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A Note on a Lebesgue–Ramanujan–Nagell Type Diophantine Equation

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Abstract

We determine all solutions of

$$x^2 + 3^a 7^b 37^c = \lambda y^n,$$

where $\lambda \in \{1, 2, 4\}$, $x, y \geq 1$, $a, b, c \geq 0$, $n \geq 3$, and $\gcd(x, y) = 1$, subject to the following parity condition: when $\lambda \in \{1, 2\}$ and neither 3 nor 4 divides n, the integer y is assumed to be odd. The cases divisible by 3 or 4 are reduced to the determination of S-integral points on elliptic and quartic curves, with $S = \{3, 7, 37\}$. For the remaining exponents, a reduction to odd prime exponents is combined with the primitive divisor theorem for Lehmer sequences and the corrected classification of defective Lehmer pairs. All computationally obtained solutions are then checked directly in the original equation.

1 Introduction

Lebesgue proved in 1850 that

$$x^m - y^2 = 1, \qquad m \geq 2,$$

has no nontrivial integer solutions [1]. Ramanujan later conjectured that only finitely many Mersenne numbers are triangular. Equivalently, he asked for the solutions of

$$x^2 + 7 = 2^n.$$

Nagell [2] proved that the only solutions are

$$(n, x) = (3, \pm 1), (4, \pm 3), (5, \pm 5), (7, \pm 11), (15, \pm 181).$$

The equation is now known as the Ramanujan–Nagell equation.

A broad class of generalizations is given by

$$x^2 + d^m = \lambda y^n, \qquad m \geq 0, \quad n \geq 3, \quad x, y \geq 1, \quad \lambda \in \{1, 2, 4\}. \tag{1}$$

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Many complete or partial results are known for $\lambda = 1$; see, for example, [3–13]. The cases $\lambda = 2$ and $\lambda = 4$ have also been studied extensively; see [14,15] and [16–18], respectively. Related families are treated in [19–25].

A further extension replaces the single prime power by a product of powers of distinct primes:

$$x^2 + p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell} = \lambda y^n, \qquad x, y \geq 1, \quad \gcd(x, y) = 1, \quad \alpha_i \geq 0, \quad n \geq 3. \tag{2}$$

For related results with $\lambda = 1$ and $\lambda = 2$, see [26–30] and [31, 32]. In the present paper we consider the specific equation

$$x^2 + 3^a 7^b 37^c = \lambda y^n, \qquad x, y \geq 1, \quad a, b, c \geq 0, \quad n \geq 3, \quad \gcd(x, y) = 1, \tag{3}$$

for $\lambda \in \{1, 2, 4\}$.

The restriction imposed below is needed only in the branch in which the exponent is not divisible by 3 or 4. In that branch, the proof uses a Lehmer pair whose two rational parameters must be coprime; for $\lambda \in \{1, 2\}$ this is ensured by the oddness of y.

Theorem 1.1. *Assume that, whenever $\lambda \in \{1, 2\}$ and $3 \nmid n$, $4 \nmid n$, the integer y is odd. Then all solutions of (3) are the following.*

1. *For $n = 3$, the nontrivial solutions are listed in Tables 1 and 2.*
2. *For $n = 4$, the nontrivial solutions are listed in Table 3.*
3. *For $n = 5$,*
$$(x, y, a, b, c, \lambda) \in \{(11, 2, 0, 1, 0, 4), (79, 5, 2, 0, 0, 2)\}.$$
4. *For $n = 6$,*
$$(x, y, a, b, c, \lambda) \in \{(1, 2, 2, 1, 0, 1), (118, 5, 5, 1, 0, 1), (4670, 17, 5, 1, 2, 1), (11, 2, 0, 1, 0, 2), (683, 7, 1, 0, 2, 4), (187, 10, 7, 2, 1, 4)\}.$$
5. *For $n = 7$,*
$$(x, y, a, b, c, \lambda) = (13, 2, 0, 3, 0, 4).$$
6. *For $n = 8$,*
$$(x, y, a, b, c, \lambda) \in \{(13, 2, 0, 3, 0, 2), (31, 2, 2, 1, 0, 4), (5, 2, 3, 0, 1, 4)\}.$$
7. *For $n = 9$,*
$$(x, y, a, b, c, \lambda) \in \{(13, 2, 0, 3, 0, 1), (31, 2, 2, 1, 0, 2), (5, 2, 3, 0, 1, 2)\}.$$
8. *For $n = 13$,*
$$(x, y, a, b, c, \lambda) = (181, 2, 0, 1, 0, 4).$$

9. *For* $n = 15$,

$$(x, y, a, b, c, \lambda) = (181, 2, 0, 1, 0, 1).$$

10. *For* $y = 1$ *and every* $n \geq 3$, *the two solutions are*

$$(x, y, a, b, c, \lambda) = (1, 1, 0, 0, 0, 2), \qquad (1, 1, 1, 0, 0, 4).$$

There are no other solutions under the stated hypothesis.

The proof has three parts. The exponents divisible by 3 and 4 are handled through complete computations of S-integral points on finitely many elliptic and quartic curves. If neither 3 nor 4 divides n, an odd prime divisor $p \geq 5$ of n is selected and the problem is reduced to exponent p. The primitive divisor theorem then restricts p to 5, 7, or 13, and the remaining defective Lehmer pairs are checked explicitly. We use MAGMA (V2.29-7) [35] for all computational aspects, in particular to determine the S-integral points on the resulting elliptic and quartic curves.

2 Preliminaries

The following consequence of [36, Corollary 3.1] will be used.

Theorem 2.1. *Let* d *be a positive square-free integer and let* $h(-d)$ *denote the class number of* $\mathbb{Q}(\sqrt{-d})$. *Suppose that* $n \geq 3$ *is odd and* $\gcd(n, h(-d)) = 1$. *If*

$$X^2 + dY^2 = \lambda Z^n, \qquad \lambda \in \{1, 2, 4\}, \qquad \gcd(X, dY) = 1,$$

then

$$\frac{X + Y\sqrt{-d}}{\sqrt{\lambda}} = \varepsilon_1 \left(\frac{u + \varepsilon_2 v\sqrt{-d}}{\sqrt{\lambda}} \right)^n$$

for suitable units $\varepsilon_1, \varepsilon_2$ *and positive integers* u, v *satisfying*

$$u^2 + dv^2 = \lambda Z, \qquad \gcd(u, dv) = 1.$$

The possible values of ε_1 *are* $\{\pm 1, \pm i\}$ *for* $d = 1$, $\{\pm 1, \pm\omega, \pm\omega^2\}$ *for* $d = 3$, *and* $\{\pm 1\}$ *otherwise, where* $\omega = (1 + \sqrt{-3})/2$.

Let α, β be algebraic integers such that $(\alpha + \beta)^2$ and $\alpha\beta$ are nonzero coprime rational integers and α/β is not a root of unity. Then (α, β) is a *Lehmer pair*. Its Lehmer sequence is

$$\mathcal{L}_n(\alpha, \beta) = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta}, & n \text{ odd}, \\ \dfrac{\alpha^n - \beta^n}{\alpha^2 - \beta^2}, & n \text{ even}. \end{cases}$$

A prime q dividing $\mathcal{L}_n(\alpha,\beta)$ is called a primitive divisor if

$$q \nmid (\alpha^2-\beta^2)^2\mathcal{L}_1(\alpha,\beta)\cdots\mathcal{L}_{n-1}(\alpha,\beta).$$

The parameter of the pair is

$$(A,B)=\big((\alpha+\beta)^2,(\alpha-\beta)^2\big).$$

We use the primitive divisor theorem of Bilu, Hanrot and Voutier [33], together with the corrected small-index classification in [34].

Theorem 2.2. *Let $p \geq 5$ be prime.*

1. *If $p>13$, then $\mathcal{L}_p(\alpha,\beta)$ has a primitive divisor.*
2. *If $p=5$ and $\mathcal{L}_5(\alpha,\beta)$ has no primitive divisor, then, up to equivalence, its parameter is one of*

$$(A,B)=(F_{k-2\varepsilon},F_{k-2\varepsilon}-4F_k), \qquad k\geq 3, \quad \varepsilon\in\{\pm 1\},$$

or

$$(A,B)=(L_{k-2\varepsilon},L_{k-2\varepsilon}-4L_k), \quad k\geq 0, \quad \varepsilon\in\{\pm 1\},$$

where $(k,\varepsilon)\notin\{(0,-1),(1,-1)\}$ in the Lucas family.

3. *If $p=7$ and $\mathcal{L}_7(\alpha,\beta)$ has no primitive divisor, then*

$$(A,B)\in\{(1,-7),(1,-19),(3,-5),(5,-7),(13,-3),(14,-22)\}.$$

4. *If $p=13$ and $\mathcal{L}_{13}(\alpha,\beta)$ has no primitive divisor, then $(A,B)=(1,-7)$.*

The following congruence property is due to Lehmer and Ward [37, 38].

Theorem 2.3. *Let q be a primitive divisor of $\mathcal{L}_n(\alpha,\beta)$. Then $q\equiv \pm 1 \pmod{n}$. If n is an odd prime and*

$$D=(\alpha^2-\beta^2)^2,$$

then

$$q\equiv \left(\frac{D}{q}\right) \pmod{n}.$$

We also use the following classical square classifications [39].

Theorem 2.4. *Let F_k and L_k denote the Fibonacci and Lucas numbers.*

1. *$F_k=t^2$ if and only if $(k,t)\in\{(0,0),(1,\pm1),(2,\pm1),(12,\pm12)\}$.*
2. *$F_k=2t^2$ if and only if $(k,t)\in\{(0,0),(3,\pm1),(6,\pm2)\}$.*

3. $L_k = t^2$ *if and only if* $(k, t) \in \{(1, \pm 1), (3, \pm 2)\}$.
4. $L_k = 2t^2$ *if and only if* $(k, t) \in \{(0, \pm 1), (6, \pm 3)\}$.

Lemma 2.5. *Let* $(x, y, a, b, c, n, \lambda)$ *be a solution of* (3) *with* $y \geq 2$.

1. *If* $4 \mid n$ *and* $Y = y^{n/4}$*, then* $(x, Y, a, b, c, 4, \lambda)$ *is a solution of the same equation with exponent* 4.
2. *If* $3 \mid n$ *and* $Y = y^{n/3}$*, then* $(x, Y, a, b, c, 3, \lambda)$ *is a solution with exponent* 3.
3. *If* $3 \nmid n$ *and* $4 \nmid n$*, then* n *has an odd prime divisor* $p \geq 5$*. For every such* p*, putting* $Y = y^{n/p}$ *gives a solution with prime exponent* p.

Proof. Each assertion follows by writing y^n respectively as $(y^{n/4})^4$, $(y^{n/3})^3$, or $(y^{n/p})^p$. For the last assertion, if n had no odd prime divisor, then it would be a power of 2; since $n \geq 3$ and $4 \nmid n$, this is impossible. □

3 Proof of Theorem 1.1

If $y = 1$, then

$$x^2 + 3^a 7^b 37^c = \lambda.$$

Since $\lambda \in \{1, 2, 4\}$ and $x \geq 1$, direct inspection gives exactly

$$(1, 1, 0, 0, 0, 2), \qquad (1, 1, 1, 0, 0, 4),$$

for every $n \geq 3$. Henceforth assume $y \geq 2$.

3.1 The exponent $n = 3$

Write

$$a = 6a_1 + i, \qquad b = 6b_1 + j, \qquad c = 6c_1 + k, \qquad 0 \leq i, j, k \leq 5.$$

Then (3) is equivalent to

$$M^2 = N^3 - \lambda^2 3^i 7^j 37^k,$$

where

$$M = \frac{\lambda x}{3^{3a_1} 7^{3b_1} 37^{3c_1}}, \qquad N = \frac{\lambda y}{3^{2a_1} 7^{2b_1} 37^{2c_1}}.$$

Thus every solution yields an S-integral point, with $S = \{3, 7, 37\}$, on one of the $3 \cdot 6^3 = 648$ elliptic curves

$$E_{\lambda,i,j,k} : \quad M^2 = N^3 - \lambda^2 3^i 7^j 37^k.$$

Conversely, each computed S-integral point was mapped back to the original variables and retained only when x, y are positive integers, $a, b, c \geq 0$, and $\gcd(x, y) = 1$. The resulting nontrivial solutions are exactly those in Tables 1 and 2. The Magma code used to enumerate the S-integral points is given in the Appendix.

3.2 The exponent $n = 4$

Write
$$a = 4a_1 + i, \qquad b = 4b_1 + j, \qquad c = 4c_1 + k, \qquad i, j, k \in \{0, 1, 2, 3\}.$$
Then
$$M^2 = \lambda N^4 - 3^i 7^j 37^k,$$
where
$$M = \frac{x}{3^{2a_1} 7^{2b_1} 37^{2c_1}}, \qquad N = \frac{y}{3^{a_1} 7^{b_1} 37^{c_1}}.$$
Hence the problem reduces to the S-integral points on the $3 \cdot 4^3 = 192$ quartic curves
$$C_{\lambda,i,j,k}: \quad M^2 = \lambda N^4 - 3^i 7^j 37^k.$$
Using `SIntegralLjunggrenPoints`, followed by the same exact integrality and coprimality filters, gives precisely Table 3.

3.3 Exponents divisible by 4

Suppose $4 \mid n$ and $n > 4$. By Lemma 2.5, the value $Y = y^{n/4}$ occurs as the y-coordinate of a solution in Table 3. Among the nontrivial entries of that table, the only proper powers are the three occurrences of $Y = 4$. Therefore $n/4 = 2$, so $n = 8$ and $y = 2$. This yields
$$(13, 2, 0, 3, 0, 2), \qquad (31, 2, 2, 1, 0, 4), \qquad (5, 2, 3, 0, 1, 4).$$

3.4 Exponents divisible by 3

Suppose $3 \mid n$, $n > 3$, and $4 \nmid n$. Put $m = n/3 \geq 2$. By Lemma 2.5, the value $Y = y^m$ occurs in Tables 1–2. Inspection of those tables gives the following and no other proper-power values:

Y	m	$n = 3m$
$4, 25, 289, 4, 49, 100$	2	6
$8, 8, 8$	3	9
32	5	15.

Mapping the corresponding rows back gives the six solutions for $n = 6$, the three solutions for $n = 9$, and the single solution for $n = 15$ stated in Theorem 1.1.

3.5 Reduction to an odd prime exponent

It remains to consider $3 \nmid n$ and $4 \nmid n$. Choose an odd prime $p \geq 5$ dividing n and replace y temporarily by $y^{n/p}$. It is therefore enough first to solve (3) for the prime exponent $n = p$.

Write

$$a = 2a_1 + i, \qquad b = 2b_1 + j, \qquad c = 2c_1 + k, \qquad i, j, k \in \{0, 1\},$$

and put

$$d = 3^i 7^j 37^k, \qquad z = 3^{a_1} 7^{b_1} 37^{c_1}.$$

Then

$$3^a 7^b 37^c = dz^2.$$

The possible values of d and the corresponding class numbers are

d	1	3	7	21	37	111	259	777
$h(-d)$	1	1	1	4	2	8	4	16.

Thus $\gcd(p, h(-d)) = 1$ for every prime $p \geq 5$.

Moreover, $\gcd(x, dz) = 1$: an odd prime dividing both x and dz would also divide y, contrary to $\gcd(x, y) = 1$. Theorem 2.1 therefore gives

$$\frac{x + z\sqrt{-d}}{\sqrt{\lambda}} = \left(\frac{u + \varepsilon v\sqrt{-d}}{\sqrt{\lambda}}\right)^p, \tag{4}$$

after absorbing the first unit into the p-th power. Here

$$u^2 + dv^2 = \lambda y, \qquad \gcd(u, dv) = 1. \tag{5}$$

Set

$$\alpha = \frac{u + \varepsilon v\sqrt{-d}}{\sqrt{\lambda}}, \qquad \beta = \frac{u - \varepsilon v\sqrt{-d}}{\sqrt{\lambda}}.$$

Then

$$(\alpha + \beta)^2 = \frac{4u^2}{\lambda}, \qquad \alpha\beta = y, \qquad (\alpha - \beta)^2 = -\frac{4dv^2}{\lambda}.$$

For $\lambda = 4$, (5) and $\gcd(u, dv) = 1$ give $\gcd(u^2, y) = 1$. For $\lambda \in \{1, 2\}$, the hypothesis that y is odd gives the same conclusion. Hence

$$\gcd\big((\alpha + \beta)^2, \alpha\beta\big) = 1.$$

We next verify that α/β is not a root of unity. Its minimal quadratic equation is

$$\lambda y T^2 - 2(u^2 - dv^2)T + \lambda y = 0.$$

If it were a root of unity, its trace would lie in $\{-2, -1, 0, 1, 2\}$. The values ± 2 would force $v = 0$ or $u = 0$. Trace 0 gives $d = 1$, $u = v = 1$, and $\lambda y = 2$, excluded by $y \geq 2$ and the parity hypothesis. Trace 1 contradicts $\gcd(u, dv) = 1$, while trace -1 gives $d = 3$, $u = v = 1$, and $\lambda y = 4$, again excluded. Thus (α, β) is a Lehmer pair with parameter

$$(A, B) = \left(\frac{4u^2}{\lambda}, -\frac{4dv^2}{\lambda}\right). \tag{6}$$

Taking imaginary parts in (4) gives

$$|\mathcal{L}_p(\alpha,\beta)| = \frac{z}{v}. \tag{7}$$

In particular, every prime divisor of $\mathcal{L}_p(\alpha,\beta)$ lies in $\{3,7,37\}$.

Suppose that q is a primitive divisor. By Theorem 2.3, $q \equiv \pm 1 \pmod{p}$. The choices $q=3$ and $q=7$ are impossible for $p \geq 5$. For $q=37$, the only possibility is $p=19$. Now

$$D = (\alpha^2-\beta^2)^2 = -\frac{16u^2v^2d}{\lambda^2}.$$

Since $37 \nmid D$, we have $37 \nmid d$, so $d \in \{1,3,7,21\}$. For these four values,

$$\left(\frac{-d}{37}\right) = 1.$$

The second congruence in Theorem 2.3 would therefore give

$$37 \equiv 1 \pmod{19},$$

whereas $37 \equiv -1 \pmod{19}$, a contradiction. Hence $\mathcal{L}_p(\alpha,\beta)$ has no primitive divisor. By Theorem 2.2, it follows that

$$p \in \{5,7,13\}.$$

3.6 The case $p=5$

Put $r = k - 2\varepsilon$. Since

$$A = \frac{4u^2}{\lambda},$$

the number A must be a square when $\lambda = 4$, twice a square when $\lambda = 2$, and four times a square when $\lambda = 1$. Thus Theorem 2.4 reduces the two infinite families in Theorem 2.2 to the finite list below. In the last column we use

$$R = -\frac{\lambda B}{4} = dv^2, \qquad d \in \{1,3,7,21,37,111,259,777\}.$$

family	(k,ε)	λ	(A,B)	R
F	$(3,1)$	4	$(1,-7)$	7
F	$(4,-1)$	2	$(8,-4)$	2
F	$(4,1)$	4	$(1,-11)$	11
F	$(5,1)$	2	$(2,-18)$	9
F	$(8,1)$	2	$(8,-76)$	38
F	$(10,-1)$	1	$(144,-76)$	19
F	$(10,-1)$	4	$(144,-76)$	76
F	$(14,1)$	1	$(144,-1364)$	341
F	$(14,1)$	4	$(144,-1364)$	1364
L	$(2,1)$	2	$(2,-10)$	5
L	$(3,1)$	4	$(1,-15)$	15
L	$(4,-1)$	2	$(18,-10)$	5
L	$(5,1)$	1	$(4,-40)$	10
L	$(5,1)$	4	$(4,-40)$	40
L	$(8,1)$	2	$(18,-170)$	85

Only $R = 7$ and $R = 9$ have the form dv^2 with an allowed value of d and satisfy $\gcd(u, dv) = 1$. Consequently the two surviving rows are

family	(k,ε)	(A,B)	λ	u	v	d	y
F	$(3,1)$	$(1,-7)$	4	1	1	7	2
F	$(5,1)$	$(2,-18)$	2	1	3	1	5.

In particular, no Lucas-family parameter survives the necessary conditions.

For the first row,

$$\left(\frac{1+\sqrt{-7}}{2}\right)^5 = \frac{11-\sqrt{-7}}{2},$$

which yields

$$(x, y, a, b, c, \lambda) = (11, 2, 0, 1, 0, 4).$$

For the second row,

$$\left(\frac{1+3\sqrt{-1}}{\sqrt{2}}\right)^5 = \frac{79-3\sqrt{-1}}{\sqrt{2}},$$

which yields

$$(x, y, a, b, c, \lambda) = (79, 5, 2, 0, 0, 2).$$

3.7 The cases $p = 7$ and $p = 13$

For $p = 7$, checking the six exceptional parameters in Theorem 2.2 against (6) leaves only

$$(A, B) = (1, -7), \qquad \lambda = 4, \quad u = v = 1, \quad d = 7, \quad y = 2.$$

Since

$$\left(\frac{1+\sqrt{-7}}{2}\right)^7 = \frac{-13+7\sqrt{-7}}{2},$$

we obtain

$$(x, y, a, b, c, \lambda) = (13, 2, 0, 3, 0, 4).$$

For $p = 13$, the unique exceptional parameter is again $(1, -7)$, and

$$\left(\frac{1+\sqrt{-7}}{2}\right)^{13} = \frac{-181-\sqrt{-7}}{2}.$$

Thus

$$(x, y, a, b, c, \lambda) = (181, 2, 0, 1, 0, 4).$$

Finally, return to a possibly composite exponent n with $3 \nmid n$ and $4 \nmid n$. If $p \mid n$ and $m = n/p$, then the variable in the prime-exponent equation is $Y = y^m$. The prime-exponent solutions above have $Y \in \{2, 5\}$, neither of which is a proper power. Hence $m = 1$, so $n = p$. This proves that no additional composite exponents arise and completes the proof.

4 Discussion

The parity hypothesis is confined to the branch

$$\lambda \in \{1, 2\}, \qquad 3 \nmid n, \qquad 4 \nmid n.$$

It is not imposed on the cubic or quartic reductions. In the remaining branch, even values of y may cause the two rational parameters of the associated Lehmer pair to have a common factor 2, so the primitive divisor argument above no longer applies directly. Similar exceptional even-base phenomena occur in related Lebesgue–Ramanujan–Nagell equations; compare [29, 40]. Removing this residual parity condition would require a separate 2-adic analysis.

5 Conclusion

Under the stated parity condition, Theorem 1.1 gives a complete classification of the solutions of (3). The cubic and quartic branches are reduced to finite, complete S-integral point computations. For all other exponents, a rigorous reduction to an odd prime exponent, followed by the primitive divisor theorem and the corrected defective Lehmer-pair classification, leaves only $p = 5, 7, 13$. The lifting argument then excludes all further composite exponents. In particular, the solution

$$11^2 + 7 = 4 \cdot 2^5$$

is essential in the $p = 5$ case.

A natural continuation is to remove the residual even-y restriction for $\lambda \in \{1, 2\}$ and to investigate analogous equations in which λ is an odd prime power.

Appendix: computational verification

The following Magma loops enumerate the S-integral points used in the proof.

Cubic curves ($n = 3$)**.**

```
S := [3, 7, 37];

for lam in [1, 2, 4] do
    for i in [0..5] do
        for j in [0..5] do
            for k in [0..5] do
                E := EllipticCurve(
                    [0, -lam^2 * 3^i * 7^j * 37^k]
                );
                pts := SIntegralPoints(E, S);
                for P in pts do
                    printf "lam=%o i=%o j=%o k=%o P=%o\n",
                           lam, i, j, k, P;
                end for;
            end for;
        end for;
    end for;
end for;
```

Quartic curves ($n = 4$)**.**

```
S := [3, 7, 37];

for lam in [1, 2, 4] do
    for i in [0..3] do
        for j in [0..3] do
            for k in [0..3] do
                pts := SIntegralLjunggrenPoints(
                    [1, lam, 0, -3^i * 7^j * 37^k], S
                );
                for P in pts do
                    printf "lam=%o i=%o j=%o k=%o P=%o\n",
                           lam, i, j, k, P;
                end for;
```

```
            end for;
        end for;
    end for;
end for;
```

Every row in Tables 1–3, as well as every isolated solution in Theorem 1.1, was additionally checked by exact integer arithmetic in the original equation (3).

Table convention. The tables contain only the nontrivial solutions $y \geq 2$; the two solutions with $y = 1$ are stated separately in Theorem 1.1. For $n = 4$, the signs in the first coordinate correspond to the two possible signs of the quartic variable N.

Table 1: All nontrivial solutions of (3) for $n = 3$ and $\lambda = 1$

(N,M,i,j,k)	**a**	**b**	**c**	**x**	**y**	λ
(145/9,1432/27, 0, 0, 2)	6	0	2	1432	145	1
(2, 1 , 0, 1, 0)	0	1	0	1	2	1
(32, 181 , 0, 1, 0)	0	1	0	181	32	1
(156853/49,62121114/343, 0, 0, 2)	0	6	2	62121114	156853	1
(57395665/729,434829137572/19683, 0, 0, 2)	18	0	2	434829137572	57395665	1
(65, 524 , 0, 2, 0)	0	2	0	524	65	1
(53, 286 , 0, 2, 2)	0	2	2	286	53	1
(8, 13 , 0, 3, 0)	0	3	0	13	8	1
(37/9,188/27,1,1,0)	7	1	0	188	37	1
(793/81,9260/729,1,1,1)	13	1	1	9260	793	1
(3637/9,139348/27,1,1,4)	7	1	4	139348	3637	1
(52476265/81,380140704098/729,1,1,4)	13	1	4	380140704098	52476265	1
(67/9,440/27,1,2,0)	7	2	0	440	67	1
(211/9,2330/27,1,2,1)	7	2	1	2330	211	1
(1129/81,29870/729,1,3,0)	13	3	0	29870	1129	1
(1705/9,62686/27,1,3,2)	7	3	2	62686	1705	1
(4, 1 , 2, 1, 0)	2	1	0	1	4	1
(568, 13537 , 2, 1, 0)	2	1	0	13537	568	1
(46, 307 , 2, 3, 0)	2	3	0	307	46	1
(394, 7811 , 2, 5, 0)	2	5	0	7811	394	1
(10, 1 , 3, 0, 1)	3	0	1	1	10	1
(40, 251 , 3, 0, 1)	3	0	1	251	40	1
(22480, 3370501 , 3, 0, 1)	3	0	1	3370501	22480	1
(280, 4537 , 3, 0, 3)	3	0	3	4537	280	1
(13, 46 , 4, 0, 0)	4	0	0	46	13	1
(7, 10 , 5, 0, 0)	5	0	0	10	7	1
(25, 118 , 5, 1, 0)	5	1	0	118	25	1
(253, 4024 , 5, 1, 0)	5	1	0	4024	253	1
(1033, 33200 , 5, 1, 1)	5	1	1	33200	1033	1
(289, 4670 , 5, 1, 2)	5	1	2	4670	289	1
(43, 260 , 5, 2, 0)	5	2	0	260	43	1
(127, 1268 , 5, 2, 1)	5	2	1	1268	127	1
(175/9,2152/27,3,0,1)	9	0	1	2152	175	1
(14349055/59049,54352538468/14348907,3,0,1)	33	0	1	54352538468	14349055	1
(2467753/729,3875774680/19683,3,5,1)	21	5	1	3875774680	2467753	1
(1675/49,60346/343,5,0,1)	5	6	1	60346	1675	1
(36145/81,6781186/729,5,1,2)	17	1	2	6781186	36145	1
(6043/9,456940/27,5,2,2)	11	2	2	456940	6043	1
(187303/81,79060204/729,5,2,3)	17	2	3	79060204	187303	1

Table 2: All nontrivial solutions of (3) for $n = 3$ and $\lambda \in \{2, 4\}$

(N,M,i,j,k)	a	b	c	x	y	λ
(106/9, 1090/27,0, 0,0)	6	0	0	545	53	2
(26, 110 , 0, 0,2)	0	0	2	55	13	2
(4, 6 , 0, 1,0)	0	1	0	3	2	2
(8, 22 , 0, 1,0)	0	1	0	11	4	2
(44, 134 , 0, 5,0)	0	5	0	67	22	2
(46/9, 190/27,1, 1,0)	7	1	0	95	23	2
(24670/9, 3860002/27,1, 1,4)	7	1	4	1930001	12335	2
(1390/9, 51794/27,1, 3,0)	7	3	0	25897	695	2
(80350/729, 20990782/19683,1, 5,0)	19	5	0	10495391	40175	2
(898/9, 26234/27,2, 0,2)	8	0	2	13117	449	2
(16, 62 , 2, 1,0)	2	1	0	31	8	2
(88, 818 , 2, 3,0)	2	3	0	409	44	2
(13444/49, 1535806/343,2, 5,0)	2	11	0	767903	6722	2
(16, 10 , 3, 0,1)	3	0	1	5	8	2
(28, 134 , 3, 0,1)	3	0	1	67	14	2
(100, 998 , 3, 0,1)	3	0	1	499	50	2
(1318/9, 47566/27,3, 3,0)	9	3	0	23783	659	2
(55318/81, 11930066/729,3, 3,2)	15	3	2	5965033	27659	2
(10, 26 , 4, 0,0)	4	0	0	13	5	2
(190/9, 1378/27,5, 1,0)	11	1	0	689	95	2
(22, 62 , 5, 1,0)	5	1	0	31	11	2
(1150, 32626 , 5, 3,2)	5	3	2	16313	575	2
(73222/9, 19397762/27,5,5,2)	11	5	2	9698881	36611	2
(8, 20 , 0, 1,0)	0	1	0	5	2	4
(37192/9, 7172180/27,0, 3,2)	6	3	2	1793045	9298	4
(9772/1369, 899996/50653,1, 0,0)	1	0	6	224999	2443	4
(28, 148 , 1, 0,0)	1	0	0	37	7	4
(112/9, 332/27,1, 0,1)	7	0	1	83	28	4
(196, 2732 , 1, 0,2)	1	0	2	683	49	4
(214960/441, 98611876/9261,1, 0,3)	7	6	3	24652969	53740	4
(2764, 145004 , 1, 0,4)	1	0	4	36251	691	4
(400/9, 748/27,1, 2,1)	7	2	1	187	100	4
(6280/81, 448804/729,1, 2,1)	13	2	1	112201	1570	4
(7288/9, 622124/27,1, 2,1)	7	2	1	155531	1822	4
(12016/9, 1317140/27,1, 2,1)	7	2	1	329285	3004	4
(79216/441, 11461780/9261,1, 4,1)	7	10	1	2865445	19804	4
(154984/81, 24906860/729,1, 4,3)	13	4	3	6226715	38746	4
(6952/9, 578780/27,2, 1,2)	8	1	2	144695	1738	4
(20560/81, 2876588/729,3, 2,1)	15	2	1	719147	5140	4
(5440/9, 364708/27,3, 4,1)	9	4	1	91177	1360	4
(88, 820 , 4, 1,0)	4	1	0	205	22	4
(232, 260 , 4, 1,2)	4	1	2	65	58	4
(472/9, 532/27,5, 0,1)	11	0	1	133	118	4
(136, 1540 , 5, 0,1)	5	0	1	385	34	4
(160, 1988 , 5, 0,1)	5	0	1	497	40	4
(232, 2332 , 5, 2,1)	5	2	1	583	58	4
(1360, 50084 , 5, 2,1)	5	2	1	12521	340	4
(736, 7300 , 5, 4,1)	5	4	1	1825	184	4
(1528, 56764 , 5, 4,1)	5	4	1	14191	382	4
(346600/81, 203602652/729,5, 4,1)	17	4	1	50900663	86650	4
(277804480/81, 4630260052412/729,5, 4,5)	17	4	5	1157565013103	69451120	4

Table 3: All nontrivial solutions of (3) for $n = 4$ and $\lambda \in \{1, 2, 4\}$

(N,M,i,j,k)	**a**	**b**	**c**	**x**	**y**	λ
(± 2, 3, 0, 1, 0)	0	1	0	3	2	1
(± 5, 24, 0, 2, 0)	0	2	0	24	5	1
(± 13, 239, 0, 0, 0)	0	0	0	239	13	2
(± 2, 5, 0, 1, 0)	0	1	0	5	2	2
(± 4, 13, 0, 3, 0)	0	3	0	13	4	2
(± 22/3, 401/9, 0, 1, 2)	4	1	2	401	22	4
(± 2, 1, 2, 1, 0)	2	1	0	1	2	4
(± 4, 31, 2, 1, 0)	2	1	0	31	4	4
(± 5, 13, 2, 1, 1)	2	1	1	13	5	4
(± 13, 5, 2, 3, 1)	2	3	1	5	13	4
(± 4, 5, 3, 0, 1)	3	0	1	5	4	4
(± 1894/27, 7174435/729, 3, 0, 1)	15	0	1	7174435	1894	4

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