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Geometric Attributes of Normalized Seven-Parameters Mittag-Leffler Function in the Unit Disk

Maryam K. Rasheed[1,*] and Abdulrahman H. Majeed[2]

[1] Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq
e-mail: mariam.khodair1103a@sc.uobaghdad.edu.iq

[2] Al-Ma'moon University, Baghdad, Iraq

Abstract

This paper dedicated to suggest a normalization of the seven-parameter Mittag-Leffler function, then assume sufficient conditions to investigates certain essential geometric properties, such as starlikness, convexity and closed-to-convexity on the unit disk. In addition, it constructs specific relations involving that normalization to look into how it belong to a developed class of normalized univalent function.

1 Introduction

The study of geometric behaviors of analytic functions is one of the most fascinating areas in Geometric Function Theory. It raised in the 20th century and persist to draw numerous attention for many researchers. Besides, it was developed to be linked with certain lines of complex analysis such as univalent theory, subordination theory, and special functions field, see [9, 11, 14, 18, 19] . Moreover, its applications played an active role in pure mathematics branches, fluid mechanics, and mathematical physics, see [3, 8, 12, 17].

Mittag-Leffler function, was appeared in the solutions of fractional differential equations, for instance while investigating the fractional generalization of the kinetic equation. Its first form was deduced by Gosta Mittag-Leffler in 1903, as

$$E_\tau = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\tau n + 1)}, \quad z \in \mathbb{C}, Re(\tau) > 0,$$

which is an entire function of order $\rho = \frac{1}{Re(\tau)}$ and type $\sigma = 1$. Due to its multiple uses in different fields of science and engineering, this function has grown in relevance and popularity over the last decades. Wherefore, it has seen many improvements from various researchers; some generalized it to investigate assorted characteristics, while others concentrate on applications and geometric behaviors, see [1, 2, 6, 7, 10, 21].

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*Corresponding author

Not long ago, Rasheed and Majeed [20], had submitted new generalization of Mittag-Leffler function named, seven-parameter Mittag-Leffler function, defined as follows

$$E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \; n!\Gamma(\tau_1 n + \lambda_1)\Gamma(\tau_2 n + \lambda_2)} z^n, \tag{1}$$

where $z \in \mathbb{C}$ and $\min\{Re(a), Re(b), Re(c), Re(\tau_1), Re(\tau_2), Re(\lambda_1), Re(\lambda_2)\} > 0$. It is notable that the function (1) assumed as entire function in the complex plane with order $\rho = \frac{1}{Re(\tau_1)+Re(\tau_2)}$, and type $\sigma = \left(\frac{1}{\rho|\tau_1|}\right)^{\rho Re(\tau_1)} \left(\frac{1}{\rho|\tau_2|}\right)^{\rho Re(\tau_2)}$.

Here, we modify the function (1) for the normalized form as follows:

$$\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = \Gamma(\lambda_1)\Gamma(\lambda_2) z E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = z + \sum_{n=2}^{\infty} \mu_n z^n, \tag{2}$$

where

$$\mu_n = \frac{(a)_{n-1}\,(b)_{n-1}\Gamma(\lambda_1)\Gamma(\lambda_2)}{(c)_{n-1}\,\Gamma(n)\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)}. \tag{3}$$

and $z \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$ and $min\,\{Re(a), Re(b), Re(c), Re(\tau_1), Re(\tau_2), Re(\lambda_1), Re(\lambda_2)\} > 0$. Our investigations of this work will assume real values for the parameters $a, b, c, \tau_1, \lambda_1, \tau_2$, and λ_2 as a special case of the proposed function defined in (2).

It is noteworthy to mention that the function (2) involving several well-known functions as its special case, such as

$$\mathcal{N}^{1,c,c}_{\tau_1,\lambda_1,0,\lambda_2}(z) = \mathbb{E}_{\tau_1,\lambda_1} = \Gamma(\lambda_2) z E_{\tau_1,\lambda_1},$$

which is the normalized Mittag-Leffler function proposed by Bansal and Prajapat in 2015 in [2], where E_{τ_1,λ_1} is Wiman's function. In addition,

$$\mathcal{N}^{1,c,c}_{\tau_1,\lambda_1,1,0}(z) = \mathbb{W}_{\tau_1,\lambda_1}(z) = \Gamma(\lambda_1) z W_{\tau_1,\lambda_1},$$

which demonstrates the normalized wright function presented by Prajapat in [16], here W_{τ_1,λ_1} denotes the Wright function. Moreover,

$$\mathcal{N}^{1,c,c}_{0,\lambda_1,0,\lambda_2}(z) = \frac{z}{1-z}, \tag{4}$$

$$\mathcal{N}^{1,c,c}_{0,\lambda_1,1,1}(z) = ze^z, \tag{5}$$

$$\mathcal{N}^{1,c,c}_{0,\lambda_1,1,2}(z) = e^z - 1, \tag{6}$$

$$\mathcal{N}^{1,c,c}_{0,\lambda_1,2,1}(z) = z\cosh(\sqrt{z}), \tag{7}$$

$$\mathcal{N}^{1,c,c}_{0,\lambda_1,2,2}(z) = \sqrt{z}\,\sinh(\sqrt{z}), \tag{8}$$

$$\mathcal{N}^{1,c,c}_{0,\lambda_1,1,4}(z) = \frac{6\,(e^z - 1 - z) - 3z^2}{z^2}, \tag{9}$$

$$\mathcal{N}^{1,c,c}_{0,1,3,4}(z) = \frac{z}{2}\left[e^{z^{\frac{1}{3}}} + 2e^{-\frac{1}{2}z^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}z^{\frac{1}{3}}\right)\right]. \tag{10}$$

The class of normalized analytic functions in the open unit disk $\mathbb{D}$, denoted by $\mathcal{A}$, such that every $f \in \mathcal{A}$ is of the form

$$f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n, \quad z \in \mathbb{D}, \tag{11}$$

also, $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ which containing univalent functions [4]. Remind that,

$$\mathcal{K} = \left\{ f \in \mathcal{A} : Re\left(\frac{z\acute{\acute{f}}(z)}{\acute{f}(z)}\right) + 1 > \zeta, 0 \le \zeta < 1 \right\},$$

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : Re\left(\frac{z\acute{f}(z)}{f(z)}\right) > 0 \right\},$$

$$\mathcal{C} = \left\{ f \in \mathcal{A} : Re\left(\frac{z\acute{f}(z)}{\acute{g}(z)}\right) > 0 : g \in \mathcal{K} \right\},$$

denotes the classes of convex, starlike and closed-to-convex functions respectively [4]. Furthermore, we mention the subclass of normalized univalent functions, which is defined by Rosy et al. [22]

$$\mathcal{UCD}(\kappa) = \left\{ f \in \mathcal{A} : Re\,(\acute{f}(z)) \ge \kappa |z\acute{\acute{f}}(z)|, \kappa > 0 \right\}.$$

Now, we recall some fundamental lemmas that will be useful to prove our results:

Lemma 1. *[5] Let $\{\alpha_n\}_{n\ge1}$ be a sequence of nonnegative real numbers such that $a_1 = 1$. If the quantities*

$$\underline{\triangle} a_n = na_n - (n+1)a_{n+1},$$

and

$$\underline{\triangle}(a_n)^2 := na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2},$$

are nonnegative, then the function $\sum_{n=1}^{\infty} a_n z^n$ is starlike in $\mathbb{D}$.

Lemma 2. *[15] Suppose that f is defined by (11) such that*

$$1 \geq 2\alpha_2 \geq ... \geq n\alpha_n \geq (n+1)\alpha_{n+1} \geq ... \geq 0, \tag{12}$$

or

$$1 \leq 2\alpha_2 \leq ... \leq n\alpha_n \leq (n+1)\alpha_{n+1} \leq ... \leq 2, \tag{13}$$

then f is closed-to-convex with respect to the function $-\log(1-z)$ in $\mathbb{D}$.

Lemma 3. *[13] Let $\vartheta \geq 0, \varrho \geq 1$ and $\alpha_n \in \mathbb{R}^+$, for all $n \geq 2$. If*

$$1 \geq 2(2+\vartheta)^{\varrho}\alpha_2 \geq ... \geq (n+\vartheta)^{\varrho} n\alpha_n \geq (n+1+\vartheta)^{\varrho}(n+1)\alpha_{n+1},$$

holds for all $n \geq 2$, then the function $f \in A$ is closed to convex of order κ with respect to the starlike function $\frac{z}{1-z^2}$.

Lemma 4. *[22] A function of the form (11) is in the class $\mathcal{UCD}(\kappa)$, if*

$$\sum_{n=2}^{\infty} \left[n(1-\kappa) + n^2\kappa\right] |\alpha_n| \leq 1.$$

2 Geometric Outcomes

This part discusses certain geometric characteristics such as stalikeness criterion, close to convexity criterion and univalency for the normalized seven parameters Mittag-Leffler function defined in (2).

Lemma 5. *If*

$$c \geq \max\left\{a+b+(ab-1)/2, ab+1\right\}, \tag{14}$$

then, for the coefficient $\mu_n, n \geq 2$ which defined in (3), the following relation holds

$$n\mu_n - (n+1)\mu_{n+1} \geq 0. \tag{15}$$

Proof. We have

$$\begin{aligned} n\mu_n - (n+1)\mu_{n+1} &= \Gamma(\lambda_1)\Gamma(\lambda_2) \\ &\Big(\frac{n^2(a)_{n-1}(b)_{n-1}(c)_n\Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2)}{n!(c)_n(c)_{n-1}\Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2)\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)} \\ &\quad - \frac{(n+1)(a)_n(b)_n(c)_{n-1}\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)}{n!(c)_n(c)_{n-1}\Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2)\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)}\Big). \end{aligned}$$

For simplifying, the above formula will be written as

$$n\mu_n - (n+1)\mu_{n+1} = \Gamma(\lambda_1)\Gamma(\lambda_2)\left(\frac{A_1B_1 - A_2B_2}{D}\right),$$

where

$$\begin{aligned} A_1 &= n^2(a)_{n-1}(b)_{n-1}(c)_n, \qquad & B_1 &= \Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2), \\ A_2 &= (n+1)(a)_n(b)_n(c)_{n-1}, \qquad & B_2 &= \Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2). \end{aligned} \tag{16}$$

Mainly, we intend to confirm that $A_1B_1 - A_2B_2 \geq 0$, but it is obvious that $B_1 > B_2$ for all $n \geq 2$, so it is enough to seek whether $A_1 \geq A_2$. Let

$$P(n) = n^2(a)_{n-1}(b)_{n-1}(c)_n - (n+1)(a)_n(b)_n(c)_{n-1}. \tag{17}$$

Using the fact that, $(a)_n = (a+n-1)(a)_{n-1}$, implies

$$\begin{aligned} P(n) &= n^2(c+n-1)(a)_{n-1}(b)_{n-1}(c)_{n-1} - (n+1)(a+n-1)(b+n-1)(a)_{n-1}(b)_{n-1}(c)_{n-1}, \\ &= n^2\left[c-(a+b)\right] + n(1-ab) - (a-1)(b-1). \end{aligned}$$

Since $c > a+b$, then the term involving n^2 is nonnegative, so we can use the fact that $n^2 \geq (2n-1)$ for all $n \geq 1$ as

$$\begin{aligned} P(n) &\geq (2n-1)\left[c-(a+b)\right] + n(1-ab) - (a-1)(b-1), \\ &= n\left[2(c-a-b)+1-ab\right] + 2(a+b) - ab - c - 1 = Q(n), \end{aligned}$$

remind the condition (14), we obtain that $Q(n)$ is nonnegative. In addition, $P(n) \geq Q(n)$, hence $P(n)$ is nonnegative. $\square$

The following theorem look into necessary conditions for the function (2) to be starlike in $\mathbb{D}$.

Theorem 1. *If $\lambda_1, \lambda_2 > 0$ and c satisfy condition (14), then the function $\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)$ is starlike in $\mathbb{D}$.*

Proof. Immediately, from Lemma 5 we have $\underline{\triangle} a_n \geq 0$. For the other condition of Lemma 1 to achieve starlikeness, we have

$$\begin{aligned} \underline{\triangle}(\mu_n)^2 := n\mu_n - 2(n+1)\mu_{n+1} + (n+2)\mu_{n+2} &= \Gamma(\lambda_1)\Gamma(\lambda_2) \\ \Bigg(\frac{n^2(a)_{n-1}(b)_{n-1}(c)_n\Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2)}{n!(c)_n(c)_{n-1}\Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2)\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)} \\ - \frac{2(n+1)(a)_n(b)_n(c)_{n-1}\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)}{n!(c)_n(c)_{n-1}\Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2)\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)} \\ + \frac{(n+2)(a)_{n+1}(b)_{n+1}}{(n-1)!(c)_{n+1}\Gamma(\tau_1(n+1)+\lambda_1)\Gamma(\tau_2(n+1)\lambda_2)} \Bigg). \end{aligned} \tag{18}$$

Using the same procedures as in Lemma 5, such that

$$n\mu_n - 2(n+1)\mu_{n+1} + (n+2)\mu_{n+2} = \Gamma(\lambda_1)\Gamma(\lambda_2)\left(\frac{A_1B_1 - 2A_2B_2}{D} + \frac{C}{G}\right),$$

where A_1, A_2, B_1 and B_2 mentioned in (16). Since $\frac{C}{G} > 0$ for all $\lambda_1, \lambda_2 > 0$ and $n \geq 2$, then it is sufficient to find that $A_1B_1 - 2A_2B_2$ is nonnegative. Assume

$$S(n) = n^2(a)_{n-1}(b)_{n-1}(c)_n - 2(n+1)(a)_n(b)_n(c)_{n-1}, \tag{19}$$

by a similar method used in Lemma 5, we imply that

$$S(n) \geq n\left[2c - 4(a+b) + 2(1-ab)\right] + 2(a+b) = T(n),$$

from the condition (14), we obtain that $T(n) > 0$, i.e., $S(n)$ is nonnegative. That is, $\underline{\triangle}(\mu_n)^2$ is nonnegative for all $\lambda_1, \lambda_2 > 0$ and $n \geq 2$. Therefore, by Lemma 5, we conclude that the function (2) is starlike in $\mathbb{D}$. □

Further, we can achieve starlikness of the function (2) in $\mathbb{D}$ by another method and more specific conditions, as in the subsequent theorem.

Theorem 2. *If $0 < a, b \leq 1, c, \lambda_1 \geq 1$ and $\lambda_2 \geq 1 + \sqrt{3}$, then the function $\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)$ is starlike in $\mathbb{D}$.*

Proof. Consider

$$G(z) = \frac{z\acute{\mathcal{N}}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)}{\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)}; \quad z \in \mathbb{D},$$

evidently, $G(z)$ is analytic in $\mathbb{D}$ with $G(0) = 1$. To certify starlikness of the function (2), we need only to confirm that $Re\{G(z)\} > 0$. One can easily observe that

$$\acute{\mathcal{N}}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n+1)(a)_n\,(b)_n\Gamma(\lambda_1)\Gamma(\lambda_2)}{n!(c)_n\,\Gamma(\tau_1 n + \lambda_1)\Gamma(\tau_2 n + \lambda_2)} z^n, \quad z \in \mathbb{D}.$$

By virtue of the given conditions we find that

$$(r)_n \leq n!\,;\ n \geq 1,\ r = a, b$$
$$\Gamma(n + \lambda_i) \leq \Gamma(\tau_i n + \lambda_i),\ i = 1, 2$$

hence,

$$\frac{\Gamma(\lambda_i)}{\Gamma(\tau_i n + \lambda_i)} \leq \frac{\Gamma(\lambda_i)}{\Gamma(n + \lambda_i)} = \frac{1}{(\lambda_i)_n},\ \ i = 1, 2.$$

In addition, we have, $\frac{\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)}{z} \neq 0$ on $\mathbb{D}$, so the following inequalities are satisfied

$$
\begin{aligned}
\left| \acute{\mathcal{N}}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) - \frac{\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)}{z} \right| &\leq \sum_{n=1}^{\infty} \frac{n(a)_n\,(b)_n\Gamma(\lambda_1)\Gamma(\lambda_2)}{n!(c)_n\,\Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2)}|z^n| \\
&< \sum_{n=1}^{\infty} \frac{n\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(n+\lambda_1)\Gamma(n+\lambda_2)} \\
&= \sum_{n=1}^{\infty} \frac{n}{(\lambda_1)_n(\lambda_2)_n} \\
&\leq \sum_{n=1}^{\infty} \frac{1}{(n-1)!(\lambda_2)_n} \\
&\leq \frac{1}{\lambda_2} + \frac{1}{\lambda_2(\lambda_2+1)} + \frac{1}{\lambda_2(\lambda_2+2)} + ... \\
&\leq \frac{1}{\lambda_2}\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_2+1}\right)^n = \frac{\lambda_2+1}{\lambda_2^2}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\left|\frac{\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)}{z}\right| &\geq 1 - \left|\sum_{n=1}^{\infty} \frac{(a)_n\,(b)_n\Gamma(\lambda_1)\Gamma(\lambda_2)}{n!(c)_n\,\Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2)} z^n\right| \\
&> 1 - \frac{\lambda_2+1}{\lambda_2^2} = \frac{\lambda_2^2-\lambda_2-1}{\lambda_2^2}.
\end{aligned}
$$

Consequently, we obtain

$$
\left|\frac{z\acute{\mathcal{N}}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)}{\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)} - 1\right| = \left|\frac{\acute{\mathcal{N}}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) - \frac{\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)}{z}}{\frac{\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)}{z}}\right| < \frac{\lambda_2+1}{\lambda_2^2-\lambda_2-1}. \tag{20}
$$

Now, by the hypothesis that $\lambda_2 \geq 1+\sqrt{3}$ we conclude that $\frac{\lambda_2+1}{\lambda_2^2-\lambda_2-1} \leq 1$, hence our required result has been proved. □

Example 1. *Set $a = \tau_2 = 1, b = c, \tau_1 = 0$ and $\lambda_2 = 4$ in the function (2), we observe that*

$$
\mathcal{N}^{1,c,c}_{0,\lambda_1,1,4}(z) = \frac{6\,(e^z - 1 - z) - 3z^2}{z^2},
$$

is starlike on $\mathbb{D}$.

Example 2. *Set $a = \lambda_1 = 1, b = c, \tau_2 = 3$ and $\lambda_2 = 4$ for the function (2), we have*

$$
\mathcal{N}^{1,c,c}_{0,1,3,4}(z) = \frac{z}{2}\left[e^{z^{\frac{1}{3}}} + 2e^{-\frac{1}{2}z^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}z^{\frac{1}{3}}\right)\right],
$$

is starlike on $\mathbb{D}$.

Further, we will verify closed-to-convexity criterion for the function (2) in the unit disk $\mathbb{D}$, under some necessary conditions to utilize Lemma 2, in the next theorem as one of the significant geometric property.

Theorem 3. *If $a, b, c, \tau_1, \tau_2, \lambda_1, \lambda_2 > 0$ with $c \geq 2ab$ and condition (14) is satisfied, then the function $\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)$ is closed-to-convex with respect to the convex function $-\log(1-z)$.*

Proof. By virtue of relation (15), for all $n \geq 2$ we have

$$n\mu_n \geq (n+1)\mu_{n+1}. \tag{21}$$

Hence, we intend to apply relation (12) to prove our required result. It is obvious that $\mu_n \geq 0$ for all $n \geq 2$. On the other hand,

$$\begin{aligned} 2\mu_2 &= \frac{2ab}{c}.\frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\tau_1+\lambda_1)\Gamma(\tau_2+\lambda_2)}, \\ &\leq \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\tau_1+\lambda_1)\Gamma(\tau_2+\lambda_2)}, \end{aligned}$$

since $c \leq 2ab$. Also, It is clear that for $(i = 1, 2)$, we have $\Gamma(\lambda_i) \leq \Gamma(\tau_i + \lambda_i)$.

So, we obtain that $2\mu_2 \leq 1$. Therefore, by Lemma 2 we conclude that the function $\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)$ is closed-to-convex with respect to the convex function $-\log(1-z)$. □

Proposition 1. *If $a, b, c, \geq 0, \tau_1 = \tau_2 = 1; \lambda_1, \lambda_2 \geq 1; c \geq 4ab$, and condition (14) holds, then the function $\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)$ is closed-to-convex of order κ with respect to the starlike function $\frac{z}{1-z^2}$.*

Proof. First, under the given conditions, it is obvious that,

$$\mu_n = \frac{(a)_{n-1}\,(b)_{n-1}}{(n-1)!(c)_{n-1}(\lambda_1)_{n-1}(\lambda_2)_{n-1}} \in \mathbb{R}^+$$

for all $n \geq 2$, with $\mu_1 = 1$. In addition, relation (15) provides the coefficient property $n\mu_n \geq (n+1)\mu_{n+1}$, for all $n \geq 2$. Furthermore, it is easy to find that $\mu_1 \geq 4\mu_2$, after

$$\mu_1 - 4\mu_2 = 1 - 4\left(\frac{ab}{c}\right) \geq 0.$$

Directly, by applying Lemma 3, we achieve the desired result. □

Ultimately, we could assume a specific conditions to derive a recurrence relation of the function $\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)$ which belong to the class $\mathcal{UCD}(\kappa)$.

Theorem 4. *If $0 \leq \omega \leq 1, \kappa > 0$ and*

$$\kappa\omega\mathcal{N}'''^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(1) + (\kappa+\omega-5\kappa\omega)\mathcal{N}''^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(1) + (1+\omega)\mathcal{N}'^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(1) - (\kappa+1) \leq 1. \tag{22}$$

Then $(1-\omega)\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) + \omega z\mathcal{N}'^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) \in \mathcal{UCD}(\kappa)$.

Proof. Initially, consider

$$\begin{aligned} G_\omega(z) &= (1-\omega)\mathcal{N}^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) + \omega z \mathcal{N}'^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) \\ &= z + \sum_{n=2}^{\infty}(n\omega - \omega + 1)\mu_n z^n, \ z \in \mathbb{D}, \end{aligned}$$

denote, $\gamma_n = (n\omega - \omega + 1)\,\mu_n$, where μ_n defined in (3). Now, in order to use Lemma 4, we take

$$\begin{aligned} \sum_{n=2}^{\infty}\left[n(1-\kappa)+n^2\kappa\right]|\gamma_n| &= \sum_{n=2}^{\infty}(n^2\kappa - n\kappa + n)\gamma_n = \sum_{n=2}^{\infty}(n^2\kappa - n\kappa + n)(n\omega - \omega + 1)\,\mu_n \\ &= \kappa\omega\sum_{n=2}^{\infty}n^3\mu_n + (\kappa+\omega-2\kappa\omega)\sum_{n=2}^{\infty}n^2\mu_n + (\kappa\omega-\kappa+1)\sum_{n=2}^{\infty}n\mu_n. \end{aligned}$$

Employ the equality, $n^2 = n(n-1)+n$ and $n^3 = n(n-1)(n-2)+3n(n-1)+n$ in above expression, we acquire

$$\begin{aligned} \sum_{n=2}^{\infty}\left[n(1-\kappa)+n^2\kappa\right]|\gamma_n| &= \kappa\omega\sum_{n=2}^{\infty}\left[n(n-1)(n-2)+3n(n-1)+n\right]\mu_n \\ &\quad + (\kappa+\omega-5\kappa\omega)\sum_{n=2}^{\infty}\left[n(n-1)+n\right]\mu_n + (\kappa+1)\sum_{n=2}^{\infty}n\mu_n. \end{aligned} \tag{23}$$

since

$$\mathcal{N}'^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(1) - 1 = \sum_{n=2}^{\infty}\frac{n(a)_{n-1}\,(b)_{n-1}\Gamma(\lambda_1)\Gamma(\lambda_2)}{(c)_{n-1}\,\Gamma(n)\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)} = \sum_{n=2}^{\infty}n\mu_n, \tag{24}$$

$$\mathcal{N}''^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(1) = \sum_{n=2}^{\infty}\frac{n(n-1)(a)_{n-1}\,(b)_{n-1}\Gamma(\lambda_1)\Gamma(\lambda_2)}{(c)_{n-1}\,\Gamma(n)\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)} = \sum_{n=2}^{\infty}n(n-1)\mu_n, \tag{25}$$

and

$$\mathcal{N}'''^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(1) = \sum_{n=2}^{\infty}\frac{n(n-1)(n-2)(a)_{n-1}\,(b)_{n-1}\Gamma(\lambda_1)\Gamma(\lambda_2)}{(c)_{n-1}\,\Gamma(n)\Gamma(\tau_1(n-1)+\lambda_1)\Gamma(\tau_2(n-1)+\lambda_2)} = \sum_{n=2}^{\infty}n(n-1)(n-2)\mu_n. \tag{26}$$

Directly from (24), (25) and (26), we can write expression (23) as

$$\begin{aligned} \sum_{n=2}^{\infty}\left[n(1-\kappa)+n^2\kappa\right]|\gamma_n| &= \kappa\omega\mathcal{N}'''^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(1) + (\kappa+\omega-2\kappa\omega)\mathcal{N}''^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(1) \\ &\quad + (\kappa+1)\mathcal{N}'^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(1) - (\kappa+1). \end{aligned}$$

By virtue of condition (22), we can immediately apply Lemma 3, which yields our vital result. □

Conclusion

The summarize idea of this work, appeared in the confirmation of the essential geometric attributes of the normalized seven-parameters Mittag-Leffler function (2), in the unit disk $\mathbb{D}$ such as starlikness under certain sufficient conditions, closed-to-convexity with respect to the convex functions $-\log(1-z)$ and closed-to-convexity with respect to the starlike function $\frac{z}{1-z^2}$. Besides, it was showed that some specific relation involving that normalized seven-parameter Mittag-Leffler function considering to be a member of a developed subclass of the well-known class $\mathcal{A}$ under some given hypothesis. On the other hand, as well as the functions $1+z$ and $z+z^2$, we remind that the normalized seven-parameter Mittag-Leffler function (2), and the standard seven-parameters Mittag-Leffler function is different in many features not only normalization property.

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