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Almost Polynomial Contraction Mapping Theorem in Metric Spaces

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Abstract

In this paper, we introduce the notion of an almost polynomial type contraction mapping operator and obtain a fixed point theorem for such mappings in the setting of metric spaces. Some consequences of the main result and a conjecture are stated. We show that the conjecture may be true by illustrating it with an example.

1 Introduction and Preliminaries

Theorem 1.1. *[1] Let (X, d) be a complete metric space, and let $T : X \mapsto X$ be an almost contraction, that is, a mapping for which there exists a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that $d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx)$ for all $x, y \in X$. Then*

(a) *$Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$.*

(b) *For any $x_0 \in X$, the Picard sequence $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots,$ converges to some $x^* \in Fix(T)$.*

(c) *The following estimates hold:*

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1})$$

for $n = 0, 1, 2, \cdots; i = 1, 2, \ldots.$

Definition 1.2. [2] Let (X, d) be a metric space. A map $T : X \mapsto X$ is said to be a $(\delta, 1-\delta)$-weak contraction mapping if it is not the identity mapping, and there exists $\delta \in (0, 1)$ such that, for all $x, y \in X$, the following holds:

$$d(Tx, Ty) \leq \delta d(x, y) + (1 - \delta)d(y, Tx).$$

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Definition 1.3. [3] A self-mapping T on a metric space (X,d) is termed a polynomial contraction if there is a collection of functions $a_1,\ldots,a_k : X\times X \mapsto \mathbb{R}_+$ and $\lambda\in[0,1)$ such that, for some natural number k, the inequality

$$\sum_{i=0}^{k} a_i(Tx,Ty)d^i(Tx,Ty) \leq \lambda \sum_{i=0}^{k} a_i(x,y)d^i(x,y)$$

holds true.

Theorem 1.4. *[3] Consider a complete metric space (X,d) endowed with $T : X \mapsto X$, a polynomial contraction with respect to a collection of functions $a_1,\ldots,a_k : X\times X\mapsto\mathbb{R}_+$. Assume further that the following conditions hold:*

(a) *The mapping T is continuous.*

(b) *There is an index $j\in\{1,2,\ldots,k\}$ and $A_j>0$ such that the corresponding coefficient function a_j satisfies $a_j(x,y)\geq A_j$ for all $x,y\in X$.*

Then T possesses exactly one fixed point x^*. Moreover, for any starting value $x_0\in X$, the Picard sequence $\{x_n\}$ converges to x^*.

Definition 1.5. [4] A mapping $T : X\mapsto X$ on a metric space (X,d) is termed a polynomial Kannan contraction if there exists λ with $0\leq\lambda<\frac{1}{2}$, a natural number k, and a collection of mappings $a_i : X\times X\mapsto\mathbb{R}_+$ for $i=1,\ldots,k$, satisfying the following:

$$\sum_{i=1}^{k} a_i(Tx,Ty)d^i(Tx,Ty) \leq \lambda\left(\sum_{i=1}^{k} a_i(x,Tx)d^i(x,Tx) + \sum_{i=1}^{k} a_i(y,Ty)d^i(y,Ty)\right)$$

for all $x,y\in X$.

Theorem 1.6. *[4] Consider a complete metric space (X,d) endowed with a mapping $T : X\mapsto X$ that satisfies a polynomial Kannan contractive condition with respect to a collection of functions $a_1,\ldots,a_k : X\times X\mapsto\mathbb{R}_+$. Assume further that the following conditions hold:*

(i) *The coefficient function a_i is symmetric in its arguments and continuous in its second argument for each $i=1,2,\ldots,k$.*

(ii) *There is an index $j\in\{1,2,\cdots,k\}$ and $A_j>0$ such that the corresponding coefficient function a_j satisfies the inequality $a_j(x,y)\geq A_j$ for all $x,y\in X$.*

Then T possesses exactly one fixed point x^*. Moreover, for any starting value $x_0\in X$, the Picard sequence $\{x_n\}$ converges to x^*.

Definition 1.7. [3] Let (X, d) be a metric space, and let $T : X \mapsto X$ be a given mapping. We say that T is an almost polynomial contraction if there exists $\lambda \in [0, 1)$, a natural number $k \geq 1$, a finite sequence $\{L_i\}_{i=0}^k \subset (0, \infty)$, and a family of mappings $a_i : X \times X \mapsto \mathbb{R}_+$, $i = 0, 1, \cdots, k$, such that

$$\sum_{i=0}^{k} a_i(Tx, Ty)d^i(Tx, Ty) \leq \lambda \sum_{i=0}^{k} a_i(x, y)[d^i(x, y) + L_i d^i(y, Tx)]$$

for every $x, y \in X$.

Motivation/Novelty: The key difference between Definition 1.8 and Definition 2.1, is that the constants L_i in Definition 1.8 lie in $(0, \infty)$, whilst the constant δ in Definition 2.1 lies in $(0, 1)$. The novelty of almost polynomial contractions lies in their ability to bridge and advance two highly influential, independent areas of metric fixed point theory, almost contractions (introduced by Berinde in 2004) and polynomial-type contractive conditions. By unifying these concepts, researchers have established a much more flexible, nonlinear algebraic framework that generalizes existing linear contraction constraints.

2 Main Result

Definition 2.1. Let (X, d) be a metric space, and let $T : X \mapsto X$ be a given mapping. We say that T is an almost polynomial type contraction if there exists $\delta \in (0, 1)$, a natural number $k \geq 1$, and a family of mappings $a_i : X \times X \mapsto [0, \infty)$, $i = 1, 2, \ldots, k$, such that

$$\sum_{i=1}^{k} a_i(Tx, Ty)d^i(Tx, Ty) \leq \delta \sum_{i=1}^{k} a_i(x, y)d^i(x, y) + (1 - \delta) \sum_{i=1}^{k} a_i(y, Tx)d^i(y, Tx)$$

for every $x, y \in X$.

Remark 2.2. The generalized $(\delta, 1 - \delta)$ weak contractive condition introduced above encompasses various new classes of contractive mappings as particular instances:

(i) Setting $k = 1$ and $a_1 \equiv 1$ yields the $(\delta, 1 - \delta)$ weak contraction originally introduced by Ampadu [2], which satisfies

$$d(Tx, Ty) \leq \delta d(x, y) + (1 - \delta)d(y, Tx)$$

for all $x, y \in X$.

(ii) Choosing $k = 2$, $a_1 \equiv 0$, and $a_2 \equiv 1$ leads to the pure quadratic $(\delta, 1 - \delta)$ weak contraction, that is, a self-mapping T on a metric space (X, d) for which there exists a constant $\delta \in (0, 1)$ such that

$$d^2(Tx, Ty) \leq \delta d^2(x, y) + (1 - \delta)d^2(y, Tx)$$

for all $x, y \in X$.

(*iii*) Similarly, setting $k = 3$, $a_1 \equiv a_2 \equiv 0$, and $a_3 \equiv 1$ gives rise to the pure cubic $(\delta, 1-\delta)$ weak contraction, where a self-mapping T on (X, d) satisfies

$$d^3(Tx, Ty) \leq \delta d^3(x, y) + (1-\delta)d^3(y, Tx)$$

for all $x, y \in X$, and for a certain $\delta \in (0, 1)$.

(*iv*) More generally, setting $k = m$, with $a_j \equiv 0$ for $1 \leq j \leq m-1$ and $a_m \equiv 1$, yields a pure $(\delta, 1-\delta)$ weak contraction of power m, defined by the condition

$$d^m(Tx, Ty) \leq \delta d^m(x, y) + (1-\delta)d^m(y, Tx)$$

for all $x, y \in X$ and a certain $\delta \in (0, 1)$.

Theorem 2.3. *Consider a complete metric space (X, d) endowed with a mapping $T : X \mapsto X$ that satisfies an almost polynomial type contractive condition with respect to a collection of functions $a_1, \cdots, a_k : X \times X \mapsto [0, \infty)$. Assume further that the following conditions hold:*

(*i*) *The coefficient function a_i is symmetric in its arguments and continuous in its second argument for each $i = 1, 2, \cdots, k$.*

(*ii*) *There is an index $j \in \{1, 2, \cdots, k\}$ and $A_j > 0$ such that the corresponding coefficient function a_j satisfies the inequality $a_j(x, y) \geq A_j$ for all $x, y \in X$.*

Then T possesses a fixed point x^*. Moreover, for any starting value x_0 in X, the Picard sequence $\{x_n\}$ converges to x^*.

Proof. Let $x_0 \in X$ be chosen arbitrarily. Define the sequence $\{x_n\}_{n=0}^{\infty}$ recursively by $x_{n+1} = Tx_n$ for all n, and also define

$$P_n = \sum_{i=1}^{k} a_i(x_n, x_{n+1})d^i(x_n, x_{n+1}).$$

Letting $x = x_n$ and $y = x_{n+1}$ in Definition 2.1 yields

$$\begin{aligned}\sum_{i=1}^{k} a_i(x_{n+1}, x_{n+2})d^i(x_{n+1}, x_{n+2}) &\leq \delta \sum_{i=1}^{k} a_i(x_n, x_{n+1})d^i(x_n, x_{n+1}) \\ &\quad + (1-\delta)\sum_{i=1}^{k} a_i(x_{n+1}, x_{n+1})d^i(x_{n+1}, x_{n+1}) \\ &= \delta \sum_{i=1}^{k} a_i(x_n, x_{n+1})d^i(x_n, x_{n+1})\end{aligned}$$

that is, $P_{n+1} \leq \delta P_n$. It follows by induction that $P_n \leq \delta^n P_0$, for all $n \geq 0$. By (ii) there is an index j such that $a_j(x_n, x_{n+1}) \geq A_j > 0$. Then we have

$$A_j d^j(x_n, x_{n+1}) \leq \sum_{i=1}^{k} a_i(x_n, x_{n+1})d^i(x_n, x_{n+1}) = P_n \leq \delta^n P_0$$

which implies that

$$d(x_n, x_{n+1}) \le \left(\frac{\delta^n P_0}{A_j}\right)^{\frac{1}{j}}.$$

Applying the triangle inequality for $m > n \ge 0$, it follows that

$$d(x_n, x_m) \le \sum_{l=n}^{m-1} d(x_l, x_{l+1}) \le \left(\frac{P_0}{A_j}\right)^{\frac{1}{j}} \sum_{l=n}^{\infty} \delta^{\frac{l}{j}} = \left(\frac{P_0}{A_j}\right)^{\frac{1}{j}} \frac{\delta^{\frac{n}{j}}}{1-\delta^{\frac{1}{j}}}.$$

Given that $\delta < 1$, the right-hand side approaches to zero. Thus, $\{x_n\}$ is a Cauchy sequence in the complete metric space X, ensuring its convergence to a point $x^* \in X$. To confirm that x^* is a fixed point, we substitute $x = x^*$ and $y = x_n$ in Definition 2.1. Then we have

$$\sum_{i=1}^{k} a_i(Tx^*, x_{n+1}) d^i(Tx^*, x_{n+1}) \le \delta \sum_{i=1}^{k} a_i(x^*, x_n) d^i(x^*, x_n) + (1-\delta) \sum_{i=1}^{k} a_i(x_n, Tx^*) d^i(x_n, Tx^*).$$

Passing to the limit as $n \to \infty$ and applying (i) we deduce that

$$\sum_{i=1}^{k} a_i(Tx^*, x^*) d^i(Tx^*, x^*) \le (1-\delta) \sum_{i=1}^{k} a_i(x^*, Tx^*) d^i(x^*, Tx^*).$$

That is,

$$P(Tx^*, x^*) \le (1-\delta) P(x^*, Tx^*),$$

where $P(u,v) = \sum_{i=1}^{k} a_i(u,v) d^i(u,v)$. Owing to $\delta \ne 0$ and the fact that each a_i is symmetric in its arguments, it follows that $Tx^* = x^*$. □

Remark 2.4. The fixed point in the above theorem is not unique. A proof by contradiction fails, showing that the fixed point is not unique.

Now, using Remark 2.2 and Theorem 2.3, we have the following.

Corollary 2.5. *Let (X, d) be a metric space. If (X, d) is complete and $T : X \mapsto X$ is a $(\delta, 1-\delta)$ weak contraction, then there is an element x^* such that $Tx^* = x^*$. For every starting value x_0 in X, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recurrence relation*

$$x_{n+1} = Tx_n, \quad n \in \mathbb{Z}_{\ge 0},$$

converges to x^* with respect to the metric d.

Corollary 2.6. *Let (X, d) be a metric space. If (X, d) is complete and $T : X \mapsto X$ is a pure quadratic $(\delta, 1-\delta)$ weak contraction, then there is an element x^* such that $Tx^* = x^*$. For every starting value x_0 in X, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recurrence relation*

$$x_{n+1} = Tx_n, \quad n \in \mathbb{Z}_{\ge 0},$$

converges to x^* with respect to the metric d.

Corollary 2.7. *Let (X, d) be a metric space. If (X, d) is complete and $T : X \mapsto X$ is a pure cubic $(\delta, 1-\delta)$ weak contraction, then there is an element x^* such that $Tx^* = x^*$. For every starting value x_0 in X, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recurrence relation*

$$x_{n+1} = Tx_n, \quad n \in \mathbb{Z}_{\geq 0},$$

converges to x^* with respect to the metric d.

Corollary 2.8. *Let (X, d) be a metric space. If (X, d) is complete and $T : X \mapsto X$ is a pure $(\delta, 1-\delta)$ weak contraction of power m, then there is an element x^* such that $Tx^* = x^*$. For every starting value x_0 in X, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recurrence relation*

$$x_{n+1} = Tx_n, \quad n \in \mathbb{Z}_{\geq 0},$$

converges to x^* with respect to the metric d.

Now, to conclude this section, we state an open problem and illustrate it.

Conjecture 2.9. *Consider a complete metric space (X, d) endowed with a mapping $T : X \mapsto X$ that satisfies an almost polynomial type contractive condition with respect to a collection of functions $a_1, \cdots, a_k : X \times X \mapsto [0, \infty)$. Assume further that the following conditions hold:*

(i) For every $1 \leq i \leq k$, there exists $M_i > 0$ such that the coefficient function a_i satisfies $a_i(x, y) \leq M_i$ for all (x, y) in $X \times X$.

(ii) There is an index $j \in \{1, 2, \cdots, k\}$ and $A_j > 0$ such that the corresponding coefficient function a_j satisfies the inequality $a_j(x, y) \geq A_j$ for all $x, y \in X$.

(iii) Each coefficient function a_i is lower semicontinuous in its first argument.

Then T possesses at least one fixed point x^*. Moreover, for any starting value x_0 in X, the Picard sequence $\{x_n\}$ converges to x^*.

Example 2.10. We consider (X, d), a metric space with $X = [1, 3]$ and the standard Euclidean metric $d(x, y) = |x - y|$. Define $T : X \mapsto X$ by

$$T(x) = \frac{x^2 + 3}{4}.$$

We show that T is an almost polynomial contraction with $k = 1$ and the weight function $a_1(x, y) = 1 + x^2 + y^2$. To verify the contraction condition, we must establish that, for some $\delta \in (0, 1)$, the inequality below holds for every pair $x, y \in X$:

$$a_1(Tx, Ty)d(Tx, Ty) \leq \delta a_1(x, y)d(x, y) + (1 - \delta)a_1(y, Tx)d(y, Tx).$$

Figure 1 shows that the "BlueGreenYellow" surface (RHS of the contraction inequality) dominates the "RustTones" surface (LHS of the contraction inequality). This demonstrates that the inequality is satisfied for all $x, y \in [1, 3]$ with $\delta = \frac{1}{2}$. Since $\frac{1}{2} < 1$, it follows that T satisfies the almost polynomial contraction condition with the given parameters. Moreover, the function $a_1(x, y)$ is bounded above by 9, that is, $a_1(x, y) \leq 9$. It also satisfies the lower bound $a_1(x, y) \geq 3 = A_1 > 0$. Therefore, the requirements of the above conjecture hold, ensuring that T has two fixed points, which are $x = 1$ and $x = 3$.

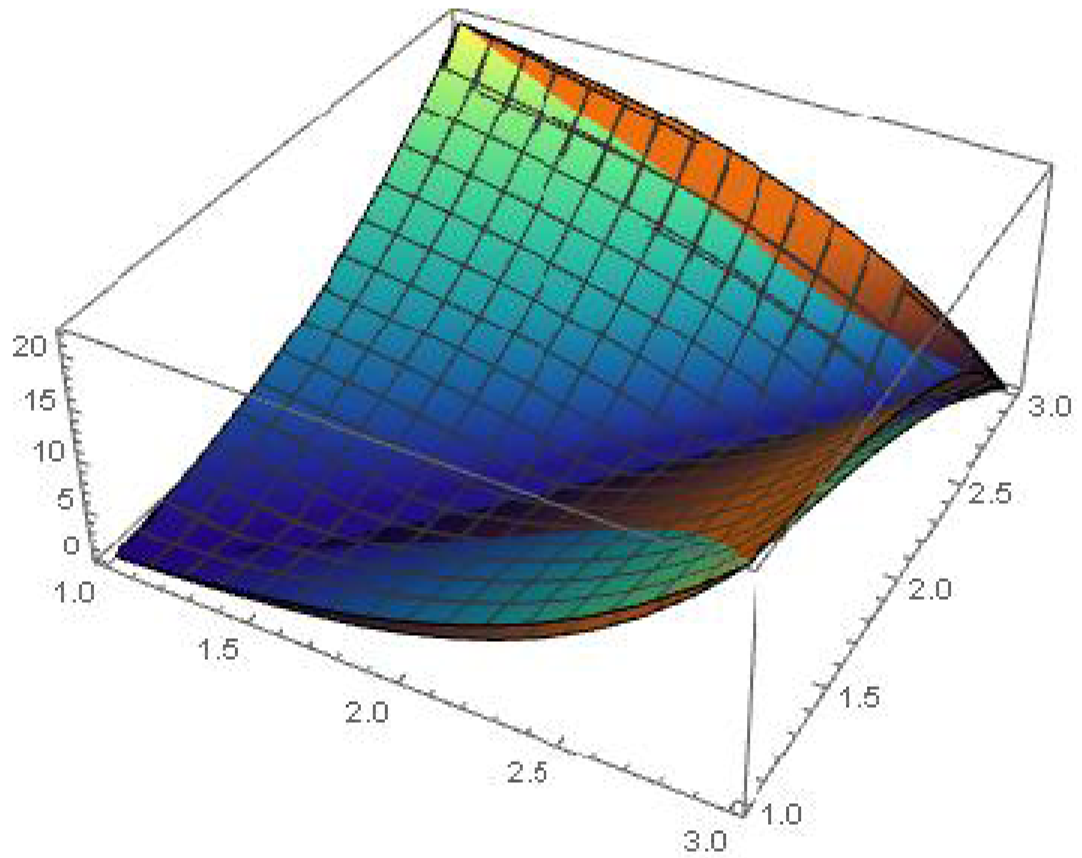

Figure 1: Comparison of both sides of the almost polynomial contraction inequality in the above example over $[1, 3]$.

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