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# Almost Path-Averaged Polynomial Contractions: A New Generalization of Almost Polynomial Contractions, Almost Contractions, and Almost Path-Averaged Contractions

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**Abstract**

In this paper, we combine the notions of polynomial contractions [3], path-averaged contractions which appeared in [5] for metric spaces, in [6, 7] for b-metric spaces, in [8] in suprametric spaces, while their combined notion appeared in [9], and almost contractions [2], to introduce almost path-averaged polynomial contractions. We obtain a fixed point theorem for such contractions in the setting of metric spaces. Finally, we give an example showing that almost path-averaged polynomial contractions are not Banach contractions.

## 1 Introduction and Preliminaries

**Theorem 1.1.** *[1] Let $(X,d)$ be a complete metric space and let $T : X \mapsto X$ be an almost contraction, that is, a mapping for which there exists a constant $\delta \in [0,1)$ and some $L \geq 0$ such that $d(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx)$ for all $x,y \in X$. Then*

(a) $Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$.

(b) *For any $x_0 \in X$, the Picard sequence $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$, $n = 0,1,2\cdots$, converges to some $x^* \in Fix(T)$.*

(c) *The following estimate holds*

$$d(x_{n+i-1},x^*) \leq \frac{\delta^i}{1-\delta}d(x_n,x_{n-1})$$

*for $n = 0,1,2,\ldots$, $i = 1,2,\ldots$.*

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**Definition 1.2.** [2] Let $(X, d)$ be a metric space. A map $T : X \mapsto X$ is said to be a $(\delta, 1-\delta)$-weak contraction mapping if it is not the identity mapping, and there exists $\delta \in (0, 1)$ such that, for all $x, y \in X$, the following holds:

$$d(Tx, Ty) \leq \delta d(x, y) + (1-\delta)d(y, Tx).$$

**Definition 1.3.** [3] A self-mapping $T$ on a metric space $(X, d)$ is termed a polynomial contraction if there is a collection of functions $a_1, \dots, a_k : X \times X \mapsto \mathbb{R}_+$ and $\lambda \in [0, 1)$ such that, for some natural number $k$, the inequality

$$\sum_{i=0}^{k} a_i(Tx, Ty)d^i(Tx, Ty) \leq \lambda \sum_{i=0}^{k} a_i(x, y)d^i(x, y)$$

holds true.

**Theorem 1.4.** *[3] Consider a complete metric space $(X, d)$ endowed with $T : X \mapsto X$, a polynomial contraction with respect to a collection of functions $a_1, \dots, a_k : X \times X \mapsto \mathbb{R}_+$. Assume further that the following conditions hold:*

(a) *The mapping $T$ is continuous.*

(b) *There is an index $j \in \{1, 2, \dots, k\}$ and $A_j > 0$ such that the corresponding coefficient function $a_j$ satisfies $a_j(x, y) \geq A_j$ for all $x, y \in X$.*

*Then $T$ possesses exactly one fixed point $x^*$. Moreover, for any starting value $x_0 \in X$, the Picard sequence $\{x_n\}$ converges to $x^*$.*

**Definition 1.5.** [3] Let $(X, d)$ be a metric space and $T : X \mapsto X$ be a given mapping. We say that $T$ is an almost polynomial contraction if there exists $\lambda \in [0, 1)$, a natural number $k \geq 1$, a finite sequence $\{L_i\}_{i=0}^{k} \subset (0, \infty)$, and a family of mappings $a_i : X \times X \mapsto \mathbb{R}_+$, $i = 0, 1, \dots, k$, such that

$$\sum_{i=0}^{k} a_i(Tx, Ty)d^i(Tx, Ty) \leq \lambda \sum_{i=0}^{k} a_i(x, y)[d^i(x, y) + L_i d^i(y, Tx)]$$

for all $x, y \in X$.

**Definition 1.6.** [4] Let $(X, d)$ be a metric space, and $T : X \mapsto X$ be a given mapping. We say that $T$ is an almost polynomial type contraction if there exists $\delta \in (0, 1)$, a natural number $k \geq 1$, and a family of mappings $a_i : X \times X \mapsto [0, \infty)$, $i = 1, 2, \dots, k$, such that

$$\sum_{i=1}^{k} a_i(Tx, Ty)d^i(Tx, Ty) \leq \delta \sum_{i=1}^{k} a_i(x, y)d^i(x, y) + (1-\delta) \sum_{i=1}^{k} a_i(y, Tx)d^i(y, Tx)$$

for every $x, y \in X$.

**Theorem 1.7.** *[4] Consider a complete metric space $(X, d)$ endowed with a mapping $T : X \mapsto X$ that satisfies an almost polynomial type contractive condition with respect to a collection of functions $a_1, \dots, a_k : X \times X \mapsto [0, \infty)$. Assume further that the following conditions hold:*

(*i*) *The coefficient function $a_i$ is symmetric in its arguments and continuous in its second argument for each $i = 1, 2, \ldots, k$.*

(*ii*) *There is an index $j \in \{1, 2, \ldots, k\}$ and $A_j > 0$ such that the corresponding coefficient function $a_j$ satisfies the inequality $a_j(x, y) \geq A_j$ for all $x, y \in X$.*

*Then $T$ possesses a fixed point $x^*$. Moreover, for any starting value $x_0 \in X$, the Picard sequence $\{x_n\}$ converges to $x^*$.*

**Definition 1.8.** [5] Let $(X, d)$ be a metric space. A mapping $T : X \mapsto X$ is called a PA-contraction (Path-Averaged Contraction) if there exists $\alpha \in (0, 1)$ and $N \in \mathbb{N}$ such that, for all $x, y \in X$ and all $n \geq N$,

$$\sum_{k=0}^{n-1} d(T^{k+1}x, T^{k+1}y) \leq \alpha \sum_{k=0}^{n-1} d(T^k x, T^k y).$$

**Theorem 1.9.** *[5] Let $(X, d)$ be a complete metric space, and let $T : X \mapsto X$ be a continuous PA-contraction. Then $T$ has a unique fixed point $x^* \in X$, and for any $x_0 \in X$, the Picard sequence $x_n = T^n x_0$ converges to $x^*$.*

## 2 Main Result

**Definition 2.1.** Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is called an *almost PA-polynomial contraction* if there exists $\delta \in (0, 1)$, natural numbers $r, N \in \mathbb{N}$, and a family of mappings $a_i : X \times X \to [0, \infty)$, $i = 0, 1, 2, \ldots, r$, such that, for all $x, y \in X$ and all $n \geq N$, we have

$$\begin{aligned}\sum_{k=0}^{n-1}\sum_{i=0}^{r} a_i(T^{k+1}x, T^{k+1}y)d^i(T^{k+1}x, T^{k+1}y) \leq \delta \sum_{k=0}^{n-1}\sum_{i=0}^{r} a_i(T^k x, T^k y)d^i(T^k x, T^k y)\\ + (1-\delta)\sum_{k=0}^{n-1}\sum_{i=0}^{r} a_i(T^k y, T^{k+1}x)d^i(T^k y, T^{k+1}x).\end{aligned}$$

*Remark* 2.2. The almost PA-polynomial contraction generalizes several contractions as follows:

(*i*) If $n = 1$, then the above contraction reduces to

$$\sum_{i=0}^{r} a_i(Tx, Ty)d^i(Tx, Ty) \leq \delta \sum_{i=0}^{r} a_i(x, y)d^i(x, y) + (1-\delta)\sum_{i=0}^{r} a_i(y, Tx)d^i(y, Tx),$$

which means that $T$ is an almost polynomial type contraction [4].

(*ii*) If $n = 1$, $r = 1$, $a_0 \equiv 0$, and $a_1 \equiv 1$, then the above contraction reduces to

$$d(Tx, Ty) \leq \delta d(x, y) + (1-\delta)d(y, Tx),$$

that is, $T$ is a $(\delta, 1-\delta)$-weak contraction [2].

$(iii)$ If $r = 1$, $a_0 \equiv 0$, and $a_1 \equiv 1$, then the above contraction reduces to

$$\sum_{k=0}^{n-1} d(T^{k+1}x, T^{k+1}y) \leq \delta \sum_{k=0}^{n-1} d(T^k x, T^k y) + (1-\delta) \sum_{k=0}^{n-1} d(T^k y, T^{k+1}x),$$

that is, $T$ is an almost path-averaged contraction.

The introduction of the path-averaged structure provides significantly greater flexibility by evaluating the contractive condition over a sequence of iterates rather than at a single step. This allows mappings that may not contract distances at every single iteration (thus failing to be almost polynomial contractions) to still exhibit an overall contracting behavior when averaged over a path of length $n \geq N$. Consequently, the class of almost PA-polynomial contractions strictly contains the classes of almost polynomial contractions, almost polynomial type contractions, and almost path-averaged contractions, offering a more robust framework for establishing fixed point results in complex dynamical systems where step-by-step contraction is too restrictive.

**Theorem 2.3.** *Consider a complete metric space $(X, d)$ endowed with $T : X \to X$, an almost PA-polynomial contraction with respect to a collection of functions $a_0, \ldots, a_r : X \times X \to [0, \infty)$. Assume further that the following conditions hold:*

(a) *The mapping $T$ is continuous.*

(b) *The index $j = 1$ is chosen to ensure metric convergence, and there exists $A_1 > 0$ such that the corresponding coefficient function $a_1$ satisfies $a_1(x, y) \geq A_1$ for all $x, y \in X$.*

*Then $T$ possesses a fixed point $x^*$. Moreover, for any starting value $x_0 \in X$, the Picard sequence $\{x_n\}$ converges to $x^*$.*

*Proof.* Let $x_0 \in X$ be arbitrary. Define the Picard sequence $x_n = T^n x_0$ for $n \geq 0$. Define the sequence of non-negative terms $Q_k$ by

$$Q_k = \sum_{i=0}^{r} a_i(x_k, x_{k+1}) d^i(x_k, x_{k+1}).$$

Applying the almost PA-polynomial contraction condition (Definition 2.1) to $x = x_0$ and $y = Tx_0 = x_1$, we observe that, for the term involving $(1 - \delta)$,

$$T^k y = T^k(Tx_0) = x_{k+1} \quad \text{and} \quad T^{k+1}x = T^{k+1}x_0 = x_{k+1}.$$

Thus, $d(T^k y, T^{k+1}x) = d(x_{k+1}, x_{k+1}) = 0$, causing the entire $(1 - \delta)$ sum to vanish. The inequality simplifies to:

$$\sum_{k=0}^{n-1} \sum_{i=0}^{r} a_i(x_{k+1}, x_{k+2}) d^i(x_{k+1}, x_{k+2}) \leq \delta \sum_{k=0}^{n-1} \sum_{i=0}^{r} a_i(x_k, x_{k+1}) d^i(x_k, x_{k+1})$$
$$\implies \sum_{k=0}^{n-1} Q_{k+1} \leq \delta \sum_{k=0}^{n-1} Q_k.$$

Let $S_n = \sum_{k=0}^{n-1} Q_k$. The inequality can be written as $S_{n+1} - Q_0 \leq \delta S_n$, which implies

$$S_{n+1} \leq \delta S_n + Q_0.$$

By induction, it follows that $S_n$ is bounded above. Specifically, $S_n \leq \frac{Q_0}{1-\delta}$ for all $n$. Since $\{S_n\}$ is a non-decreasing sequence of partial sums of non-negative terms and is bounded, the series $\sum_{k=0}^{\infty} Q_k$ converges. Consequently, $\lim_{k\to\infty} Q_k = 0$. From condition $(b)$ with $j = 1$, we have $a_1(x_k, x_{k+1}) \geq A_1 > 0$. Thus,

$$A_1 d(x_k, x_{k+1}) \leq Q_k.$$

Since $Q_k \to 0$, it follows that $d(x_k, x_{k+1}) \to 0$ as $k \to \infty$. To show that $\{x_n\}$ is a Cauchy sequence, we utilize the convergence of the series. Since $\sum Q_k < \infty$ and $Q_k \geq A_1 d(x_k, x_{k+1})$, we have $\sum_{k=0}^{\infty} d(x_k, x_{k+1}) < \infty$. For any $m \geq 1$, by the triangle inequality:

$$d(x_n, x_{n+m}) \leq \sum_{i=0}^{m-1} d(x_{n+i}, x_{n+i+1}).$$

Since the series $\sum d(x_k, x_{k+1})$ converges, the tail sum approaches 0 as $n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence. Since $(X, d)$ is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. By the continuity of $T$:

$$Tx^* = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus, $x^*$ is a fixed point of $T$. This completes the proof. □

*Remark* 2.4. We note that the fixed point established in Theorem 2.3 is not necessarily unique. This is consistent with the theory of almost contractions, where the presence of the $d(y, Tx)$ term in the contractive condition does not generally guarantee uniqueness without additional assumptions (see, e.g., Berinde [1]).

**Proposition 2.5.** *Every almost polynomial type contraction is an almost path-averaged polynomial contraction.*

*Proof.* Suppose $T$ satisfies the almost polynomial type contraction condition:

$$\sum_{i=0}^{r} a_i(Tx, Ty) d^i(Tx, Ty) \leq \delta \sum_{i=0}^{r} a_i(x, y) d^i(x, y) + (1-\delta) \sum_{i=0}^{r} a_i(y, Tx) d^i(y, Tx)$$

for all $x, y \in X$ and $\delta \in (0, 1)$. Replacing $x$ with $T^k x$ and $y$ with $T^k y$, we obtain:

$$\begin{aligned}\sum_{i=0}^{r} a_i(T^{k+1}x, T^{k+1}y) d^i(T^{k+1}x, T^{k+1}y) &\leq \delta \sum_{i=0}^{r} a_i(T^k x, T^k y) d^i(T^k x, T^k y) \\ &\quad + (1-\delta) \sum_{i=0}^{r} a_i(T^k y, T^{k+1}x) d^i(T^k y, T^{k+1}x).\end{aligned}$$

Summing this inequality from $k = 0$ to $n-1$, we have:

$$\begin{aligned}\sum_{k=0}^{n-1}\sum_{i=0}^{r} a_i(T^{k+1}x, T^{k+1}y)d^i(T^{k+1}x, T^{k+1}y) \le \delta \sum_{k=0}^{n-1}\sum_{i=0}^{r} a_i(T^k x, T^k y)d^i(T^k x, T^k y) \\ + (1-\delta)\sum_{k=0}^{n-1}\sum_{i=0}^{r} a_i(T^k y, T^{k+1}x)d^i(T^k y, T^{k+1}x).\end{aligned}$$

Thus, $T$ is an almost PA-polynomial contraction with $N = 1$. □

**Example 2.6** (Almost PA-polynomial contraction not Banach)**.** Let $X = \{0, 1, 2\}$ with the discrete metric:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Define $T(0) = 1,\ T(1) = 2,\ T(2) = 2$. $T$ is not a Banach contraction since $d(T0, T1) = d(1, 2) = 1 = d(0, 1)$, so no $\alpha < 1$ satisfies $d(Tx, Ty) \le \alpha d(x, y)$ for all $x, y$. However, $T$ is an almost PA-polynomial contraction with $\delta = \frac{1}{2}$, $N = 2$, $r = 2$, $a_0 \equiv 0$, $a_1 \equiv a_2 \equiv 1$. For any $x, y \in X$ and $n \ge 2$, we have:

$$\begin{aligned}\sum_{k=0}^{n-1}[d(T^{k+1}x, T^{k+1}y) + d^2(T^{k+1}x, T^{k+1}y)] \le \frac{1}{2}\sum_{k=0}^{n-1}[d(T^k x, T^k y) + d^2(T^k x, T^k y)] \\ + \frac{1}{2}\sum_{k=0}^{n-1}[d(T^k y, T^{k+1}x) + d^2(T^k y, T^{k+1}x)].\end{aligned}$$

**Verification for the pair $(0, 1)$ at $n = 2$:**

$$\begin{aligned}\text{LHS} &= d(T0, T1) + d^2(T0, T1) + d(T^2 0, T^2 1) + d^2(T^2 0, T^2 1) \\ &= d(1, 2) + d^2(1, 2) + d(2, 2) + d^2(2, 2) = 1 + 1 + 0 + 0 = 2.\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{2}[d(0, 1) + d^2(0, 1) + d(T0, T1) + d^2(T0, T1)] \\ &\quad + \frac{1}{2}[d(1, T0) + d^2(1, T0) + d(T1, T^2 0) + d^2(T1, T^2 0)] \\ &= \frac{1}{2}[1 + 1 + 1 + 1] + \frac{1}{2}[0 + 0 + 0 + 0] = 2.\end{aligned}$$

Thus, $2 \le 2$. For the pair $(0, 2)$, the calculations are identical to $(0, 1)$ since $d(0, 2) = d(0, 1) = 1$ and the iterates follow the same path to the fixed point 2. For the pair $(1, 2)$, since 2 is a fixed point ($T(2) = 2$), $T^k(2) = 2$ for all $k$. Thus, $d(T^k(1), T^k(2)) = d(T^k(1), 2)$. For $k = 0$, $d(1, 2) = 1$; for $k \ge 1$, $T^k(1) = 2$, so the distance is 0. The LHS and RHS both evaluate to 1 at $n = 2$, satisfying the inequality. For $n > 2$, all distances become 0, thereby trivially satisfying the condition for all pairs. Hence, $T$ is an almost PA-polynomial contraction but not a Banach contraction.

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