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A Four-Parameter Dass–Gupta–Reich Fixed Point Theorem in (α, β)-Complex-Valued b-Metric Spaces

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Abstract

In this paper, we establish a four-parameter fixed point theorem for self-mappings on complete (α, β)-complex-valued b-metric spaces. The proposed contractive condition combines a Dass–Gupta rational term with two Reich-type self-distance terms and allows the coefficients of $d(x, Tx)$ and $d(y, Ty)$ to be different. The principal parameter restriction is expressed in the compact form

$$\beta(\lambda + \gamma_1) + \mu + \gamma_2 < 1, \qquad \alpha\gamma_1 < 1,$$

which simultaneously guarantees the geometric decay required in the Picard iteration and the final fixed-point verification. A detailed example on $[0, 1]$ verifies all four contractive components directly, including the rational term in both real and imaginary components. An application to a nonlinear Fredholm integral equation is obtained on an invariant space of constant functions. Several classical contractions arise as parameter specializations, including Banach-, Kannan-, Reich-, asymmetric Reich-, and Dass–Gupta-type conditions.

1 Introduction

Fixed point theory provides a fundamental framework for studying nonlinear equations, integral equations and iterative processes. The Banach contraction principle [3] is one of its foundational results, and numerous extensions have subsequently been obtained by weakening the underlying metric structure or modifying the contractive condition.

The notion of a b-metric was introduced to replace the ordinary triangle inequality by a relaxed inequality involving a constant $s \geq 1$; see Bakhtin [2] and Czerwik [10].

Complex-valued metric spaces provide another useful generalization. Azam et al. [1] introduced complex-valued metric spaces using the componentwise order on $\mathbb{C}$, and subsequent authors combined this structure with b-metrics. Rao et al. [5], among others, investigated fixed point problems in such spaces.

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These spaces are advantageous in modeling physical systems in electrical engineering, wave mechanics, and signal processing [7, 12].

Singh et al. [6] considered a relaxed complex-valued b-metric type in which two coefficients occur separately in the generalized triangle inequality:

$$d(x,y) \preceq \alpha d(x,z) + \beta d(z,y), \quad \alpha, \beta \geq 1.$$

It is important, however, not to interpret this formulation as a strict enlargement of the complex-valued b-metric class. Indeed, if $s = \max\{\alpha, \beta\}$, then

$$d(x,y) \preceq s[d(x,z) + d(z,y)].$$

The importance of the two-parameter formulation is therefore that it retains the individual coefficients α and β in the fixed point estimates.

A substantial part of fixed point theory concerns contractive conditions beyond Banach's linear condition. Reich [15] introduced inequalities involving the self-distance terms $d(x, Tx)$ and $d(y, Ty)$, typically of the form:

$$d(Tx, Ty) \leq ad(x,y) + bd(x,Tx) + cd(y,Ty), \quad \text{where} \quad a, b, c \geq 0.$$

Hardy and Rogers [13] subsequently obtained a five-term generalization of this type.

Dass and Gupta [11] introduced a rational extension of the Banach contraction principle:

$$d(Tx, Ty) \leq ad(x,y) + b\frac{d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)}, \quad \text{where} \quad a, b \geq 0.$$

Recent work has continued along these two directions. In particular, Barman et al. [4] established fixed point results for Hardy–Rogers-type in complex-valued b-metric spaces. Their work confirms the continuing relevance of Reich- and Hardy–Rogers-type conditions in complex-valued generalized metric structures. More recently, Bouker et al. [9] investigated generalized contractions and applications in complex-valued b-metric spaces.

The authors have also recently studied a Dass–Gupta-type rational contraction in complete (α, β)-complex-valued b-metric spaces [8]. That paper considered the two-parameter rational condition

$$d(Tx, Ty) \preceq \lambda d(x,y) + \mu\frac{d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)}.$$

The present paper is deliberately different from that earlier result. Rather than merely modifying the parameter restrictions of the same rational contraction, we incorporate two additional Reich-type self-distance terms with independent coefficients:

$$\gamma_1 d(x,Tx) + \gamma_2 d(y,Ty).$$

Thus, the present condition permits an asymmetric treatment of the two self-distance terms, while simultaneously retaining the nonlinear Dass–Gupta rational component. The resulting theorem is therefore a hybrid Dass–Gupta–Reich framework rather than a reformulation of the earlier two-parameter result. In particular, the proof contains a new fixed-point verification in which the structural coefficient α interacts with γ_1 through the additional requirement $\alpha\gamma_1 < 1$.

Motivated by these observations, we consider a self-mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \preceq \lambda d(x, y) + \mu \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \gamma_1 d(x, Tx) + \gamma_2 d(y, Ty),$$

for all $x, y \in X$, where $\lambda, \mu, \gamma_1, \gamma_2 \geq 0$. The principal sufficient condition is

$$\beta(\lambda + \gamma_1) + \mu + \gamma_2 < 1, \quad \alpha\gamma_1 < 1.$$

The first inequality $\beta(\lambda + \gamma_1) + \mu + \gamma_2 < 1$ is a compact form of the convergence requirement generated by the Picard iteration. Indeed, it implies

$$\mu + \gamma_2 < 1$$

and

$$q := \frac{\lambda + \gamma_1}{1 - \mu - \gamma_2} < \frac{1}{\beta}.$$

The second inequality is used in the verification that the limit of the Picard sequence is actually a fixed point.

The main features of the result are therefore as follows:

(i) the Dass–Gupta rational term and Reich-type self-distance terms occur simultaneously;

(ii) the coefficients γ_1 and γ_2 may be different;

(iii) the coefficient β enters explicitly into the convergence condition;

(iv) the coefficient α enters the final fixed-point verification through $\alpha\gamma_1 < 1$;

(v) several familiar contractive conditions arise as parameter specializations.

The remainder of the paper is organized as follows. Section 2 recalls the required definitions and establishes a geometric Cauchy estimate. Section 3 contains the main fixed point theorem. Section 4 presents a nontrivial example in which all four contractive parameters are nonzero. Section 5 gives an application to a nonlinear Fredholm integral equation on an invariant space of constant functions. Section 6 records important special cases and remarks. Section 7 concludes the paper.

2 Preliminaries

Let $\mathbb{C}$ denote the set of complex numbers. For $z_1, z_2 \in \mathbb{C}$, define the partial order $\preceq$ by

$$z_1 \preceq z_2 \quad \text{if and only if} \quad \Re(z_1) \leq \Re(z_2) \quad \text{and} \quad \Im(z_1) \leq \Im(z_2).$$

Consequently, if $0 \preceq z_1 \preceq z_2$ then $|z_1| \leq |z_2|$ (see Azam et al. [1]).

Definition 2.1. *Let X be a nonempty set and let $\alpha, \beta \geq 1$. A mapping $d : X \times X \to \mathbb{C}$ is an (α, β)-complex-valued b-metric if, for all $x, y, z \in X$,*

(D1) $0 \preceq d(x, y)$ *and* $d(x, y) = 0 \iff x = y$;

(D2) $d(x, y) = d(y, x)$;

(D3) $d(x, y) \preceq \alpha d(x, z) + \beta d(z, y)$.

The pair (X, d) is called an (α, β)-complex-valued b-metric space (see Singh et al. [6]).

Remark 2.2. *Although the distance function d is symmetric, the generalized triangle inequality need not be coefficient-symmetric when $\alpha \neq \beta$. Interchanging x and y gives*

$$d(y, x) \preceq \beta d(y, z) + \alpha d(z, x),$$

which should not be confused with the original inequality.

Definition 2.3 (Singh et al. [6])**.** *A sequence $\{x_n\}$ in (X, d) converges to $x \in X$ if*

$$|d(x_n, x)| \longrightarrow 0 \qquad (n \to \infty).$$

The sequence $\{x_n\}$ is called Cauchy if

$$|d(x_n, x_m)| \longrightarrow 0 \qquad (n, m \to \infty).$$

The space (X, d) is complete if every Cauchy sequence converges to a point of X.

Proposition 2.4. *Let $\{x_n\}$ be a sequence in an (α, β)-complex-valued b-metric space. Suppose that, for some $q \geq 0$,*

$$d(x_n, x_{n+1}) \preceq q^n d(x_0, x_1), \quad n \geq 0, \quad \text{and} \quad 0 \leq \beta q < 1.$$

Then $\{x_n\}$ is Cauchy.

Proof. Since $0 \preceq d(x_n, x_{n+1}) \preceq q^n d(x_0, x_1)$, we have:

$$|d(x_n, x_{n+1})| \leq q^n |d(x_0, x_1)|.$$

For $m \geq 1$, repeated use of the generalized triangle inequality gives

$$\begin{aligned}
d(x_n, x_{n+m}) &\preceq \alpha d(x_n, x_{n+1}) + \beta d(x_{n+1}, x_{n+m}) \\
&\preceq \alpha d(x_n, x_{n+1}) + \alpha\beta d(x_{n+1}, x_{n+2}) + \cdots + \beta^{m-1} d(x_{n+m-1}, x_{n+m}) \\
&\preceq \alpha \sum_{j=0}^{m-2} \beta^j d(x_{n+j}, x_{n+j+1}) + \beta^{m-1} d(x_{n+m-1}, x_{n+m}).
\end{aligned}$$

Since $\alpha, \beta \geq 1$ and $\beta^{m-1} \leq \alpha\beta^{m-1}$, we have that

$$d(x_n, x_{n+m}) \preceq \alpha \sum_{j=0}^{m-1} \beta^j d(x_{n+j}, x_{n+j+1})$$

Hence

$$\begin{aligned}
|d(x_n, x_{n+m})| &\leq \alpha \sum_{j=0}^{m-1} \beta^j |d(x_{n+j}, x_{n+j+1})| \\
&\leq \alpha q^n |d(x_0, x_1)| \sum_{j=0}^{m-1} (\beta q)^j \\
&\leq \frac{\alpha q^n}{1 - \beta q} |d(x_0, x_1)|.
\end{aligned}$$

Since $\beta q < 1$, the right-hand side tends to zero as $n \to \infty$, uniformly in m.

Thus $\{x_n\}$ is Cauchy. □

3 Main Result

Theorem 3.1. *Let (X, d) be a complete (α, β)-complex-valued b-metric space, where $\alpha, \beta \geq 1$. Suppose that a self mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \preceq \lambda d(x, y) + \mu \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \gamma_1 d(x, Tx) + \gamma_2 d(y, Ty), \tag{1}$$

for all $x, y \in X$, where $\lambda, \mu, \gamma_1, \gamma_2 \geq 0$. If

$$\beta(\lambda + \gamma_1) + \mu + \gamma_2 < 1 \quad \textit{and} \tag{2}$$

$$\alpha\gamma_1 < 1, \tag{3}$$

then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$ and define

$$x_{n+1} = Tx_n, \qquad n \geq 0.$$

Set

$$q = \frac{\lambda + \gamma_1}{1 - \mu - \gamma_2}. \tag{4}$$

Condition (2) implies

$$\mu + \gamma_2 < 1 \quad \text{and} \quad \beta(\lambda + \gamma_1) < 1 - \mu - \gamma_2.$$

Consequently,

$$q < \frac{1}{\beta}. \tag{5}$$

Putting $x = x_{n-1}$ and $y = x_n$ in (1), we obtain

$$d(x_n, x_{n+1}) \preceq \lambda d(x_{n-1}, x_n) + \mu d(x_n, x_{n+1}) + \gamma_1 d(x_{n-1}, x_n) + \gamma_2 d(x_n, x_{n+1}),$$

because the rational term reduces (which relies on the non-zero nature of $1 + d(x_{n-1}, x_n)$ in $\mathbb{C}$, ensuring division is well-defined) exactly to

$$\frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} = d(x_n, x_{n+1}).$$

Therefore,

$$d(x_n, x_{n+1}) \preceq q\, d(x_{n-1}, x_n),$$

By induction,

$$|d(x_n, x_{n+1})| \leq q^n |d(x_0, x_1)|.$$

Since $q < 1/\beta$, Proposition 2.4 implies that $\{x_n\}$ is Cauchy.

Completeness of the space (X, d) gives some $p \in X$ such that

$$x_n \to p \quad \text{as} \quad n \to \infty.$$

We next prove that p is a fixed point. Taking $x = p$ and $y = x_n$ in (1),

$$d(Tp, x_{n+1}) \preceq \lambda d(p, x_n) + \mu \frac{d(x_n, x_{n+1})[1 + d(p, Tp)]}{1 + d(p, x_n)} + \gamma_1 d(p, Tp) + \gamma_2 d(x_n, x_{n+1}).$$

$$|d(Tp, x_{n+1})| \leq \lambda |d(p, x_n)| + \mu \frac{|d(x_n, x_{n+1})||1 + d(p, Tp)|}{|1 + d(p, x_n)|} + \gamma_1 |d(p, Tp) + \gamma_2 |d(x_n, x_{n+1})|.$$

Since $|d(p, x_n)| \to 0, \quad |d(x_n, x_{n+1})| \to 0$, we obtain

$$\limsup_{n \to \infty} |d(Tp, x_{n+1})| \leq \gamma_1 |d(p, Tp)|.$$

The generalized triangle inequality gives

$$d(Tp, p) \preceq \alpha d(Tp, x_{n+1}) + \beta d(x_{n+1}, p).$$

Therefore,

$$|d(Tp, p)| \leq \alpha \gamma_1 |d(p, Tp)|.$$

Since $d(Tp,p) = d(p,Tp)$ and $\alpha\gamma_1 < 1$, we obtain $d(Tp,p) = 0$.

Thus

$$Tp = p.$$

Finally, suppose p, r are fixed points of T. Applying (1) with $x = p$ and $y = r$ gives

$$d(p,r) = d(Tp,Tr) \preceq \lambda d(p,r),$$

because $d(p,Tp) = d(r,Tr) = 0$. Consequently, $|d(p,r)| \leq \lambda|d(p,r)|$. Since

$$\lambda \leq \lambda + \gamma_1 < 1 - \mu - \gamma_2 \leq 1,$$

we have $\lambda < 1$. Hence $d(p,r) = 0$, so $p = r$. Therefore the fixed point is unique. □

Remark 3.2. *The condition*

$$\frac{\lambda + \gamma_1}{1 - \mu - \gamma_2} < \frac{1}{\beta}$$

is a sufficient condition generated by the Picard iteration. It is not claimed to be necessary. The restriction $\alpha\gamma_1 < 1$ *is required for the fixed-point verification used in the proof.*

3.1 Corollaries of Theorem 3.1

The following consequences are obtained directly from Theorem 3.1 by suitable choices of the parameters.

Corollary 3.3 (Dass–Gupta-type contraction)**.** *Let* (X,d) *be a complete* (α,β)*-complex-valued b-metric space, where* $\alpha, \beta \geq 1$*, and let* $T : X \to X$ *satisfy*

$$d(Tx,Ty) \preceq \lambda d(x,y) + \mu\frac{d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)}$$

for all $x, y \in X$*, where* $\lambda, \mu \geq 0$*. If*

$$\beta\lambda + \mu < 1,$$

then T *has a unique fixed point in* X*.*

Proof. This is obtained from Theorem 3.1 by taking

$$\gamma_1 = \gamma_2 = 0.$$

The principal condition becomes

$$\beta(\lambda + \gamma_1) + \mu + \gamma_2 = \beta\lambda + \mu < 1,$$

while $\alpha\gamma_1 = 0 < 1$. Hence Theorem 3.1 applies and yields a unique fixed point. □

Corollary 3.4 (Reich-type contraction)**.** *Let (X, d) be a complete (α, β)-complex-valued b-metric space, where $\alpha, \beta \geq 1$, and let $T : X \to X$ satisfy*

$$d(Tx, Ty) \preceq \lambda d(x, y) + \gamma_1 d(x, Tx) + \gamma_2 d(y, Ty)$$

for all $x, y \in X$, where $\lambda, \gamma_1, \gamma_2 \geq 0$. If

$$\beta(\lambda + \gamma_1) + \gamma_2 < 1, \quad \alpha\gamma_1 < 1,$$

then T has a unique fixed point in X.

Proof. Set $\mu = 0$ in Theorem 3.1. The rational term disappears, and the resulting contractive condition is precisely the stated Reich-type condition. The two required parameter restrictions reduce to

$$\beta(\lambda + \gamma_1) + \gamma_2 < 1, \qquad \alpha\gamma_1 < 1.$$

The conclusion follows from Theorem 3.1. □

Corollary 3.5 (Kannan-type contraction)**.** *Let (X, d) be a complete (α, β)-complex-valued b-metric space, where $\alpha, \beta \geq 1$, and let $T : X \to X$ satisfy*

$$d(Tx, Ty) \preceq \gamma\big[d(x, Tx) + d(y, Ty)\big]$$

for all $x, y \in X$, where $\gamma \geq 0$. If

$$(\beta + 1)\gamma < 1, \quad \alpha\gamma < 1,$$

then T has a unique fixed point in X.

Proof. Take

$$\lambda = \mu = 0, \qquad \gamma_1 = \gamma_2 = \gamma$$

in Theorem 3.1. The principal condition becomes

$$\beta\gamma + \gamma = (\beta + 1)\gamma < 1,$$

and the fixed-point verification condition becomes

$$\alpha\gamma < 1.$$

Therefore Theorem 3.1 gives the existence and uniqueness of a fixed point. □

Corollary 3.6 (Banach-type contraction)**.** *Let (X, d) be a complete (α, β)-complex-valued b-metric space, where $\alpha, \beta \geq 1$, and let $T : X \to X$ satisfy*

$$d(Tx, Ty) \preceq \lambda d(x, y)$$

for all $x, y \in X$, where $\lambda \geq 0$. If

$$\beta\lambda < 1,$$

then T has a unique fixed point in X.

Proof. Set

$$\mu = \gamma_1 = \gamma_2 = 0$$

in Theorem 3.1. The main parameter condition becomes

$$\beta\lambda < 1,$$

while $\alpha\gamma_1 = 0 < 1$. Hence the conclusion follows. □

Corollary 3.7 (Symmetric (α, β)-structure)**.** *Let (X, d) be a complete (s, s)-complex-valued b-metric space, $s \geq 1$, and let $T : X \to X$ satisfy*

$$d(Tx, Ty) \preceq \lambda d(x, y) + \mu\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \gamma_1 d(x, Tx) + \gamma_2 d(y, Ty)$$

for all $x, y \in X$. If

$$s(\lambda + \gamma_1) + \mu + \gamma_2 < 1, \qquad s\gamma_1 < 1,$$

then T has a unique fixed point in X.

In particular, when $\gamma_1 = \gamma_2 = \gamma$,

$$s(\lambda + \gamma) + \mu + \gamma < 1$$

together with

$$s\gamma < 1$$

is sufficient for the existence and uniqueness of the fixed point.

Proof. Put

$$\alpha = \beta = s$$

in Theorem 3.1. The result follows immediately. □

Remark 3.8. *The preceding corollaries should be understood as* ***parameter specializations of the unified four-parameter theorem****, rather than as separate fixed-point theories. The principal result is the four-parameter Dass–Gupta–Reich theorem, while the Banach-, Kannan-, Reich-, and Dass–Gupta-type results follow by suppressing selected terms in the contractive condition.*

In particular, the corollaries show that the parameters α and β play different roles in the proof: β controls the geometric decay in the Picard iteration, whereas α appears in the final verification that the limit is a fixed point.

4 Illustrative Example

Example 4.1. *Let* $X = [0,1]$ *and define* $d : X \times X \to \mathbb{C}$ *by*

$$d(x,y) = |x-y|(1+i), \quad x,y \in X.$$

We take $\alpha = 2, \quad \beta = \frac{3}{2}$.

Then (X,d) *is a complete* $(2, \frac{3}{2})$*-complex-valued b-metric space.*

Indeed, $d(x,y) = |x-y|(1+i)$ *is symmetric and positive definite, while*

$$|x-y| \le |x-z| + |z-y| \le 2|x-z| + \frac{3}{2}|z-y|,$$

and hence

$$d(x,y) \preceq 2d(x,z) + \frac{3}{2}d(z,y).$$

Completeness follows from the completeness of $[0,1]$ *under the usual metric.*

Now define $T : X \longrightarrow X$ *by*

$$Tx = \frac{1}{10} + \frac{3}{10}\arctan x.$$

Since

$$0 \le \arctan x \le \frac{\pi}{4},$$

we have

$$\frac{1}{10} \le Tx \le \frac{1}{10} + \frac{3\pi}{40} < 1,$$

so $T(X) \subseteq X$.

By the mean value theorem,

$$|Tx - Ty| = \frac{3}{10(1+\xi^2)}|x-y| \le \frac{3}{10}|x-y|$$

for some ξ *between* x *and* y. *Therefore*

$$d(Tx,Ty) \preceq \frac{3}{10}d(x,y). \tag{6}$$

We now verify the four-parameter contractive condition of Theorem 3.1.

Choose

$$\lambda = \frac{1}{5}, \qquad \mu = \frac{1}{20}, \qquad \gamma_1 = \frac{1}{5}, \qquad \gamma_2 = \frac{7}{50}.$$

Clearly,

$$\mu + \gamma_2 = \frac{1}{20} + \frac{7}{50} = \frac{19}{100} < 1,$$

and

$$q = \frac{\lambda + \gamma_1}{1 - \mu - \gamma_2} = \frac{2/5}{81/100} = \frac{40}{81} < \frac{2}{3} = \frac{1}{\beta}.$$

Moreover,

$$\alpha\gamma_1 = 2\left(\frac{1}{5}\right) = \frac{2}{5} < 1.$$

For convenience, put

$$a = |x - y|, \qquad b = |x - Tx|, \qquad c = |y - Ty|.$$

Then

$$d(x, y) = a(1 + i), \qquad d(x, Tx) = b(1 + i), \qquad d(y, Ty) = c(1 + i).$$

A crucial estimate follows directly from the Lipschitz property of T. Indeed,

$$\begin{aligned} a &= |x - y| \\ &\le |x - Tx| + |Tx - Ty| + |Ty - y| \\ &\le b + \frac{3}{10}a + c. \end{aligned}$$

Hence

$$b + c \ge \frac{7}{10}a. \tag{7}$$

Next, consider the rational term

$$R(x, y) := \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}.$$

A direct calculation gives

$$R = \frac{c(2ab + 2a + 1)}{1 + 2a + 2a^2} + i\frac{c(2ab + 2b + 1)}{1 + 2a + 2a^2}. \tag{8}$$

Consequently,

$$\Re R = \frac{c(2ab + 2a + 1)}{1 + 2a + 2a^2}, \quad \textit{and} \quad \Im R = \frac{c(2ab + 2b + 1)}{1 + 2a + 2a^2}.$$

We now prove

$$\frac{1}{5}d(x, y) + \frac{1}{20}R + \frac{1}{5}d(x, Tx) + \frac{7}{50}d(y, Ty) \succeq \frac{3}{10}a(1 + i). \tag{9}$$

Case 1: *$c < a/5$.*

By (7),

$$b \ge \frac{7}{10}a - c > \frac{7}{10}a - \frac{1}{5}a = \frac{1}{2}a.$$

Therefore

$$\frac{1}{5}b > \frac{1}{10}a, \quad \textit{and hence} \quad \frac{1}{5}a + \frac{1}{5}b > \frac{3}{10}a.$$

Since R and $d(y, Ty)$ have nonnegative real and imaginary parts,

$$\frac{1}{5}d(x, y) + \frac{1}{20}R + \frac{1}{5}d(x, Tx) + \frac{7}{50}d(y, Ty) \succeq \frac{3}{10}a(1 + i).$$

Case 2: $c \geq a/5$.

From (7),

$$\frac{1}{5}b + \frac{7}{50}c \geq \frac{7}{50}(b+c) \geq \frac{49}{500}a.$$

For the real component of R*, since* $a, b, c \geq 0$,

$$\Re R \geq c\frac{2a+1}{1+2a+2a^2}.$$

Because $0 \leq a \leq 1$,

$$1 + 2a + 2a^2 \leq 5(2a+1),$$

and therefore

$$\Re R \geq \frac{1}{5}c \geq \frac{1}{25}a.$$

Thus

$$\frac{1}{20}\Re R \geq \frac{1}{500}a.$$

Similarly,

$$\Im R \geq \frac{c}{1+2a+2a^2} \geq \frac{1}{5}c \geq \frac{1}{25}a,$$

and consequently

$$\frac{1}{20}\Im R \geq \frac{1}{500}a.$$

Hence

$$\frac{1}{5}d(x,Tx) + \frac{7}{50}d(y,Ty) + \frac{1}{20}R \succeq \left(\frac{49}{500} + \frac{1}{500}\right)a(1+i) = \frac{1}{10}a(1+i).$$

Adding

$$\frac{1}{5}d(x,y) = \frac{1}{5}a(1+i)$$

gives

$$\frac{1}{5}d(x,y) + \frac{1}{20}R + \frac{1}{5}d(x,Tx) + \frac{7}{50}d(y,Ty) \succeq \frac{3}{10}a(1+i).$$

Together with (6)*, this yields*

$$d(Tx,Ty) \preceq \frac{1}{5}d(x,y) + \frac{1}{20}\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \frac{1}{5}d(x,Tx) + \frac{7}{50}d(y,Ty).$$

Thus all hypotheses of Theorem 3.1 are satisfied. Consequently, the theorem guarantees that T *has a unique fixed point* $p \in [0,1]$*. Equivalently,*

$$p = \frac{1}{10} + \frac{3}{10}\arctan p.$$

Remark 4.2. *The example is nontrivial in two respects. First, all four parameters in the proposed contractive condition are nonzero. Second, the coefficients* γ_1 *and* γ_2 *are unequal. The rational term is verified in the complex order rather than being replaced by a purely real estimate.*

5 Application to a Nonlinear Fredholm Integral Equation

Consider

$$u(t) = \frac{1}{10} + \frac{3}{10}\int_0^1 \arctan(u(s))ds, \quad t \in [0,1]. \tag{10}$$

Let

$$X_c = \{u_x : u_x(t) = x, x \in [0,1]\}$$

be the space of constant functions, equipped with

$$d(u_x, u_y) = |x - y|(1 + i).$$

It is a complete $(2, \frac{3}{2})$-complex-valued b-metric space.

Define

$$T : X_c \to X_c$$

by

$$(Tu_x)(t) = \frac{1}{10} + \frac{3}{10}\int_0^1 \arctan(u_x(s))\, ds.$$

Since $u_x(s) = x$,

$$(Tu_x)(t) = \frac{1}{10} + \frac{3}{10}\arctan x = u_{f(x)}(t),$$

where

$$f(x) = \frac{1}{10} + \frac{3}{10}\arctan x.$$

By Example 4.1, the induced operator T satisfies all hypotheses of Theorem 3.1. Hence Theorem 3.1 guarantees that T has a unique fixed point $u_p \in X_c$.

The corresponding solution of (10) is

$$u(t) = p, \quad p = \frac{1}{10} + \frac{3}{10}\arctan p.$$

Remark 5.1. *The application is formulated on the invariant constant-function space X_c. A corresponding theorem on the full space $C([0,1])$ would require a separate verification of the four-parameter contractive inequality for arbitrary continuous functions and is not claimed here.*

6 Special Cases and Remarks

The main condition contains several familiar forms as parameter specializations. These are reductions of the proposed framework and are not intended to establish strict inclusion relations between contractive classes.

Table 1: Important specializations of the main contractive condition.

Type	Parameter choice	Contractive condition
Banach	$\mu = \gamma_1 = \gamma_2 = 0$	$d(Tx, Ty) \preceq \lambda d(x, y)$
Kannan	$\lambda = \mu = 0, \gamma_1 = \gamma_2 = \gamma$	$d(Tx, Ty) \preceq \gamma[d(x, Tx) + d(y, Ty)]$
Reich	$\mu = 0, \gamma_1 = \gamma_2 = \gamma$	$d(Tx, Ty) \preceq \lambda d(x, y) + \gamma[d(x, Tx) + d(y, Ty)]$
Asymmetric Reich	$\mu = 0, \gamma_1 \neq \gamma_2$	$d(Tx, Ty) \preceq \lambda d(x, y) + \gamma_1 d(x, Tx) + \gamma_2 d(y, Ty)$
Dass–Gupta	$\gamma_1 = \gamma_2 = 0$	$d(Tx, Ty) \preceq \lambda d(x, y) + \mu \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}$

For the Banach specialization, the sufficient condition is

$$\beta\lambda < 1 \quad \text{or equivalently} \quad \lambda < \frac{1}{\beta}, \quad \alpha\gamma_1 = 0 < 1.$$

For the Kannan specialization,

$$\beta\gamma + \gamma < 1 \quad \text{or equivalently} \quad \gamma < \frac{1}{\beta + 1}, \quad \alpha\gamma < 1.$$

For the Reich specialization,

$$\beta(\lambda + \gamma) + \gamma < 1 \quad \text{or equivalently} \quad \lambda + \gamma < \frac{1 - \gamma}{\beta}, \quad \alpha\gamma < 1.$$

In the ordinary metric case $\alpha = \beta = 1$, this becomes

$$\lambda + 2\gamma < 1, \quad \gamma < 1.$$

For the asymmetric Reich-type specialization,

$$\beta(\lambda + \gamma_1) + \gamma_2 < 1, \quad \alpha\gamma_1 < 1.$$

Finally, for the Dass–Gupta specialization,

$$\beta\lambda + \mu < 1, \quad \alpha\gamma_1 = 0 < 1.$$

These reductions illustrate the unified role of the four parameters. In particular, β controls the geometric convergence restriction generated by the generalized triangle inequality, whereas α enters the final fixed-point verification through the coefficient of the $d(p, Tp)$ term.

It is again emphasized that the (α, β)-complex-valued b-metric framework is not claimed to be a strict enlargement of the complex-valued b-metric class. Its usefulness in the present work lies in preserving the separate coefficients α and β throughout the estimates and thereby allowing their distinct roles to be identified.

7 Conclusion

We have established a four-parameter Dass–Gupta–Reich-type fixed point theorem for self-mappings on complete (α, β)-complex-valued b-metric spaces. The contractive condition combines a nonlinear Dass–Gupta rational term with two independently weighted Reich-type self-distance terms:

$$d(Tx, Ty) \preceq \lambda d(x, y) + \mu \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \gamma_1 d(x, Tx) + \gamma_2 d(y, Ty).$$

The coefficients γ_1 and γ_2 are allowed to be different, so the framework accommodates an asymmetric treatment of the two self-distance terms. The principal sufficient parameter condition has been written in the compact form

$$\beta(\lambda + \gamma_1) + \mu + \gamma_2 < 1, \qquad \alpha\gamma_1 < 1.$$

The first inequality yields

$$q = \frac{\lambda + \gamma_1}{1 - \mu - \gamma_2} < \frac{1}{\beta},$$

which provides the geometric decay required for the Cauchy argument. The second inequality guarantees the final fixed-point step.

The illustrative example demonstrates that the four components of the contractive condition can be simultaneously active and that the two self-distance coefficients can genuinely differ. The rational term is verified directly in both complex components, thereby showing that the theorem is not being applied merely through a simpler Banach-type estimate.

An application to a nonlinear Fredholm integral equation has also been presented on an invariant space of constant functions. This provides a concrete setting in which the abstract fixed point theorem yields existence and uniqueness without requiring a separate theorem on the full space of continuous functions.

The present result differs from the authors' earlier Dass–Gupta-type theorem by incorporating independent Reich-type self-distance terms into the rational contraction. This creates a unified four-parameter framework that simultaneously retains the nonlinear rational contribution and permits asymmetric self-distance control.

Possible directions for further study include common and coupled fixed points, multivalued mappings, ordered complex-valued spaces, stability and data-dependence results, and applications to nonlinear integral and fractional differential equations.

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