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On Generalization of Centred Odd Cube of Length $2r+1$ $$\sum_{i=1}^{2r+1} w_i^3 = \left(\sum_{i=1}^{2r+1} w_i\right)\left(m^2 + r(r+1)d^2\right)$$

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Abstract

The study of integer I for which I is a sum of cubes is a subject of study in the areas of Diophantine equations, perhaps this is due to the fact that Diophantine equations have direct applications in coding theory. Let r, w, d and m be positive integers and suppose that d is a common difference between any two integers in the sequence w_{i+1} and w_i, where $m = w_{r+1}$ denotes the middle term of the sequence. The current paper is therefore set to develop and introduce the general formula for the sum of an odd number of centred odd cubes. In particular, the formula $\sum_{i=1}^{2r+1} w_i^3 = \left(\sum_{i=1}^{2r+1} w_i\right)\left(m^2 + r(r+1)d^2\right)$ is introduced. The methodology involves expressing the arithmetic sequence in its standard form, expanding the cube terms, and matching coefficients. Several examples are presented to verify the identity and illustrate its computational efficiency.

Introduction

The study of the representation of integers as a sum of powers occupies a central place in the field of additive number theory and Diophantine equations. Sankei et al. [1,2] introduced some novel formulations on the set of even and odd numbers and proved that every even number has an equivalent representation through an algebraic expression, thereby developing an alternative framework for research on additive number theory involving even integers. These scholars argued that this formulation facilitates the systematic partitioning of even numbers and serves as a foundation for subsequent investigations of classical number-theoretic conjectures.

Sankei et al. [3] initiated the study of residual sets in the areas of Goldbach representations. The analysis of the authors focused on the density of Goldbach partitions, which provides verifiable computational evidence on the frequency and distribution of prime representations of even integers and

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density-related questions, thereby improving their earlier work on computational investigations. Some recent investigations can also be mentioned. Lao et al. [7] have developed polynomial identities and parametric families arising from arithmetic progressions and provided a systematic formula for sums of cubes in arithmetic progression in the paper, "Some Generalized Formula for Sums of Cubes." The work considers integers $a_1, a_2, \cdots, a_n$ satisfying the condition $a_{i+1} - a_i = d$ and $a_1^3 + a_2^3 + a_3^3 + \cdots + a_n^3 = (a_1 + a_2 + a_3 + \cdots + a_n)L$, where L is an integer. The paper demonstrated that an arithmetic progression provides a framework for deriving compact cubic identities and generalized representations of sums of powers.

Subsequently, Kimtai and Lao [9] extended these studies to some generalized sums of six, seven, and nine cubes. In this work, the authors expressed the sums of six, seven, and nine cubes in arithmetic progression and obtained corresponding identities involving sums of cubes. The approach heavily relied on analogies involving integer decomposition and factorization techniques. For comprehensive research on sums of related powers of degree less than 3, the readers are encouraged to survey [5, 8, 11, 13], and for studies of sums of powers of degree higher than 3, the readers can see [10, 12, 14, 15].

The current study presents a centred identity for the sum of an odd number of terms in an arithmetic progression. In contrast to earlier treatments, in which the investigation primarily focuses on fixing the number of terms and determining the integer L, the current formulation expresses the sum of odd cubes in arithmetic progression in terms of the middle term m and the common difference d. In particular,

$$\sum_{i=1}^{2r+1} w_i^3 = \left(\sum_{i=1}^{2r+1} w_i\right)\left(m^2 + r(r+1)d^2\right).$$

Main Results

Theorem 0.1. *Let $r \geq 1$, and let*

$$w_1, w_2, \ldots, w_{2r+1}$$

be an arithmetic sequence with common difference d, that is,

$$w_{i+1} - w_i = d, \qquad i = 1, 2, \ldots, 2r.$$

Let

$$m = w_{r+1}$$

denote the middle term of the sequence. Then

$$\sum_{i=1}^{2r+1} w_i^3 = \left(\sum_{i=1}^{2r+1} w_i\right)\left(m^2 + r(r+1)d^2\right).$$

Since

$$\sum_{i=1}^{2r+1} w_i = (2r+1)m,$$

the identity may be equivalently written as

$$\sum_{i=1}^{2r+1} w_i^3 = (2r+1)m\left(m^2 + r(r+1)d^2\right).$$

Proof. Suppose

$$w_i = w + (i-1)d, \qquad i = 1, 2, \ldots, k,$$

where

$$k = 2r+1.$$

Let

$$S = \sum_{i=1}^{k} w_i$$

and

$$C = \sum_{i=1}^{k} w_i^3.$$

First,

$$S = \sum_{i=1}^{k}(w + (i-1)d) = kw + d\sum_{j=0}^{k-1} j.$$

Using

$$\sum_{j=0}^{k-1} j = \frac{k(k-1)}{2},$$

we obtain

$$S = kw + \frac{k(k-1)}{2}d. \tag{1}$$

Next,

$$C = \sum_{i=1}^{k}(w + (i-1)d)^3.$$

Expanding and summing,

$$C = kw^3 + 3w^2d\sum_{j=0}^{k-1} j + 3wd^2\sum_{j=0}^{k-1} j^2 + d^3\sum_{j=0}^{k-1} j^3.$$

Using

$$\sum_{j=0}^{k-1} j = \frac{k(k-1)}{2},$$

$$\sum_{j=0}^{k-1} j^2 = \frac{(k-1)k(2k-1)}{6},$$

and

$$\sum_{j=0}^{k-1} j^3 = \left(\frac{k(k-1)}{2}\right)^2,$$

we obtain

$$C = kw^3 + \frac{3}{2}k(k-1)w^2d + \frac{1}{2}k(k-1)(2k-1)wd^2 + \frac{1}{4}k^2(k-1)^2d^3. \tag{2}$$

From (1),

$$S = k\left(w + \frac{k-1}{2}d\right).$$

By direct multiplication,

$$C = S\left(w^2 + (k-1)wd + \frac{k(k-1)}{2}d^2\right). \tag{3}$$

Now let

$$k = 2r + 1.$$

The middle term is

$$m = w + rd. \tag{4}$$

Observe that

$$m^2 = w^2 + 2rwd + r^2d^2,$$

and hence

$$m^2 + r(r+1)d^2 = w^2 + 2rwd + r(2r+1)d^2. \tag{5}$$

Since

$$k - 1 = 2r,$$

and

$$\frac{(k-1)(k+1)}{4} = \frac{(2r)(2r+2)}{4} = r(r+1),$$

formula (3) becomes

$$C = S\left(m^2 + r(r+1)d^2\right).$$

Finally,

$$S = (2r+1)m.$$

Therefore,

$$\sum_{i=1}^{2r+1} w_i^3 = (2r+1)m\left(m^2 + r(r+1)d^2\right).$$

This completes the proof. □

Example 0.1. *Let*

$$r = 1, \qquad m = 5, \qquad d = 2.$$

Then the sequence is

$$3, 5, 7.$$

The left-hand side is

$$3^3 + 5^3 + 7^3 = 27 + 125 + 343 = 495.$$

The right-hand side is

$$(3 + 5 + 7)(5^2 + 1 \cdot 2 \cdot 2^2) = 15(25 + 8) = 495.$$

Hence,

$$3^3 + 5^3 + 7^3 = 495.$$

Example 0.2. *Let*

$$r = 2, \qquad m = 6, \qquad d = 1.$$

Then the sequence is

$$4, 5, 6, 7, 8.$$

The left-hand side is

$$4^3 + 5^3 + 6^3 + 7^3 + 8^3 = 1260.$$

The right-hand side is

$$(4 + 5 + 6 + 7 + 8)(6^2 + 2 \cdot 3 \cdot 1^2) = 30(36 + 6) = 1260.$$

Hence,

$$4^3 + 5^3 + 6^3 + 7^3 + 8^3 = 1260.$$

Example 0.3. *Let*

$$r = 3, \qquad m = 7, \qquad d = 2.$$

Then the sequences is

$$1, 3, 5, 7, 9, 11, 13.$$

The left-hand side is

$$1^3 + 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 = 4753.$$

The right-hand side is

$$(1 + 3 + 5 + 7 + 9 + 11 + 13)\left(7^2 + 3(4)(2^2)\right) = 49(49 + 48) = 49 \times 97 = 4753.$$

Hence,

$$1^3 + 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 = 4753.$$

Example 0.4. *Let*

$$r = 3, \qquad m = 7, \qquad d = 2.$$

Then the sequences is

$$-1, -3, -5, -7, -9, -11, -13.$$

The left-hand side is

$$(-1)^3 + (-3^3 + (-5)^3 + (-7)^3 + (-9)^3 + (-11)^3 + (-13)^3 = -4753.$$

The right-hand side is

$$(-1 + -3 + -5 + -7 + -9 + -11 + -13)\left(7^2 + 3(4)(2^2)\right) = -49(49 + 48) = -49 \times 97 = -4753.$$

Hence,

$$(-1)^3 + (-3)^3 + (-5)^3 + (-7)^3 + (-9)^3 + (-11)^3 + (-13)^3 = -4753.$$

Conclusion

In summary, this paper has established the general formula for sums of an odd number of cubes of length $2r + 1$ in arithmetic progression. By expressing the identities in terms of the middle term and common difference, a unified general formula has been obtained that works for all odd numbers. The results extend already existing results on sums of cubes in arithmetic progression. In particular, previously established identities for the sums of three, five, seven, and other odd numbers of cubes serve naturally as special cases of the general theorem. Consequently, the present work sharpens the existing results in the literature on sums of an odd number of cubes. Although explicit cases of sums of even numbers of cubes are known, including the sums of two, four, and six cubes, a general formula that unifies all sums of even numbers of cubes in arithmetic progression is still lacking. Thus, future investigations should focus on generalizing the sums of an even number of cubes of length $2n$.

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