\operatorname{Re}\{h'(z) + \lambda zh''(z) + \delta z^2 h'''(z)\} > |g'(z) + \lambda zg''(z) + \delta z^2 g'''(z)|$, whose members are known to be close-to-convex in $\mathbb{D}$. In this paper, the image of harmonic mappings under the Cesàro operator $\mathcal{C}f = \mathcal{C}h + \overline{\mathcal{C}g}$ is studied. Under four natural families of coefficient conditions, among them the sharp coefficient bounds of the class $\mathcal{K}_{\mathcal{H}}^0$ of convex harmonic mappings and a coefficient bound of the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ itself, sufficient radii $r \in (0, 1)$ are determined such that the normalized Cesàro operator $2\mathcal{C}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the disk $\mathbb{D}_r$. Since the underlying coefficient condition is only sufficient, the radii obtained are lower bounds for the exact radii. In particular, it is shown that the normalized Cesàro operator $f \mapsto 2\mathcal{C}f$ maps $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ into itself in an explicitly determined disk. Each radius is characterized as the unique root in $(0, 1)$ of an explicit polynomial or logarithmic-polynomial equation, and the radii corresponding to the four coefficient conditions are completely ordered. The results extend recent radius results for the Cesàro operator to a class generated by a third-order differential inequality.

1 Introduction

A continuous complex-valued function $f = u + iv$ defined in a domain $B \subset \mathbb{C}$ is said to be *harmonic* in B if both u and v are real harmonic in B . In any simply connected domain we may write $f = h + \bar{g}$, where h and g are analytic in B ; the functions h and g are called, respectively, the analytic and the co-analytic part of f . By a classical result of Lewy, f is locally univalent and sense-preserving in B if and only if $|h'(z)| > |g'(z)|$ in B ; see [1, 2].

Let $\mathcal{H}$ denote the class of harmonic functions $f = h + \bar{g}$ in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $h(0) = f_z(0) - 1 = 0$, and let $\mathcal{H}^0 = \{f \in \mathcal{H} : f_{\bar{z}}(0) = 0\}$. Every $f = h + \bar{g} \in \mathcal{H}^0$ admits the

Received: September 10, 2026; Revised & Accepted: October 17, 2026; Published online: October 23, 2026

2020 Mathematics Subject Classification: Primary 30C45; Secondary 30C50, 31A05.

Keywords and phrases: harmonic mappings; close-to-convex functions; Cesàro operator; radius problem; third-order differential inequality.

*Corresponding author

Copyright 2026 the Authors

representation

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D}. \quad (1)$$

As usual, $\mathcal{S}_{\mathcal{H}}$ denotes the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving mappings, $\mathcal{S}_{\mathcal{H}}^0 = \{f \in \mathcal{S}_{\mathcal{H}} : f_{\bar{z}}(0) = 0\}$, and $\mathcal{S}_{\mathcal{H}}^{0,*}$, $\mathcal{K}_{\mathcal{H}}^0$, $\mathcal{C}_{\mathcal{H}}^0$ are the subclasses of $\mathcal{S}_{\mathcal{H}}^0$ mapping $\mathbb{D}$ onto starlike, convex and close-to-convex domains, respectively. With $g \equiv 0$, these classes reduce to the classical families $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ of analytic univalent functions.

Ponnusamy, Yamamoto and Yanagihara [3] introduced the class of mappings $f = h + \bar{g} \in \mathcal{H}^0$ satisfying $\operatorname{Re} f_z(z) > |f_{\bar{z}}(z)|$ in $\mathbb{D}$ and proved that such mappings are univalent and close-to-convex; partial sums of this class were studied by Li and Ponnusamy [4]. Ghosh and Vasudevarao [5] investigated the class defined by the second-order inequality $\operatorname{Re}\{h'(z) + \alpha z h''(z)\} > |g'(z) + \alpha z g''(z)|$. Motivated by the class $\mathcal{R}_S(\lambda, \delta)$ of analytic functions satisfying $\operatorname{Re}\{f'(z) + \lambda z f''(z) + \delta z^2 f'''(z)\} > 0$, which was studied by Rosihan Ali et al. [6] (see also Al-Refai [7]), Yaşar and Yalçın [8] introduced and studied the harmonic class defined by a third-order differential inequality.

Definition 1.1 ([8]). Let $\lambda \geq \delta \geq 0$. A mapping $f = h + \bar{g} \in \mathcal{H}^0$ of the form (1) belongs to the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ if

$$\operatorname{Re}\{h'(z) + \lambda z h''(z) + \delta z^2 h'''(z)\} > |g'(z) + \lambda z g''(z) + \delta z^2 g'''(z)|, \quad z \in \mathbb{D}. \quad (2)$$

The members of $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ are close-to-convex (hence univalent) in $\mathbb{D}$, and the class contains the classes of [3] and [5] as the special cases $\mathcal{R}_{\mathcal{H}}^0(0, 0)$ and $\mathcal{R}_{\mathcal{H}}^0(\lambda, 0)$, respectively; see [8, Theorem 2.5 and Remark 2.12]. In particular, $\mathcal{R}_{\mathcal{H}}^0(0, 0)$ coincides with the class $\mathcal{P}_{\mathcal{H}}^0$ of [3, 4], consisting of the mappings $f = h + \bar{g} \in \mathcal{H}^0$ with $\operatorname{Re} h'(z) > |g'(z)|$ in $\mathbb{D}$.

The Cesàro operator, a classical tool in summability theory, acts on an analytic function $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$ by

$$\mathcal{C}\varphi(z) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n = \int_0^1 \frac{\varphi(tz)}{1-tz} dt. \quad (3)$$

Its boundedness and spectral properties on classical spaces of analytic functions were investigated by Siskakis [9] and Stempak [10], among many others. Recently, Hu, Srivastava and Wang [11] initiated the study of *radius constants* for the Cesàro operator: although $\mathcal{C}f$ need not inherit a geometric property of f in the whole disk $\mathbb{D}$, one may ask for the largest $r \in (0, 1)$ such that $\mathcal{C}f$ possesses this property in $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. In [11], such radii were obtained for starlikeness and convexity of order α of harmonic mappings in the sense of Jahangiri [12]; in the conference contribution [13] the corresponding problem was considered for the Goodman–Rønning-type class $\mathcal{G}_{\mathcal{H}}^0(\alpha)$ of [14].

The purpose of the present paper is to carry this problem over, by the method of [11, 13], to the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ generated by the third-order differential inequality (2); what is new here is this target class, whose coefficient weight Ψ_n in (4) is cubic in n , together with the applications in Corollaries 3.2, 3.4

and 3.8 to the classes $\mathcal{S}$, $\mathcal{K}$, $\mathcal{R}_{\mathcal{H}}^0(\lambda', \delta')$ and to stable harmonic mappings. Under four natural families of coefficient conditions, namely

$$\begin{aligned} & \text{(i) } |a_n|, |b_n| \leq n, \quad \text{(ii) } |a_n|, |b_n| \leq 1, \\ & \text{(iii) } |a_n| \leq \frac{n+1}{2}, \quad |b_n| \leq \frac{n-1}{2}, \quad \text{(iv) } |a_n| + |b_n| \leq \frac{2}{n}, \end{aligned}$$

we determine in Section 3 sufficient radii $r_1, r_2, r_3, r_4 \in (0, 1)$ such that the normalized Cesàro operator $2\mathcal{C}f$ of the corresponding harmonic mappings belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in $\mathbb{D}_{r_i}$; the necessity of the normalizing factor 2 is explained in Remark 2.1.

The four coefficient conditions above arise naturally in geometric function theory. Condition (i) is satisfied by the stable univalent harmonic mappings studied by Hernández and Martín [15]; in the analytic case $g \equiv 0$ it holds for the full class $\mathcal{S}$ in view of de Branges' theorem, and already for every close-to-convex function by a classical result of Reade [16]. Condition (ii) is fulfilled by considerably more mappings than its formal appearance suggests: by MacGregor's theorem [17], every analytic function f with $\operatorname{Re} f'(z) > 0$ in $\mathbb{D}$ satisfies $|a_n| \leq \frac{2}{n} \leq 1$ for $n \geq 2$, and the members of every class $\mathcal{R}_{\mathcal{H}}^0(\lambda', \delta')$ with $\lambda' \geq \delta' \geq 0$ obey the stronger condition (iv) by [8, Theorem 2.7]; hence (ii) covers, in particular, the harmonic classes of [3] and [5]. Condition (iii) consists of the sharp coefficient bounds of the convex class $\mathcal{K}_{\mathcal{H}}^0$ due to Clunie and Sheil-Small [1]; and condition (iv) holds for the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ itself by [8, Theorem 2.7]. As a consequence we obtain that the normalized Cesàro operator $f \mapsto 2\mathcal{C}f$ maps $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ into itself in a disk $\mathbb{D}_{r_4^*}$ with $r_4^* \geq r_4$ (Corollary 3.8), and we prove the strict ordering $r_1 < r_3 < r_2 < r_4$ for all admissible parameters (Proposition 3.9).

We emphasize that Lemma 2.2 is only a sufficient condition for membership in $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$; hence the radii obtained in this paper are sufficient radii, that is, lower bounds for the exact radii (see Remark 3.10). Graphs of the radius functions and tables of numerical values are provided.

2 Preliminaries

Throughout the paper $\lambda \geq \delta \geq 0$,

$$\begin{aligned} \Psi_n &= \Psi_n(\lambda, \delta) := n[1 + \lambda(n-1) + \delta(n-1)(n-2)] \\ &= \delta n^3 + (\lambda - 3\delta)n^2 + (1 - \lambda + 2\delta)n, \end{aligned} \tag{4}$$

and

$$M = M(\lambda, \delta) := 1 - 2\lambda + 6\delta. \tag{5}$$

Observe that $\Psi_n \geq n$ for all $n \geq 1$ and that the elementary division identity

$$\frac{\Psi_n}{n+1} = \delta n^2 + (\lambda - 4\delta)n + M - \frac{M}{n+1} \tag{6}$$

holds; it will be used repeatedly.

In view of (3), the Cesàro operator of a harmonic mapping $f = h + \bar{g} \in \mathcal{H}^0$ of the form (1) is defined, following [11], by

$$\mathcal{C}f(z) := \mathcal{C}h(z) + \overline{\mathcal{C}g(z)} = \int_0^1 \frac{h(tz)}{1-tz} dt + \overline{\int_0^1 \frac{g(tz)}{1-tz} dt}. \quad (7)$$

Observe that the analytic part of $\mathcal{C}f$ has first Taylor coefficient $\frac{1}{2}$, so that $\mathcal{C}f$ itself does not belong to $\mathcal{H}^0$. Since $a_0 = b_0 = b_1 = 0$ and $a_1 = 1$, the function $2\mathcal{C}f$ does belong to $\mathcal{H}^0$; we call

$$\begin{aligned} \tilde{\mathcal{C}}f(z) &:= 2\mathcal{C}f(z) = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=2}^{\infty} B_n z^n}, \\ A_n &= \frac{2}{n+1} \sum_{k=1}^n a_k, \quad B_n = \frac{2}{n+1} \sum_{k=2}^n b_k, \end{aligned} \quad (8)$$

the normalized Cesàro operator of f . All results of this paper are formulated for $\tilde{\mathcal{C}}f$.

Remark 2.1. Both sides of the inequality (2) are positively homogeneous of degree one in the pair (h, g) : if $c > 0$, then f satisfies (2) if and only if cf does. Consequently, $\mathcal{C}f$ satisfies the differential inequality (2) at a point of $\mathbb{D}$ if and only if the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ does. Membership in the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$, however, additionally requires the normalization $f \in \mathcal{H}^0$, which $\mathcal{C}f$ fails; for this reason all statements below are made for $\tilde{\mathcal{C}}f$ rather than for $\mathcal{C}f$. The analogous convention is adopted in [11].

The radius problem is made precise by means of the following dilation identity. For $F = H + \bar{G} \in \mathcal{H}^0$ and $0 < r < 1$ put $F_r(z) := r^{-1}F(rz) = H_r(z) + \bar{G}_r(z)$. Writing $w = rz$, a direct computation gives, for $\varphi \in \{H, G\}$,

$$\varphi'_r(z) + \lambda z \varphi''_r(z) + \delta z^2 \varphi'''_r(z) = \varphi'(w) + \lambda w \varphi''(w) + \delta w^2 \varphi'''(w), \quad w = rz. \quad (9)$$

Hence $F_r \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ if and only if the inequality (2) holds for F at every point of $\mathbb{D}_r$. Accordingly, we say that F belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the disk $\mathbb{D}_\rho$ if $F_r \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ for every $0 < r \leq \rho$.

The following sufficient coefficient condition and sharp coefficient bounds are due to Yaşar and Yalçın [8].

Lemma 2.2 ([8, Theorem 2.8]). *Let $f = h + \bar{g} \in \mathcal{H}^0$ be of the form (1). If*

$$\sum_{n=2}^{\infty} \Psi_n (|a_n| + |b_n|) \leq 1, \quad (10)$$

then $f \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$; in particular, f is close-to-convex in $\mathbb{D}$.

Lemma 2.3 ([8, Theorems 2.6 and 2.7]). *If $f = h + \bar{g} \in \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ is of the form (1), then*

$$|a_n| + |b_n| \leq \frac{2}{\Psi_n} \quad (n \geq 2),$$

and the bound is sharp for $f(z) = z + \sum_{n=2}^{\infty} \frac{2}{\Psi_n} z^n$.

Lemma 2.4 ([1]). If $f = h + \bar{g} \in \mathcal{K}_{\mathcal{H}}^0$, then for all $n \geq 1$,

$$|a_n| \leq \frac{n+1}{2}, \quad |b_n| \leq \frac{n-1}{2}.$$

We shall also need the following elementary summation formulas; here and in what follows $Q_n := \sum_{k=1}^n \frac{1}{k}$ denotes the n -th harmonic number and $\log$ is the principal branch.

Lemma 2.5. For $0 < r < 1$ the following identities hold:

$$\begin{aligned} \sum_{n=2}^{\infty} r^{n-1} &= \frac{r}{1-r}, \\ \sum_{n=2}^{\infty} nr^{n-1} &= \frac{1}{(1-r)^2} - 1, \\ \sum_{n=2}^{\infty} n^2 r^{n-1} &= \frac{1+r}{(1-r)^3} - 1, \\ \sum_{n=2}^{\infty} n^3 r^{n-1} &= \frac{1+4r+r^2}{(1-r)^4} - 1, \\ \sum_{n=2}^{\infty} n^4 r^{n-1} &= \frac{1+11r+11r^2+r^3}{(1-r)^5} - 1, \\ \sum_{n=2}^{\infty} \frac{r^{n-1}}{n+1} &= -\frac{\log(1-r)}{r^2} - \frac{1}{r} - \frac{1}{2}, \\ \sum_{n=2}^{\infty} Q_n r^{n-1} &= -\frac{\log(1-r)}{r(1-r)} - 1, \\ \sum_{n=2}^{\infty} nQ_n r^{n-1} &= \frac{1-\log(1-r)}{(1-r)^2} - 1, \\ \sum_{n=2}^{\infty} n^2 Q_n r^{n-1} &= \frac{(1+r)(1-\log(1-r))+r}{(1-r)^3} - 1, \\ \sum_{n=2}^{\infty} \frac{Q_n}{n+1} r^{n-1} &= \frac{\log^2(1-r)}{2r^2} - \frac{1}{2}. \end{aligned}$$

Proof. The first five identities follow from the geometric series $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ by repeated application of the operator $r \frac{d}{dr}$ and removal of the term $n = 1$. For the sixth, since $\sum_{m=1}^{\infty} \frac{r^m}{m} = -\log(1-r)$, we have $\sum_{n=2}^{\infty} \frac{r^{n+1}}{n+1} = \sum_{m=3}^{\infty} \frac{r^m}{m} = -\log(1-r) - r - \frac{r^2}{2}$, and dividing by r^2 gives the stated value. The generating function $\sum_{n=1}^{\infty} Q_n r^n = -\frac{\log(1-r)}{1-r}$, obtained as the Cauchy product of $\frac{1}{1-r}$ and $-\log(1-r)$, gives the seventh identity after division by r and removal of the term $n = 1$; differentiating it once yields $\sum_{n=1}^{\infty} nQ_n r^{n-1}$, and multiplying by r and differentiating once more yields $\sum_{n=1}^{\infty} n^2 Q_n r^{n-1}$, whence the eighth and ninth identities follow by discarding the term $n = 1$. Finally, integrating the generating function of Q_n ,

$$\sum_{n=1}^{\infty} \frac{Q_n}{n+1} r^{n+1} = \int_0^r \frac{-\log(1-t)}{1-t} dt = \frac{\log^2(1-r)}{2},$$

and dividing by r^2 before removing the term $n = 1$ gives the last identity. $\square$

3 Main Results

Throughout this section $f = h + \bar{g} \in \mathcal{H}^0$ is of the form (1),

$$F(z) := \tilde{\mathcal{C}}f(z) = 2\mathcal{C}f(z),$$

$$F_r(z) = \frac{F(rz)}{r} = z + \sum_{n=2}^{\infty} A_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} B_n r^{n-1} z^n}, \quad (11)$$

with A_n, B_n as in (8). In view of Remark 2.1, the identity (9) and Lemma 2.2, in order to show that the normalized Cesàro operator $\tilde{\mathcal{C}}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in $\mathbb{D}_{\rho}$ it suffices to verify that

$$S(r) := \sum_{n=2}^{\infty} \Psi_n(|A_n| + |B_n|) r^{n-1} \leq 1 \quad \text{for } r = \rho, \quad (12)$$

because the left-hand side of (12) is nondecreasing in r , so that (12) then holds for every $0 < r \leq \rho$ as well.

All numerical values of the radii reported in Tables 1, 2, 3 and 4 below were computed by applying the bisection method to the equations $F_i(r) = 0$ (for r_4^* , to the series equation of Corollary 3.8), and were verified independently by direct partial summation of the corresponding series; the values are stated rounded to four decimal places.

3.1 The coefficient condition $|a_n|, |b_n| \leq n$

Theorem 3.1. *Let $f = h + \bar{g} \in \mathcal{H}^0$ be of the form (1) and suppose that $|a_n| \leq n$ and $|b_n| \leq n$ for all $n \geq 2$. Then the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the disk $\mathbb{D}_{r_1}$, where $r_1 = r_1(\lambda, \delta)$ is the unique root in $(0, 1)$ of the equation $F_1(r) = 0$ with*

$$F_1(r) = M(1-r)^5 \log(1-r) - r^7 + 5r^6 - (10 + \lambda - 5\delta)r^5 \\ + (7 + 13\lambda - 55\delta)r^4 + (2 - 19\lambda + 47\delta)r^3 \\ - (4 - 9\lambda + 27\delta)r^2 + Mr, \quad (13)$$

and $M = 1 - 2\lambda + 6\delta$. Moreover, $F_1(r) > 0$ for $0 < r < r_1$ and $F_1(r) < 0$ for $r_1 < r < 1$.

Proof. By (8) and the hypothesis (recall $a_1 = 1$),

$$|A_n| \leq \frac{2}{n+1} \sum_{k=1}^n k = n, \quad |B_n| \leq \frac{2}{n+1} \sum_{k=2}^n k = n - \frac{2}{n+1},$$

so that $S(r) \leq \sum_{n=2}^{\infty} \Psi_n(2n - \frac{2}{n+1})r^{n-1}$, and by (12) it suffices to prove that

$$\sum_{n=2}^{\infty} \Psi_n\left(2n - \frac{2}{n+1}\right)r^{n-1} \leq 1. \quad (14)$$

Using (4) and (6),

$$\begin{aligned} \Psi_n\left(2n - \frac{2}{n+1}\right) &= 2[\delta n^4 + (\lambda - 3\delta)n^3 + (1 - \lambda + 2\delta)n^2] \\ &\quad - 2\left[\delta n^2 + (\lambda - 4\delta)n + M - \frac{M}{n+1}\right], \end{aligned}$$

so that, summing term by term by means of Lemma 2.5, the condition (14) takes the explicit form

$$\begin{aligned} &2\delta\left[\frac{1 + 11r + 11r^2 + r^3}{(1-r)^5} - 1\right] \\ &\quad + 2(\lambda - 3\delta)\left[\frac{1 + 4r + r^2}{(1-r)^4} - 1\right] \\ &\quad + 2(1 - \lambda + \delta)\left[\frac{1 + r}{(1-r)^3} - 1\right] \\ &\quad - 2(\lambda - 4\delta)\left[\frac{1}{(1-r)^2} - 1\right] - \frac{2Mr}{1-r} \\ &\quad - 2M\left[\frac{\log(1-r)}{r^2} + \frac{1}{r} + \frac{1}{2}\right] \leq 1. \end{aligned}$$

Multiplying by $\frac{r^2(1-r)^5}{2} > 0$ and rearranging, (14) becomes the equivalent inequality $F_1(r) \geq 0$, with F_1 as in (13); indeed

$$F_1(r) = \frac{1}{2}r^2(1-r)^5\left(1 - \sum_{n=2}^{\infty} \Psi_n\left(2n - \frac{2}{n+1}\right)r^{n-1}\right).$$

The series here has positive coefficients, hence increases strictly from 0 to $+\infty$ on $(0, 1)$ and equals 1 at a single point $r_1 \in (0, 1)$; consequently $F_1(r) > 0$ for $0 < r < r_1$, $F_1(r_1) = 0$, and $F_1(r) < 0$ for $r_1 < r < 1$. The claim now follows from (12). $\square$

As recalled in Section 1, the coefficients of a stable univalent harmonic mapping satisfy $|a_n| \leq n$ and $|b_n| \leq n$; in fact, de Branges' theorem applied to the univalent functions $h + \varepsilon g$, $|\varepsilon| = 1$ [15], gives $|a_n| + |b_n| \leq n$, whence $|A_n| + |B_n| \leq \frac{2}{n+1} \sum_{k=1}^n k = n$ by (8). Since this is the estimate used in the proof of Theorem 3.5 below, we obtain the following consequence.

Corollary 3.2. *If $f = h + \bar{g} \in \mathcal{H}^0$ is a stable univalent harmonic mapping in the sense of Hernández and Martín [15], then the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the disk $\mathbb{D}_{r_3}$, where $r_3 = r_3(\lambda, \delta) > r_1(\lambda, \delta)$ is as in Theorem 3.5. In particular, the same conclusion holds for every analytic univalent function $f \in \mathcal{S}$ (the case $g \equiv 0$), by de Branges' theorem.*

The graphs of F_1 for several admissible pairs (λ, δ) are shown in Figure 1, and the corresponding values of r_1 are listed in Table 1.

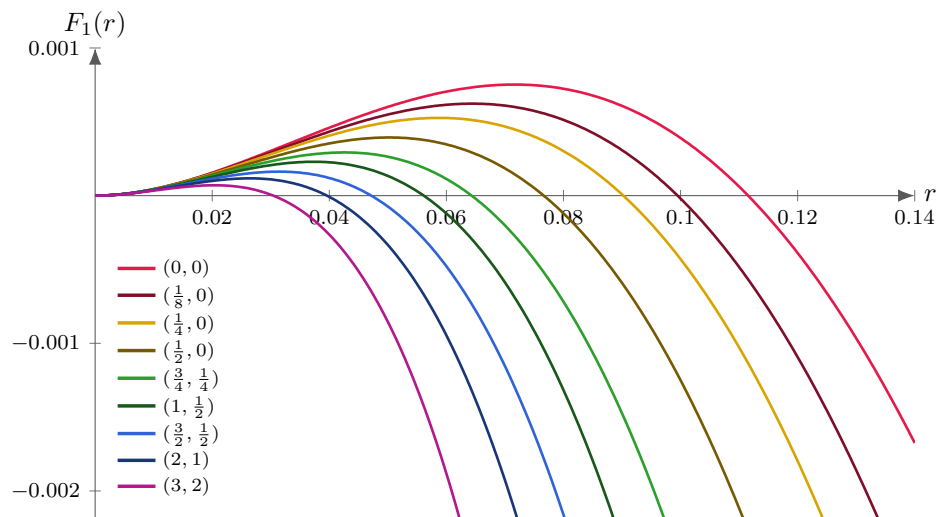


Figure 1: The graphs of $F_1(r)$ for several admissible pairs (λ, δ) ; each curve meets the level $F_1 = 0$ at the corresponding radius $r_1(\lambda, \delta)$.

Table 1: The values of $r_1(\lambda, \delta)$ (rounded to four decimal places).

(λ, δ)	$(0, 0)$	$(\frac{1}{8}, 0)$	$(\frac{1}{4}, 0)$	$(\frac{1}{2}, 0)$	$(\frac{3}{4}, \frac{1}{4})$	$(1, \frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2})$	$(2, 1)$	$(3, 2)$
r_1	0.1115	0.0995	0.0902	0.0766	0.0643	0.0561	0.0472	0.0396	0.0304

3.2 The coefficient condition $|a_n|, |b_n| \leq 1$

Theorem 3.3. Let $f = h + \bar{g} \in \mathcal{H}^0$ be of the form (1) and suppose that $|a_n| \leq 1$ and $|b_n| \leq 1$ for all $n \geq 2$. Then the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the disk $\mathbb{D}_{r_2}$, where $r_2 = r_2(\lambda, \delta)$ is the unique root in $(0, 1)$ of the equation $F_2(r) = 0$ with

$$F_2(r) = 6M(1-r)^4 \log(1-r) + 2r^6 - 8r^5 + 2(1 + 13\lambda - 45\delta)r^4 + 2(9 - 28\lambda + 78\delta)r^3 - 2(10 - 21\lambda + 63\delta)r^2 + 6Mr. \quad (15)$$

Moreover, $F_2(r) > 0$ for $0 < r < r_2$ and $F_2(r) < 0$ for $r_2 < r < 1$.

Proof. The hypothesis gives

$$|A_n| \leq \frac{2}{n+1} \sum_{k=1}^n 1 = \frac{2n}{n+1}, \quad |B_n| \leq \frac{2}{n+1} \sum_{k=2}^n 1 = \frac{2(n-1)}{n+1},$$

hence $|A_n| + |B_n| \leq \frac{2(2n-1)}{n+1} = 4 - \frac{6}{n+1}$, and by (12) it suffices to establish

$$\sum_{n=2}^{\infty} \Psi_n \left(4 - \frac{6}{n+1} \right) r^{n-1} \leq 1. \quad (16)$$

By (4) and (6),

$$\Psi_n\left(4 - \frac{6}{n+1}\right) = 4\delta n^3 + (4\lambda - 18\delta)n^2 + (4 - 10\lambda + 32\delta)n - 6M + \frac{6M}{n+1},$$

so that, by Lemma 2.5, the condition (16) takes the explicit form

$$\begin{aligned} & 4\delta\left[\frac{1+4r+r^2}{(1-r)^4} - 1\right] + (4\lambda - 18\delta)\left[\frac{1+r}{(1-r)^3} - 1\right] \\ & + (4 - 10\lambda + 32\delta)\left[\frac{1}{(1-r)^2} - 1\right] - \frac{6Mr}{1-r} \\ & - 6M\left[\frac{\log(1-r)}{r^2} + \frac{1}{r} + \frac{1}{2}\right] \leq 1. \end{aligned}$$

Multiplying by $r^2(1-r)^4 > 0$ and simplifying, (16) becomes the equivalent inequality $F_2(r) \geq 0$, with F_2 as in (15); indeed

$$F_2(r) = r^2(1-r)^4\left(1 - \sum_{n=2}^{\infty} \Psi_n\left(4 - \frac{6}{n+1}\right)r^{n-1}\right).$$

The series has positive coefficients, hence increases strictly from 0 to $+\infty$ on $(0, 1)$ and equals 1 at a single point $r_2 \in (0, 1)$; therefore $F_2(r) > 0$ for $0 < r < r_2$, $F_2(r_2) = 0$, and $F_2(r) < 0$ for $r_2 < r < 1$, exactly as in Theorem 3.1. $\square$

Since every analytic convex univalent function has Taylor coefficients bounded by 1, Theorem 3.3 applies in particular to the class $\mathcal{K}$; since moreover $b_n = 0$ for $f \in \mathcal{K}$, a larger radius is obtained.

Corollary 3.4. *If $f \in \mathcal{K}$ is an analytic convex univalent function (the case $g \equiv 0$), then the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the disk $\mathbb{D}_{r_2^*}$, where $r_2^* = r_2^*(\lambda, \delta) > r_2(\lambda, \delta)$ is the unique root in $(0, 1)$ of the equation $\sum_{n=2}^{\infty} \Psi_n \frac{2n}{n+1} r^{n-1} = 1$.*

Proof. If $f \in \mathcal{K}$, then $|a_n| \leq 1$ for all $n \geq 2$, a classical bound for analytic convex univalent functions, while $b_n = 0$; hence $B_n = 0$ and $|A_n| \leq \frac{2n}{n+1}$, and the claim follows from (12) as in the proof of Theorem 3.3. Since $\frac{2n}{n+1} < \frac{2(2n-1)}{n+1}$ for $n \geq 2$, we have $r_2^* > r_2$. $\square$

The graphs of F_2 for several admissible pairs (λ, δ) are shown in Figure 2, and the corresponding values of r_2 are listed in Table 2.

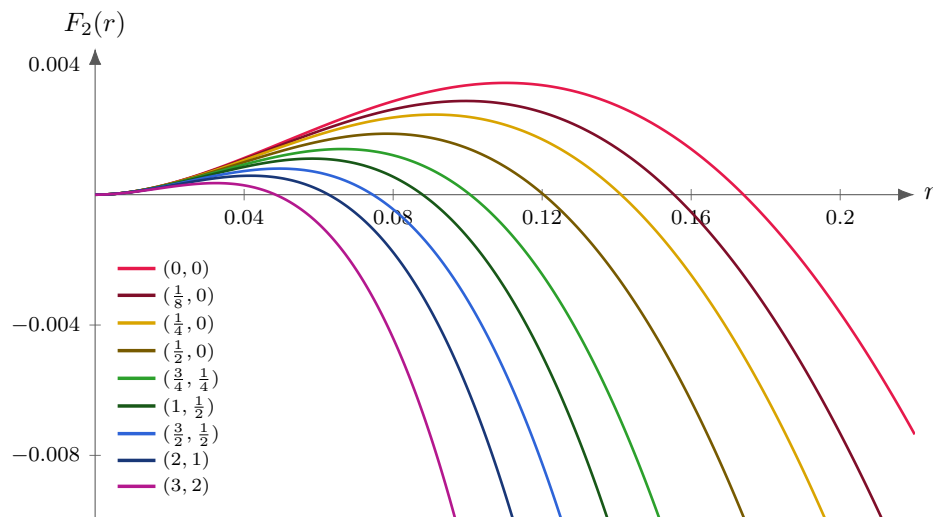


Figure 2: The graphs of $F_2(r)$ for several admissible pairs (λ, δ) ; each curve meets the level $F_2 = 0$ at the corresponding radius $r_2(\lambda, \delta)$.

Table 2: The values of $r_2(\lambda, \delta)$ (rounded to four decimal places).

(λ, δ)	$(0, 0)$	$(\frac{1}{8}, 0)$	$(\frac{1}{4}, 0)$	$(\frac{1}{2}, 0)$	$(\frac{3}{4}, \frac{1}{4})$	$(1, \frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2})$	$(2, 1)$	$(3, 2)$
r_2	0.1740	0.1554	0.1412	0.1204	0.1007	0.0879	0.0745	0.0627	0.0484

3.3 The Clunie–Sheil–Small bounds

Theorem 3.5. Let $f = h + \bar{g} \in \mathcal{H}^0$ be of the form (1) and suppose that

$$|a_n| \leq \frac{n+1}{2} \quad (n \geq 2), \quad |b_n| \leq \frac{n-1}{2} \quad (n \geq 2).$$

Then the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the disk $\mathbb{D}_{r_3}$, where $r_3 = r_3(\lambda, \delta)$ is the unique root in $(0, 1)$ of the equation $F_3(r) = 0$ with

$$F_3(r) = 1 - (9 + 4\lambda)r + (21 + 2\lambda - 18\delta)r^2 - (21 - 2\lambda + 6\delta)r^3 + 10r^4 - 2r^5. \quad (17)$$

Moreover, $F_3(r) > 0$ for $0 \leq r < r_3$ and $F_3(r) < 0$ for $r_3 < r < 1$.

Proof. Since $a_1 = 1 = \frac{1+1}{2}$, the hypothesis yields

$$|A_n| \leq \frac{2}{n+1} \sum_{k=1}^n \frac{k+1}{2} = \frac{n(n+3)}{2(n+1)}, \quad |B_n| \leq \frac{2}{n+1} \sum_{k=2}^n \frac{k-1}{2} = \frac{n(n-1)}{2(n+1)},$$

and therefore the remarkably simple estimate

$$|A_n| + |B_n| \leq \frac{n(n+3) + n(n-1)}{2(n+1)} = n.$$

Thus, by (12), it suffices to prove that

$$\sum_{n=2}^{\infty} n \Psi_n r^{n-1} \leq 1. \quad (18)$$

By (4), $n\Psi_n = \delta n^4 + (\lambda - 3\delta)n^3 + (1 - \lambda + 2\delta)n^2$; substituting the closed forms of Lemma 2.5, multiplying by $(1-r)^5 > 0$ and using $\delta + (\lambda - 3\delta) + (1 - \lambda + 2\delta) = 1$, the inequality (18) becomes the equivalent inequality $F_3(r) \geq 0$, with F_3 as in (17); indeed $F_3(r) = (1-r)^5(1 - \sum_{n=2}^{\infty} n\Psi_n r^{n-1})$. The series has positive coefficients, hence increases strictly from 0 to $+\infty$ on $(0, 1)$ and equals 1 at a single point $r_3 \in (0, 1)$; thus $F_3(r) > 0$ for $0 \leq r < r_3$, $F_3(r_3) = 0$, and $F_3(r) < 0$ for $r_3 < r < 1$, as before. $\square$

Lemma 2.4 immediately yields the following consequence.

Corollary 3.6. *If $f \in \mathcal{K}_{\mathcal{H}}^0$, then the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the disk $\mathbb{D}_{r_3}$, where $r_3 = r_3(\lambda, \delta)$ is as in Theorem 3.5.*

The graphs of F_3 for several admissible pairs (λ, δ) are shown in Figure 3, and the corresponding values of r_3 are listed in Table 3.

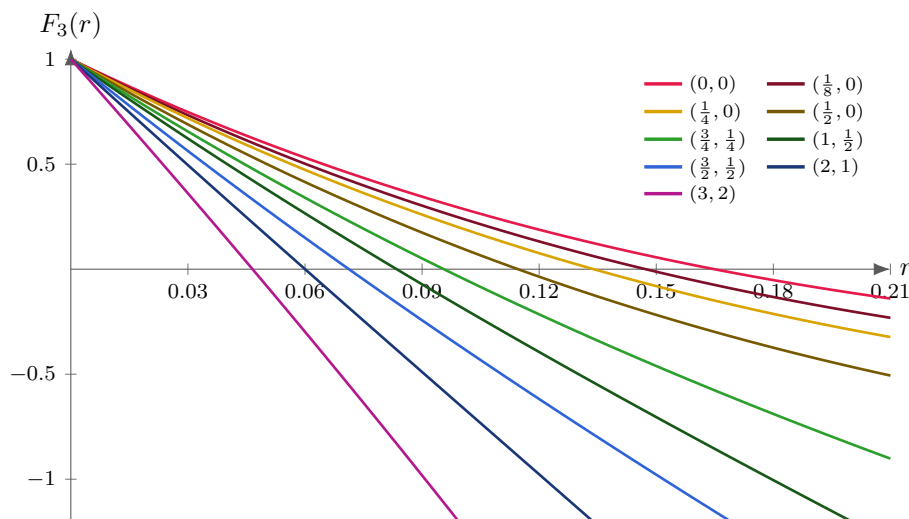


Figure 3: The graphs of $F_3(r)$ for several admissible pairs (λ, δ) ; each curve meets the level $F_3 = 0$ at the corresponding radius $r_3(\lambda, \delta)$.

Table 3: The values of $r_3(\lambda, \delta)$ (rounded to four decimal places).

(λ, δ)	(0, 0)	$(\frac{1}{8}, 0)$	$(\frac{1}{4}, 0)$	$(\frac{1}{2}, 0)$	$(\frac{3}{4}, \frac{1}{4})$	$(1, \frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2})$	(2, 1)	(3, 2)
r_3	0.1649	0.1473	0.1340	0.1146	0.0956	0.0835	0.0713	0.0600	0.0466

3.4 The Cesàro operator on $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ itself

We finally treat the coefficient condition satisfied by the members of the classes $\mathcal{R}_{\mathcal{H}}^0(\lambda', \delta')$ themselves (Lemma 2.3); the resulting radius involves the harmonic numbers Q_n through Lemma 2.5.

Theorem 3.7. *Let $f = h + \bar{g} \in \mathcal{H}^0$ be of the form (1) and suppose that*

$$|a_n| + |b_n| \leq \frac{2}{n} \quad (n \geq 2).$$

Then the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the disk $\mathbb{D}_{r_4}$, where $r_4 = r_4(\lambda, \delta)$ is the unique root in $(0, 1)$ of the equation $F_4(r) = 0$ with

$$\begin{aligned} F_4(r) = & M(1-r)^3 \log^2(1-r) \\ & + [(1-4\lambda+16\delta)r^3 - (1-4\lambda+12\delta)r^2 - Mr + M] \log(1-r) \\ & - r^5 + 3r^4 - (2+\lambda+\delta)r^3 - (1-3\lambda+9\delta)r^2 + Mr, \end{aligned} \quad (19)$$

and $M = 1 - 2\lambda + 6\delta$. Moreover, $F_4(r) > 0$ for $0 < r < r_4$ and $F_4(r) < 0$ for $r_4 < r < 1$.

Proof. By (8) and the hypothesis,

$$|A_n| + |B_n| \leq \frac{2}{n+1} \left[1 + \sum_{k=2}^n (|a_k| + |b_k|) \right] \leq \frac{2}{n+1} \left[1 + \sum_{k=2}^n \frac{2}{k} \right] = \frac{2(2Q_n - 1)}{n+1},$$

so that, by (12), it suffices to show that

$$\sum_{n=2}^{\infty} \frac{2\Psi_n(2Q_n - 1)}{n+1} r^{n-1} \leq 1. \quad (20)$$

Writing $\frac{2\Psi_n(2Q_n-1)}{n+1} = \frac{4\Psi_n}{n+1}Q_n - \frac{2\Psi_n}{n+1}$ and applying the division identity (6) to $\frac{\Psi_n}{n+1}$ in both terms expresses the summand as a linear combination of $n^2Q_n, nQ_n, Q_n, \frac{Q_n}{n+1}$ and $n^2, n, 1, \frac{1}{n+1}$; a direct computation with the corresponding closed forms of Lemma 2.5 shows that, after multiplication by $\frac{r^2(1-r)^3}{2} > 0$ and rearrangement, the condition (20) is equivalent to $F_4(r) \geq 0$, with F_4 as in (19); indeed

$$F_4(r) = \frac{r^2(1-r)^3}{2} \left(1 - \sum_{n=2}^{\infty} \frac{2\Psi_n(2Q_n - 1)}{n+1} r^{n-1} \right).$$

The series has positive coefficients, hence increases strictly from 0 to $+\infty$ on $(0, 1)$ and equals 1 at a single point $r_4 \in (0, 1)$; therefore $F_4(r) > 0$ for $0 < r < r_4$, $F_4(r_4) = 0$, and $F_4(r) < 0$ for $r_4 < r < 1$, as in the previous proofs. $\square$

Combining the proof of Theorem 3.7 with the sharp bounds of Lemma 2.3 and the inequality $\Psi_n(\lambda', \delta') \geq n$, valid for all $\lambda' \geq \delta' \geq 0$, we obtain the announced self-improvement property of the Cesàro operator.

Corollary 3.8. Let $\lambda' \geq \delta' \geq 0$. If $f \in \mathcal{R}_{\mathcal{H}}^0(\lambda', \delta')$, then the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the disk $\mathbb{D}_{r_4^*}$, where $r_4^* = r_4^*(\lambda, \delta; \lambda', \delta')$ is the unique root in $(0, 1)$ of the equation

$$\sum_{n=2}^{\infty} \Psi_n(\lambda, \delta) c_n^* r^{n-1} = 1, \quad c_n^* = \frac{2}{n+1} \left[1 + \sum_{k=2}^n \frac{2}{\Psi_k(\lambda', \delta')} \right],$$

and $r_4^* \geq r_4(\lambda, \delta)$, with equality if and only if $(\lambda', \delta') = (0, 0)$. In particular, the normalized Cesàro operator $f \mapsto 2\mathcal{C}f$ maps the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ into itself in the disk $\mathbb{D}_{r_4^*(\lambda, \delta; \lambda, \delta)}$, in the sense described in Section 2.

Proof. By Lemma 2.3, $|a_k| + |b_k| \leq \frac{2}{\Psi_k(\lambda', \delta')}$ for $k \geq 2$, so that $|A_n| + |B_n| \leq c_n^*$ by (8), and the first claim follows from (12) as in the proof of Theorem 3.7. Since $\Psi_k(\lambda', \delta') \geq k$, with strict inequality for $k \geq 2$ when $(\lambda', \delta') \neq (0, 0)$, we have $c_n^* \leq \frac{2(2Q_n-1)}{n+1}$, the factor in (20), with strict inequality in that case; hence $r_4^* \geq r_4$, with equality if and only if $(\lambda', \delta') = (0, 0)$. $\square$

The graphs of F_4 for several admissible pairs (λ, δ) are shown in Figure 4, and the corresponding values of r_4 are listed in Table 4, together with the values of the self-map radius $r_4^*(\lambda, \delta; \lambda, \delta)$ of Corollary 3.8.

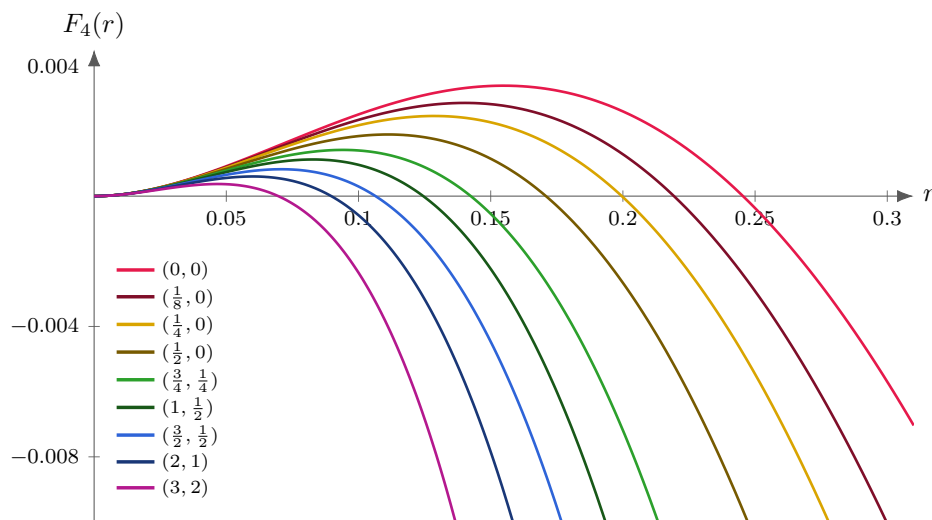


Figure 4: The graphs of $F_4(r)$ for several admissible pairs (λ, δ) ; each curve meets the level $F_4 = 0$ at the corresponding radius $r_4(\lambda, \delta)$.

Table 4: The values of $r_4(\lambda, \delta)$ and $r_4^*(\lambda, \delta; \lambda, \delta)$ (rounded to four decimal places).

(λ, δ)	(0, 0)	$(\frac{1}{8}, 0)$	$(\frac{1}{4}, 0)$	$(\frac{1}{2}, 0)$	$(\frac{3}{4}, \frac{1}{4})$	$(1, \frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2})$	(2, 1)	(3, 2)
r_4	0.2454	0.2193	0.1996	0.1709	0.1425	0.1245	0.1064	0.0897	0.0697
r_4^*	0.2454	0.2307	0.2187	0.1997	0.1756	0.1596	0.1446	0.1271	0.1046

3.5 Comparison of the radii

Proposition 3.9. *For all $\lambda \geq \delta \geq 0$, the radii of Theorems 3.1, 3.3, 3.5 and 3.7 satisfy*

$$r_1(\lambda, \delta) < r_3(\lambda, \delta) < r_2(\lambda, \delta) < r_4(\lambda, \delta).$$

Proof. For $i = 1, 2, 3, 4$ and $r \in (0, 1)$, the sufficient condition established in the proof of the corresponding theorem has the form $\sum_{n=2}^{\infty} \Psi_n c_n^{(i)} r^{n-1} \leq 1$, where

$$c_n^{(1)} = 2n - \frac{2}{n+1}, \quad c_n^{(2)} = \frac{2(2n-1)}{n+1}, \quad c_n^{(3)} = n, \quad c_n^{(4)} = \frac{2(2Q_n-1)}{n+1}$$

are the coefficient factors in (14)–(20); each of these sums increases strictly from 0 to $+\infty$ on $(0, 1)$, and r_i is the unique point at which it attains the value 1. For every $n \geq 2$,

$$\begin{aligned} c_n^{(1)} - c_n^{(3)} &= n - \frac{2}{n+1} > 0, & c_n^{(3)} - c_n^{(2)} &= \frac{(n-1)(n-2)}{n+1} \geq 0, \\ c_n^{(2)} - c_n^{(4)} &= \frac{4(n-Q_n)}{n+1} > 0, \end{aligned}$$

with strict inequality $c_n^{(3)} > c_n^{(2)}$ for all $n \geq 3$. Hence, at each fixed $r \in (0, 1)$,

$$\sum_{n=2}^{\infty} \Psi_n c_n^{(1)} r^{n-1} > \sum_{n=2}^{\infty} \Psi_n c_n^{(3)} r^{n-1} > \sum_{n=2}^{\infty} \Psi_n c_n^{(2)} r^{n-1} > \sum_{n=2}^{\infty} \Psi_n c_n^{(4)} r^{n-1},$$

and since each of these strictly increasing sums attains the value 1 the later the smaller it is, the corresponding roots satisfy $r_1 < r_3 < r_2 < r_4$. $\square$

Remark 3.10. Since Lemma 2.2 is only a sufficient condition for membership in $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$, the radii $r_1, \dots, r_4$ are lower bounds for the exact radii; they are, however, the largest radii that can be obtained from Lemma 2.2 together with the triangle inequality. Indeed, choosing nonnegative coefficients which saturate the respective hypotheses (namely $a_n = b_n = n$ in Theorem 3.1, $a_n = b_n = 1$ in Theorem 3.3, $a_n = \frac{n+1}{2}$ and $b_n = \frac{n-1}{2}$ in Theorem 3.5, and $a_n = \frac{2}{n}$, $b_n = 0$ in Theorem 3.7), all estimates used in the proofs become equalities, so that $S(r)$ coincides with the left-hand side of the corresponding condition (14)–(20) and hence exceeds 1 for every $r > r_i$; for such coefficients the condition (10) consequently fails for F_r whenever $r > r_i$, which does not mean that $F_r \notin \mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$.

To illustrate the four radii geometrically, Figure 5 shows the images of the disks $\mathbb{D}_{r_i}$, at the parameter values $(\lambda, \delta) = (0, 0)$, under the normalized Cesàro operators $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ of the four functions of Remark 3.10. In each case the image is a close-to-convex Jordan domain, in agreement with Theorems 3.1, 3.3, 3.5 and 3.7; the analytic function of Theorem 3.7 (for which $g \equiv 0$) produces the most nearly circular image.

Remark 3.11. (i) Since $\Psi_n(\lambda, \delta)$ is nondecreasing in each of the parameters λ and δ , every sum in (14)–(20) increases and hence every radius $r_i(\lambda, \delta)$ decreases as λ or δ grows; this is apparent in Tables 1–4. The largest radii occur for $(\lambda, \delta) = (0, 0)$, i.e. for the class of [3], in which case, for instance, $r_4(0, 0) \approx 0.2454$.

(ii) For $\delta = 0$ the results of this section yield sufficient radii for the second-order class of Ghosh and Vasudevarao [5], and for $\lambda = \delta = 0$ for the class of Ponnusamy, Yamamoto and Yanagihara [3]; these special cases are also new.

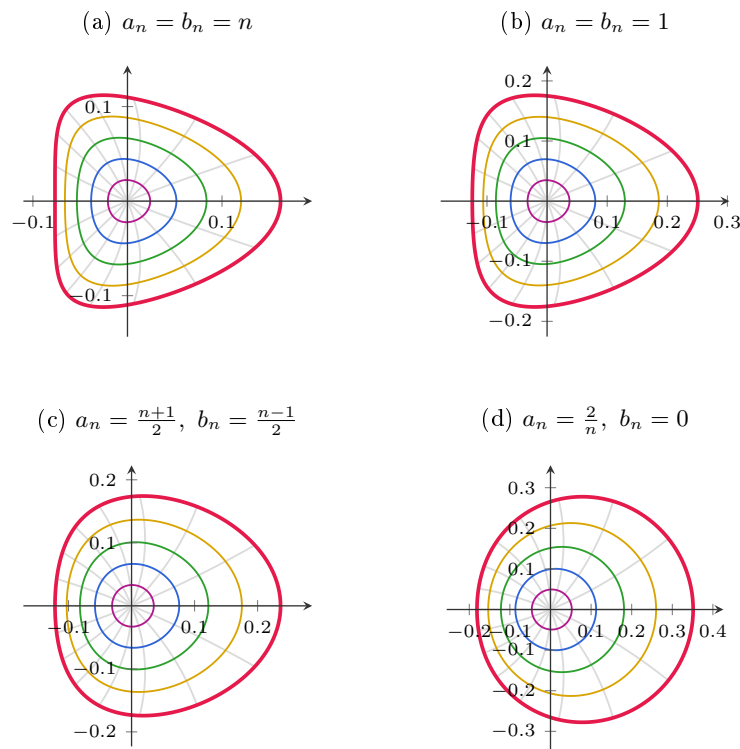


Figure 5: Images of the disks $\mathbb{D}_{r_i}$, for $(\lambda, \delta) = (0, 0)$, under the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ of the four functions that saturate the coefficient conditions (i)–(iv): (a) $a_n = b_n = n$, $r_1 = 0.1115$; (b) $a_n = b_n = 1$, $r_2 = 0.1740$; (c) $a_n = \frac{n+1}{2}$, $b_n = \frac{n-1}{2}$, $r_3 = 0.1649$; (d) $a_n = \frac{2}{n}$, $b_n = 0$, $r_4 = 0.2454$. The concentric curves are the images of the circles $|z| = \frac{m}{5} r_i$ ($m = 1, \dots, 5$), the outermost (bold) being the image of the boundary $\partial\mathbb{D}_{r_i}$, and the radial curves are the images of equally spaced diameters.

4 Conclusion

In this paper, the action of the classical Cesàro operator on harmonic mappings has been investigated in connection with the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ defined by the third-order differential inequality (2). For four natural families of coefficient conditions (including the sharp bounds of the convex harmonic class $\mathcal{K}_{\mathcal{H}}^0$ and a coefficient bound of $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ itself), we determined sufficient radii $r_1 < r_3 < r_2 < r_4$ such that the normalized Cesàro operator $\tilde{\mathcal{C}}f = 2\mathcal{C}f$ belongs to $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ in the corresponding subdisk, each radius being the unique root in $(0, 1)$ of an explicit polynomial or logarithmic-polynomial equation. In particular, the normalized Cesàro operator $f \mapsto 2\mathcal{C}f$ maps $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$ into itself in the disk $\mathbb{D}_{r_4^*(\lambda, \delta; \lambda, \delta)}$, where

$r_4^*(\lambda, \delta; \lambda, \delta) \geq r_4(\lambda, \delta)$; in each of the disks obtained, the operator $\tilde{C}f$ is univalent and close-to-convex, since so are all members of $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$.

Since they are derived from the sufficient coefficient condition of Lemma 2.2 combined with the triangle inequality, the radii obtained here are lower bounds for the exact radii (Remark 3.10). A natural direction for further research is to replace the Cesàro operator by its generalizations of order β , or by other integral means, and to ask for the analogous radii for the class $\mathcal{R}_{\mathcal{H}}^0(\lambda, \delta)$.

Declarations

Conflict of Interest Disclosure: The authors declare that they have no conflict of interest.

Acknowledgements

The authors would like to thank the editor and the anonymous reviewers.

References

- [1] Clunie, J., & Sheil-Small, T. (1984). Harmonic univalent functions. *Annales Academiae Scientiarum Fennicae Series A I Mathematica*, 9, 3–25. <https://doi.org/10.5186/aasfm.1984.0905>
- [2] Duren, P. (2004). *Harmonic mappings in the plane* (Cambridge Tracts in Mathematics, Vol. 156). Cambridge University Press. <https://doi.org/10.1017/CB09780511546600>
- [3] Ponnusamy, S., Yamamoto, H., & Yanagihara, H. (2013). Variability regions for certain families of harmonic univalent mappings. *Complex Variables and Elliptic Equations*, 58(1), 23–34. <https://doi.org/10.1080/17476933.2010.551200>
- [4] Li, L., & Ponnusamy, S. (2013). Disk of convexity of sections of univalent harmonic functions. *Journal of Mathematical Analysis and Applications*, 408, 589–596. <https://doi.org/10.1016/j.jmaa.2013.06.021>
- [5] Ghosh, N., & Vasudevarao, A. (2019). On a subclass of harmonic close-to-convex mappings. *Monatshefte für Mathematik*, 188, 247–267. <https://doi.org/10.1007/s00605-017-1138-7>
- [6] Ali, R. M., Lee, S. K., Subramanian, K. G., & Swaminathan, A. (2011). A third-order differential equation and starlikeness of a double integral operator. *Abstract and Applied Analysis*, 2011, Article 901235. <https://doi.org/10.1155/2011/901235>
- [7] Al-Refai, O. (2019). Some properties for a class of analytic functions defined by a higher-order differential inequality. *Turkish Journal of Mathematics*, 43, 2473–2493. <https://doi.org/10.3906/mat-1906-65>

- [8] Yaşar, E., & Yalçın, S. (2021). Close-to-convexity of a class of harmonic mappings defined by a third-order differential inequality. *Turkish Journal of Mathematics*, 45(2), 678–694. <https://doi.org/10.3906/mat-2004-50>
- [9] Siskakis, A. G. (1987). Composition semigroups and the Cesàro operator on H^p . *Journal of the London Mathematical Society* (2), 36, 153–164. <https://doi.org/10.1112/jlms/s2-36.1.153>
- [10] Stempak, K. (1994). Cesàro averaging operators. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 124, 121–126. <https://doi.org/10.1017/S030821050002922X>
- [11] Hu, Z. Y., Srivastava, H. M., & Wang, X. Y. (2025). Radius constants for Cesàro operator of analytic and harmonic functions in the unit disk. *Russian Journal of Mathematical Physics*, 32(3), 485–497. <https://doi.org/10.1134/S1061920824601484>
- [12] Jahangiri, J. M. (1999). Harmonic functions starlike in the unit disk. *Journal of Mathematical Analysis and Applications*, 235, 470–477. <https://doi.org/10.1006/jmaa.1999.6377>
- [13] Kılıç, U. C., Yalçın, S., & Bayram, H. (2026). Harmonik fonksiyonların Cesàro operatörü için yarıçap sabitleri [Radius constants for the Cesàro operator of harmonic functions]. In *Proceeding book of the 5th International Conference on Pioneer and Innovative Studies (ICPIS 2026)* (pp. 356–363). All Sciences Academy. ISBN 978-625-8993-66-0.
- [14] Rosy, T., Stephen, B. A., Subramanian, K. G., & Jahangiri, J. M. (2001). Goodman–Rønning-type harmonic univalent functions. *Kyungpook Mathematical Journal*, 41(1), 45–54.
- [15] Hernández, R., & Martín, M. J. (2013). Stable geometric properties of analytic and harmonic functions. *Mathematical Proceedings of the Cambridge Philosophical Society*, 155, 343–359. <https://doi.org/10.1017/S0305004113000340>
- [16] Reade, M. O. (1955). On close-to-convex univalent functions. *Michigan Mathematical Journal*, 3, 59–62. <https://doi.org/10.1307/mmj/1031710535>
- [17] MacGregor, T. H. (1962). Functions whose derivative has a positive real part. *Transactions of the American Mathematical Society*, 104, 532–537. <https://doi.org/10.1090/S0002-9947-1962-0140674-7>

This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted, use, distribution and reproduction in any medium, or format for any purpose, even commercially provided the work is properly cited.
