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On Quasi-Generalized $\phi-$recurrent Para-Kenmotsu Manifolds

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Abstract

The object of this paper is to study and reveal the various geometric properties of quasi-generalized $\phi-$recurrent (briefly, $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$) para-Kenmotsu manifold. Among the results established here it is shown that a $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ para-Kenmotsu manifold is an Einstein manifold. Moreover, we prove that a Ricci soliton in a $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ para-Kenmotsu manifold is an expanding. Further, we study quasi-generalized $\mathcal{T}-\phi-$recurrent para-Kenmotsu manifold and obtain the results which reveal the nature of its associated 1-forms.

1 Introduction

In 1985, Kaneyuki and Wiiliams [7] defined the notion of almost paracontact structure on pseudo-Riemannian manifolds, and then a systematic study of these manifolds was undertaken by Zamkovoy [19] in 2009, introducing all the technical apparatus which is needed for further investigations. Since then the study of almost paracontact manifolds and there associated structures have gained lot of attention due to its role in mathematical physics. Within this framework, para-Kenmotsu manifolds form an important subclass of almost paracontact manifolds, and they were first introduced by Welyezko [18] in the year 2013. This manifold is a counterpart of Kenmotsu manifold [8] within paracontact geometry, and is intrinsically connected to almost product manifolds. Moreover, these manifolds belong to the class $\mathbb{G}_6$ in the classification given in [22], which remained a central focus of research for an extended period. Para-Kenmotsu (p-Kenmotsu) and special para-Kenmotsu (sp-Kenmotsu) manifolds have been extensively studied under sevearl points of view in the papers [2, 13, 14, 17] and the references therein.

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Nowadays, the study of spaces of constant curvature and symmetric spaces are important sub-fields of differential geometry. In 1932, Cartan studied the issue of studying locally symmetric Riemannian manifolds, i.e., Riemannian manifolds whose curvature tensor is parallel such that satisfies $\nabla R = 0$. Since then, the perception of local symmetry on different spaces has been weakened by many geometers in distinct extent. As a gentle version of local symmetry, the concept of local $\phi-$symmetry on a Sasakian manifold has been introduced and studied by Takahashi [16]. Later, Dubey [5] proposed the notion of generalized recurrent manifolds (briefly, $\mathcal{GK}_{2n+1}$). As a generalized version of local $\phi-$symmetry, in 2003, De et al. [4] gave the idea of ϕ-recurrency on Sasakian manifolds. The concept of generalized $\phi-$recurrency (briefly, $\mathcal{G}(\phi\mathcal{K}_{2n+1})$) was introduced and studied by Basari and Murathan [1] on Kenmotsu manifolds. In addition, the concept of $\mathcal{G}(\phi\mathcal{K}_{2n+1})$ Sasakian manifolds and $\mathcal{G}(\phi\mathcal{K}_{2n+1})$ Lorentzian $\alpha-$Sasakian manifolds have been investigated in [10] and [12], respectively, yielding several significant results.

A pseudo-Riemanninan manifold M is defined as generalized recurrent (briefly, $\mathcal{GK}_{2n+1}$), when its curvature tensor R fulfills the below condition:

$$(\nabla_W \mathcal{R})(X,Y)Z = A(W)\mathcal{R}(X,Y)Z + B(W)I(X,Y)Z \tag{1}$$

for the mathematical operators, i.e., vector fields $X, Y, Z, W \in \chi(M)$, and the two non-vanishing 1-forms A and B are defined by $A(X) = g(X,\gamma_1)$, $B(X) = g(X,\gamma_2)$ and the I tensor is demonstrated as

$$I(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{2}$$

for vector fields, $X < Y, Z \in \chi(M)$. The vector fields related with non-vanishing 1-forms A and B are denoted as γ_1 and γ_2, respectively. When 1-form B is zero, then (1) reduces to the concept of $\mathcal{K}_{2n+1}$ which was proposed by Walker [23].

Let R be the curvature tensor satisfying the following condition. Then the pseudo-Riemannian manifold M is called $\mathcal{G}(\phi\mathcal{K}_{2n+1})$ and it is given by

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)I(X,Y)Z, \tag{3}$$

for vector fields $X, Y, Z, W \in \chi(\mathcal{M})$, with I defined as in (2). If X, Y and Z are vector fields that are orthogonal to ξ, then the concept of locally $\mathcal{G}(\phi\mathcal{K}_{2n+1})$ has been derived by the relation (2).

Let S be the non-vanishing Ricci tensor S of type $(0,2)$, and fulfills following equation. Then the pseudo-Riemannian manifold M is said to admit generalized Ricci recurrent (briefly, $\mathcal{GR}_{2n+1}$) [3], and it is demonstrated as

$$(\nabla_W S)(X,Y) = A(W)S(X,Y) + B(W)g(X,Y) \tag{4}$$

for vector fields $X, Y, W \in \chi(M)$, where non-vanishing 1-forms A and B are Interpreted in (1). Specifically, the relation (4) reduces to the notion of Ricci recurrent manifolds (briefly, $\mathcal{R}_{2n+1}$), when $B = 0$, that was introduced in [11].

A Pseudo-Riemanninan manifold M is defined as super generalized Ricci-recurrent, when its non-vanishing Ricci tensor $\mathcal{S}$ fulfills the below condition:

$$(\nabla_W S)(X,Y) = A(W)S(X,Y) + B(W)g(X,Y) + C(W)\eta(X)\eta(Y) \tag{5}$$

for vector fields $X, Y, W \in \chi(M)$, where A, B and C are non-vanishing unique 1-forms. In particular $B = C$, then (5) reduces to the notion of quasi-generalized Ricci-recurrent manifold (briefly, $\mathcal{QGR}_{2n+1}$) introduced by Shaikh and Roy [15].

In recent works, a new class of $\mathcal{GK}_{2n+1}$, termed quasi-generalized recurrent manifolds (briefly, $\mathcal{QGK}_{2n+1}$), was introduced in [15]. By this development, the notion of quasi generalized ϕ-recurrency (briefly, $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$) on Sasakian manifolds and $(\kappa, \mu)-$contact manifolds were further respectively explored in [17] and [9], where several of its geometric properties were established.

The paper is organized as follows: Section 2 consist of the basic definitions of para-Kenmotsu manifolds. In Section 3, we define and study the notion of quasi-generalized ϕ-recurrent para-Kenmotsu manifold and prove that it is an Einstein manifold and a Ricci soliton admitting such a type of manifold is an expanding. Also, we prove that a quasi-generalized ϕ-recurrent para-Kenmotsu manifold is of constant curvature -1. In the last section, we study quasi-generalized $\mathcal{T} - \phi-$recurrent para-Kenmotsu manifold and obtain the results which reveal the nature of its associated 1-forms.

2 Preliminaries

A smooth manifold M of $2n+1-$dimension is said to have an almost paracontact structure if it admits a $(1,1)-$tensor field ϕ, a vector field ξ, and a 1-form η satisfying the following conditions:

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0 \text{ and } \operatorname{rank}(\phi) = 2n. \tag{6}$$

and the tensor field ϕ induces an almost paracomplex structure on each fiber of $\mathcal{D} = \ker(\eta)$, i.e., the $\pm$1-eigen distributions $\mathcal{D}^{\pm} := \mathcal{D}_{\phi}(\pm)$ of ϕ have equal dimension n.

An almost paracontact structure is said to be normal [7] if and only if the $(1,2)$ type torsion tensor $N_{\phi} := [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. If an almost paracontact manifold is endowed with a pseudo-Riemannian metric g, such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \text{ and } g(X, \xi) = \eta(X), \tag{7}$$

where signature of g is necessarily $(n+1, n)$ for vector fields X, Y on M, then M is called an almost paracontact metric manifold. The fundamental 2-form Φ of an almost paracontact metric structure is defined by $\Phi(X, Y) = g(X, \phi Y)$. If $\Phi = d\eta$, then the manifold M is called a paracontact metric manifold and g is an associated metric.

If an almost paracontact metric manifold satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{8}$$

for X, Y vector fields on M, then M is called an almost para-Kenmotsu manifold. A normal almost para-Kenmotsu manifold is a para-Kenmotsu manifold.

From the above equation, it follows that

$$\nabla_X \xi = X - \eta(X)\xi \tag{9}$$

and

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y). \tag{10}$$

Moreover, in a (2n+1)-dimensional para-Kenmotsu manifold M the following relations hold:

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{11}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{12}$$

$$S(X,\xi) = -2n\eta(X), \tag{13}$$

$$Q\xi = -2n\xi, \tag{14}$$

$$(\nabla_Z R)(X,Y)\xi = g(X,Z)Y - g(Y,Z)X - R(X,Y)Z \tag{15}$$

for X, Y, Z vector fields on M, where Q denotes the Ricci operator of $\mathcal{M}$ given by $S(X,Y) = g(QX,Y)$.

Definition 2.1. *A (2n+1)-dimensional para-Kenmotsu manifold M is said to be $\eta-$Einstein if there exist smooth functions a and b, such that*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{16}$$

for any X, Y vector fields on M. If $b = 0$, then M becomes an Einstein manifold.

3 $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ Para-Kenmotsu Manifold

Definition 3.1. *A para-Kenmotsu manifold $\mathcal{M}$ is defined as $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$, when its curvature tensor $\mathcal{R}$ fulfills the below condition:*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)F(X,Y)Z, \tag{17}$$

for X, Y, Z, W vector fields on M. Here, A and B are two non-vanishing 1-forms given by $A(X) = g(X,\rho_1)$, $B(X) = g(X,\rho_2)$ and the tensor F is defined by

$$\begin{aligned} F(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &+ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \end{aligned} \tag{18}$$

for all vector fields $X, Y, Z \in TM$. Here, ρ_1 and ρ_2 are the vector fields associated with 1-forms A and B respectively. Particularly, if the 1-form B vanishes, then (17) becomes $\phi-$recurrent manifold.

Let us consider a quasi-generalized $\phi-$recurrent para-Kenmotsu manifold. In view of (17) and (6), we obtain

$$\begin{aligned} & (\nabla_W R)(X,Y)Z - \eta((\nabla_W R)(X,Y)Z)\xi \\ = \; & A(W)R(X,Y)Z + B(W)F(X,Y)Z. \end{aligned} \tag{19}$$

Taking the inner product with U in (19), we get

$$\begin{aligned} & g((\nabla_W R)(X,Y)Z,U) - \eta((\nabla_W R)(X,Y)Z)\eta(U) \\ = \; & A(W)g(R(X,Y)Z,U) + B(W)g(F(X,Y)Z,U). \end{aligned} \tag{20}$$

Let $\{e_i\}, i = 1,2,...,(2n+1)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (20) and taking summation over $i, 1 \le i \le 2n+1$, we get

$$\begin{aligned} & (\nabla_W S)(Y,Z) - \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) \\ = \; & A(W)S(Y,Z) + (2n+1)B(W)g(Y,Z) \\ + \; & (2n-1)B(W)\eta(Y)\eta(Z). \end{aligned} \tag{21}$$

Using (11), (15) and the relation $g((\nabla_W R)(X,Y)Z,U) = -g((\nabla_W R)(X,Y)U,Z)$ in the second term of the left hand side of (21), we have

$$\sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) = g((\nabla_W R)(\xi,Y)Z,\xi) = 0. \tag{22}$$

By virtue of (22), it follows from (21) that

$$\begin{aligned} (\nabla_W S)(Y,Z) &= A(W)S(Y,Z) + (2n+1)B(W)g(Y,Z) \\ &+ (2n-1)B(W)\eta(Y)\eta(Z). \end{aligned} \tag{23}$$

Thus, equation (23) leads to the following:

Theorem 3.2. *A $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ para-Kenmotsu manifold is a super generalized Ricci-recurrent manifold.*

Suppose that $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ para-Kenmotsu manifold is $\mathcal{QGR}_{2n+1}$ [15]. Then from (23) we have $2n-1 = 2n+1$, which is not possible. Therefore, we can state the following:

Corollary 3.3. *A $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ para-Kenmotsu manifold cannot be $\mathcal{QGR}_{2n+1}$.*

Setting $Z = \xi$ in (23) and using (13), we obtain

$$(\nabla_W S)(Y,\xi) = A(W)S(Y,\xi) + 4nB(W)\eta(Y). \tag{24}$$

Also, we have

$$\begin{aligned} (\nabla_W S)(Y,\xi) &= \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi) \\ &= -2ng(W,Y) - S(Y,W), \end{aligned} \tag{25}$$

using (9) and (13). Next, from (24) and (25), we get

$$-2ng(W,Y) - S(Y,W) = -2n\{A(W) + 2B(W)\}\eta(Y). \tag{26}$$

Replace Y by ξ in (26), we have

$$A(W) = 2B(W) \tag{27}$$

for all W. By virtue of (27) it follows from (26) that

$$S(Y,W) = -2ng(Y,W). \tag{28}$$

This leads us to state the following:

Theorem 3.4. *A $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ para-Kenmotsu manifold is an Einstein manifold and moreover the associated 1-forms A and B are related by $A = 2B$.*

It is known that, a para-Kenmotsu manifold is Ricci-semisymmetric if and only if it is an Einstein manifold. In fact, by Theorem 3.4, we have the following:

Corollary 3.5. *A $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ para-Kenmotsu manifold is Ricci-semisymmetric.*

Next, by virtue of (12), it follows from (15) that

$$\eta((\nabla_W R)(X,Y)\xi) = 0. \tag{29}$$

Again, in view of (15) and (29), we obtain from (19) that

$$(\nabla_W R)(X,Y)\xi = A(W)R(X,Y)\xi + B(W)F(X,Y)\xi, \tag{30}$$

from which it follows that

$$\begin{aligned} g(X,W)Y - g(Y,W)X - R(X,Y)W &= A(W)\{\eta(X)Y - \eta(Y)X\} \\ &+ 2B(W)\{\eta(Y)X - \eta(X)Y\}. \end{aligned} \tag{31}$$

Using (27) in (31), we get

$$R(X,Y)Z = -\{g(Y,W)X - g(X,W)Y\}. \tag{32}$$

Thus, (M,g) is a manifold of constant curvature -1. Hence we have the following:

Theorem 3.6. *A $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ para-Kenmotsu manifold is a manifold of constant curvature -1.*

Further, from (17), we get

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi + A(W)R(X,Y)Z + B(W)F(X,Y)Z \tag{33}$$

By using second Bianchi identity, the relation (33) gives

$$\begin{aligned} &A(W)\eta(R(X,Y)Z) + B(W)\eta(F(X,Y)Z) + A(X)\eta(R(Y,W)Z) \\ &+B(X)\eta(F(Y,W)Z) + A(Y)\eta(R(W,X)Z) + B(Y)\eta(F(W,X)Z) = 0. \end{aligned} \tag{34}$$

Using (12) and (18), the above relation becomes

$$\begin{aligned} & \{A(W)-2B(W)\}[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)] \\ + \; & \{A(X)-2B(X)\}[g(Y,Z)\eta(W)-g(W,Z)\eta(Y)] \\ + \; & \{A(Y)-2B(Y)\}[g(W,Z)\eta(X)-g(X,Z)\eta(W)]=0. \end{aligned} \tag{35}$$

Putting $Y=Z=e_i$ in (35) and taking summation over i, $1\leq i\leq 2n+1$, we get

$$\{-A(W)+2B(W)\}\eta(X)=\{-A(X)+2B(X)\}\eta(W). \tag{36}$$

Again replacing X by ξ in (36), we get

$$\{-A(W)+2B(W)\}=\{-\eta(\rho_1)+2\eta(\rho_2))\}\eta(W). \tag{37}$$

for any vector field W, where $A(\xi)=g(\rho_1,\xi)=\eta(\rho_1)$ and $B(\xi)=g(\rho_2,\xi)=\eta(\rho_2)$. Thus, in view of equations (36) and (37), we state the following theorem:

Theorem 3.7. *In a $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ para-Kenmotsu manifold, the characteristic vector field ξ and the vector field $-\rho_1+2\rho_2$ associated to the 1-form $-A+2B$ are co-directional.*

Ricci solitons, introduced by Hamilton [6], are natural generalizations of Einstein metrics, and is defined on a Riemannian manifold (M,g). A Ricci soliton (g,V,λ) is defined on (M,g) as

$$\pounds_V g(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0, \tag{38}$$

where $\pounds_V g$ denotes the Lie derivative of Riemannian metric g along a vector field V, λ is a constant, and X, Y are arbitrary vector fields on M. A Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero, and positive, respectively. In this connection we can mention the works of Blaga [2] for η-Ricci solitons on para-Kenmotsu manifolds.

Let (M^n,g) be an n-dimensional para-Kenmotsu manifold and let (g,V,λ) be a Ricci soliton in (M^n,g). Let V be pointwise collinear with ξ, i.e., $V=\xi$ on M^n. Then the relation (38) implies

$$(\pounds_\xi g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0,$$

or

$$2S(X,Y)=-(\pounds_\xi g)(X,Y)-2\lambda g(X,Y) \tag{39}$$

for any X,$Y\in\Gamma(M)$.

On a para-Kenmotsu manifold (M,g) from (9), we obtain

$$\begin{aligned}(\pounds_\xi g)(X,Y) &= g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi), \\ &= 2\{g(X,Y)-\eta(X)\eta(Y)\}.\end{aligned} \tag{40}$$

By plugging (40) in (39), we have

$$S(X,Y) = -(\lambda+1)g(X,Y) + \eta(X)\eta(Y). \tag{41}$$

Let a generalized recurrent para-Kenmotsu manifold (M,g) admits a Ricci soliton (g,ξ,λ). Then by (28) and (41), we get

$$(2n-1-\lambda)g(X,Y) + \eta(X)\eta(Y) = 0. \tag{42}$$

Substitution of $X=\xi$, the above leads to the relation:

$$\lambda = 2n. \tag{43}$$

Therefore, λ is positive. Hence, by the above discussion we are able to state:

Theorem 3.8. *A Ricci soliton (g,ξ,λ) in a $\mathcal{QG}(\phi\mathcal{K}_{2n+1})$ para-Kenmotsu manifold is an expanding.*

4 Extended Generalized $\mathcal{T}-\phi$-recurrent Para-Kenmotsu Manifolds

In a $(2n+1)-$dimensional Riemannian manifold M^{2n+1}, the $R-$curvature tensor [20], [21] is given by:

$$\begin{aligned}\mathcal{T}(X,Y)Z &= a_0R(X,Y)Z + a_1S(Y,Z)X + a_2S(X,Z)Y \\ &+ a_3S(X,Y)Z + a_4g(Y,Z)QX + a_5g(X,Z)QY \\ &+ a_6g(X,Y)QZ + a_7r(g(Y,Z)X - g(X,Z)Y),\end{aligned} \tag{44}$$

where R,S,Q and r are the curvature tensor, the Ricci tensor, the Ricci operator, and the scalar curvature, respectively. In particular, $\mathcal{T}-$curvature tensor is reduced to be quasi-conformal curvature tensor $\mathcal{C}^*$, conformal curvature tensor $\mathcal{C}$, conharmonic curvature tensor $\mathcal{L}$, concircular curvature tensor $\mathcal{V}$, pseudo-projective curvature tensor $\mathcal{P}^*$, projective curvature tensor $\mathcal{P}$, $\mathcal{M}-$projective curvature tensor, $\mathcal{W}_i-$curvature tensors $(i=0,...,9)$ and $\mathcal{W}^*_j-$curvature tensors $(j=0,1)$.

Here we define the following:

Definition 4.1. *A para-Kenmotsu manifold M is said to be quasi-generalized $\mathcal{T}-\phi-$recurrent if its $\tau-$curvature tensor satisfies the relation*

$$\phi^2((\nabla_W\mathcal{T})(X,Y)Z) = A(W)\mathcal{T}(X,Y)Z + B(W)F(X,Y)Z, \tag{45}$$

where A and B are defined as in (1).

In particular, a quasi-generalized $\mathcal{T}-\phi$-recurrent para-Kenmotsu $M^{2n+1}(\phi,\xi,\eta,g)$, $n\geq 1$, manifold is reduced to be

1. *a generalized $\mathcal{C}^*-\phi-$recurrent if*

$$a_1 = -a_2 = a_4 = -a_5, \quad a_3 = a_6 = 0, \quad a_7 = -\frac{1}{2n+1}\left(\frac{a_0}{2n} + 2a_1\right),$$

2. *a generalized $\mathcal{C}-\phi-$recurrent if*

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2n-1}, \quad a_3 = a_6 = 0, \quad a_7 = \frac{1}{2n(2n-1)},$$

3. *a generalized $\mathcal{L}-\phi-$recurrent if*

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2n-1}, \quad a_3 = a_6 = 0, \quad a_7 = 0,$$

4. *a generalized $\mathcal{V}-\phi-$recurrent if*

$$a_0 = 1, \quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{2n(2n+1)},$$

5. *a generalized $\mathcal{P}^*$-ϕ-recurrent if*

$$a_0 = 1, \quad a_1 = -a_2, \quad a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{2n(2n+1)}\left(\frac{a_0}{2n} + a_1\right),$$

6. *a generalized $\mathcal{P}$-ϕ-recurrent if*

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{2n}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

7. *a generalized $\mathcal{M}-\phi-$recurrent if*

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{4n}, \quad a_3 = a_6 = a_7 = 0,$$

8. *a generalized $\mathcal{W}_0-\phi-$recurrent if*

$$a_0 = 1, \quad a_1 = -a_5 = -\frac{1}{2n}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

9. *a generalized $\mathcal{W}_0^*-$recurrent if*

$$a_0 = 1, \quad a_1 = -a_5 = \frac{1}{2n}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

10. *a generalized $\mathcal{W}_1-\phi-$recurrent if*

$$a_0 = 1, \quad a_1 = -a_2 = \frac{1}{2n}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

11. *a generalized $\mathcal{W}_1^*-\phi-$recurrent if*

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{2n}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

12. *a generalized $\mathcal{W}_2-\phi-$recurrent if*

$$a_0 = 1, a_4 = -a_5 = \frac{1}{2n}, a_1 = a_2 = a_3 = a_6 = a_7 = 0,$$

13. *a generalized $\mathcal{W}_3 - \phi-$recurrent if*

$$a_0 = 1, \quad a_2 = -a_4 = -\frac{1}{2n}, \quad a_1 = a_3 = a_5 = a_6 = a_7 = 0,$$

14. *a generalized $\mathcal{W}_4 - \phi-$recurrent if*

$$a_0 = 1, \quad a_5 = -a_6 = -\frac{1}{2n}, \quad a_1 = a_2 = a_3 = a_4 = a_7 = 0,$$

15. *a generalized $\mathcal{W}_5 - \phi-$recurrent if*

$$a_0 = 1, \quad a_2 = -a_5 = -\frac{1}{2n}, \quad a_1 = a_3 = a_4 = a_6 = a_7 = 0,$$

16. *a generalized $\mathcal{W}_6 - \phi-$recurrent if*

$$a_0 = 1, \quad a_1 = -a_6 = -\frac{1}{2n}, \quad a_2 = a_3 = a_4 = a_5 = a_7 = 0,$$

17. *a generalized $\mathcal{W}_7 - \phi-$recurrent if*

$$a_0 = 1, \quad a_1 = -a_4 = -\frac{1}{2n}, \quad a_2 = a_3 = a_5 = a_6 = a_7 = 0,$$

18. *a generalized $\mathcal{W}_8 - \phi-$recurrent if*

$$a_0 = 1, \quad a_1 = -a_3 = \frac{1}{2n}, \quad a_2 = a_4 = a_5 = a_6 = a_7 = 0,$$

19. *a generalized $\mathcal{W}_9 - \phi-$recurrent if*

$$a_0 = 1, \quad a_3 = -a_4 = \frac{1}{2n}, \quad a_1 = a_2 = a_5 = a_6 = a_7 = 0.$$

Theorem 4.2. *If a $(2n+1)-$dimensional para-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geq 1$, is quasi-generalized $\mathcal{T} - \phi-$recurrent such that $a_0 + 2na_1 + a_2 + a_3 \neq 0$, then M^{2n+1} is generalized Ricci-recurrent if and only if the following relation holds:*

$$\begin{aligned}
& \left[\frac{(2n-1)B(W) + a_7 dr(W)}{a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6}\right] \eta(Y)\eta(Z) \\
- & \frac{(a_2 + a_6)}{(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)} \{2ng(Z, W) + S(Z, W)\}\eta(Y) \\
- & \frac{(a_3 + a_5)}{(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)} \{2ng(Y, W) + S(Y, W)\}\eta(Z) \\
= & \ 0.
\end{aligned} \tag{46}$$

Proof. Let us consider a quasi-generalized $\mathcal{T} - \phi-$recurrent Para-Kenmotsu manifold. Then by virtue of (6), it follows from (45) that

$$(\nabla_W \mathcal{T})(X, Y)Z - \eta((\nabla_W \mathcal{T})(X, Y)Z)\xi = A(W)\mathcal{T}(X, Y)Z + B(W)\mathcal{F}(X, Y)Z,$$

from which it follows that

$$\begin{aligned} & g((\nabla_W \mathcal{T})(X,Y)Z,U) - \eta((\nabla_W \mathcal{T})(X,Y)Z)\eta(U) \\ = \; & A(W)g(\mathcal{T}(X,Y)Z,U) + B(W)g(\mathcal{F}(X,Y)Z,U). \end{aligned} \tag{47}$$

Let $\{e_i, i = 1, 2, ..., 2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = U = e_i$ in (47) and taking summation over $i, 1 \leq i \leq 2n+1$, then using (7) and (44), we get

$$\begin{aligned} & \sum_{i=1}^{2n+1} g((\nabla_W \mathcal{T})(e_i,Y)Z,e_i) - \sum_{i=1}^{2n+1} \eta((\nabla_W \mathcal{T})(e_i,Y)Z)\eta(e_i) \\ & = A(W)\sum_{i=1}^{2n+1} g(\mathcal{T}(e_i,Y)Z,e_i) + B(W)\sum_{i=1}^{2n+1} g(\mathcal{F}(e_i,Y)Z,e_i). \end{aligned} \tag{48}$$

With the help of (44), we obtain

$$\begin{aligned} \sum_{i=1}^{2n+1} g((\nabla_W \mathcal{T})(e_i,Y)Z,e_i) &= [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6](\nabla_W S)(Y,Z) \\ &+ [a_4 + 2na_7]dr(W)g(Y,Z) \qquad (49) \\ \sum_{i=1}^{2n+1} \eta((\nabla_W \mathcal{T})(e_i,Y)Z)\eta(e_i) &= a_1(\nabla_W \mathcal{S})(Y,Z) - a_7 dr(W)g(Y,Z)\{g(Y,Z) + \eta(Y)\eta(Z)\} \\ &- (a_2 + a_6)\{(n-1)g(Z,W) - \mathcal{S}(Z,W)\}\eta(Y) \\ &- (a_3 + a_5)\{(n-1)g(Y,W) - \mathcal{S}(Y,W)\}\eta(Z), \qquad (50) \\ \sum_{i=1}^{2n+1} g(\mathcal{T}(e_i,Y)Z,e_i) &= [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]S(Y,Z) \\ &+ [a_4 + 2na_7]rg(Y,Z) \qquad (51) \\ \sum_{i=1}^{2n+1} g(\mathcal{G}(e_i.Y)Z,e_i) &= (2n+1)g(Y,Z) + (2n-1)\eta(Y)\eta(Z). \qquad (52) \end{aligned}$$

Making use of (49)-(52), we get

$$\begin{aligned} & [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6](\nabla_W S)(Y,Z) - \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) \\ = \; & [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]A(W).S(Y,Z) \\ + \; & [a_4 + 2na_7](A(W)r - dr(W) + (2n+1)B(W))g(Y,Z) \\ + \; & (2n-1)B(W)\eta(Y)\eta(Z). \end{aligned} \tag{53}$$

Using (11), (12), (44) and the relation $g((\nabla_W R)(X,Y)Z,U) = -g((\nabla_W R)(X,Y)U,Z)$ in the second term of the left hand side of (53), we have

$$\begin{aligned} -\sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) &= a_1(\nabla_W S)(Y,Z) + a_7 dr(W)[g(Y,Z) - \eta(Y)\eta(Z)] \\ &- (a_2 + a_6)[2ng(Z,W) + S(Z,W)]\eta(Y) \\ &- (a_3 + a_5)[2ng(Y,W) + S(Y,W)]\eta(Z). \end{aligned} \tag{54}$$

Making use of (54) in (53), we obtain

$$\begin{aligned}(\nabla_W S)(Y,Z) &= \frac{[a_0+(2n+1)a_1+a_2+a_3+a_5+a_6]}{[a_0+2na_1+a_2+a_3+a_5+a_6]}A(W).S(Y,Z)\\ &+ \frac{[a_4+2na_7]r.A(W)+(2n+1)B(W)-[a_4+(2n+1)a_7]dr(W)}{[a_0+2na_1+a_2+a_3+a_5+a_6]}g(Y,Z)\\ &+ \frac{(a_2+a_6)}{[a_0+2na_1+a_2+a_3+a_5+a_6]}\{2n.g(Z,W)+S(Z,W)\}\eta(Y)\\ &+ \frac{(a_3+a_5)}{[a_0+2na_1+a_2+a_3+a_5+a_6]}\{2n.g(Y,W)+S(Y,W)\}\eta(Z). \qquad (55)\end{aligned}$$

If the relation (46) holds,then the above relation can be reduced to

$$\nabla S = A_1 \otimes S + B_1 \otimes g,$$

where

$$A_1(W) = A(W)\Big[\frac{a_0+(2n+1)a_1+a_2+a_3+a_5+a_6}{a_0+2na_1+a_2+a_3+a_5+a_6}\Big]$$

and

$$B_1(W) = \Big[\frac{\{a_4+2na_7\}r.A(W)+(2n+1)B(W)-\{a_4+(2n+1)a_7\}dr(W)}{a_0+2na_1+a_2+a_3+a_5+a_6}\Big].$$

This implies that M^{2n+1} is generaized Ricci-recurrent. □

Substituting $Z=\xi$ in (55) and then using (7) and (13), we get

$$\begin{aligned}&-[a_0+2na_1+a_2+a_3+a_5+a_6]\{2ng(Y,W)+S(Y,W)\}\\ =\ &[\{a_4+2na_7\}r-2n\{a_0+(2n+1)a_1+a_2+a_3+a_5+a_6\}]A(W)\eta(Y)\\ &+4n^2B(W)\eta(Y)-\{a_4+2na_7\}dr(W)\eta(Y). \qquad (56)\end{aligned}$$

By making use of (58) in (56), we obtain that

$$S(Y,W) = -2ng(Y,W). \qquad (57)$$

This implies that M^{2n+1} is an Einstein manifold. Hence we state the following:

Theorem 4.3. *A quasi-generalized $\mathcal{T}-\phi$-recurrent para-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n\geq 1$, such that*

$$\frac{2(a_0+2na_1)+a_2+a_3}{2(a_0+2na_1+a_2+a_3)} \neq 0$$

is an Einstein manifold.

Replacing Y by ξ in (56), we obtain

$$\begin{aligned}B(W) &= \frac{1}{2n}[a_0+(2n+1)a_1+a_2+a_3+a_5+a_6]A(W)\\ &+ \frac{1}{4n^2}[a_4+2na_7]\{dr(W)-r.A(W)\}. \qquad (58)\end{aligned}$$

Consequently, we have the following table to reveal the nature of 1-forms A and B:

Para-Kenmotsu Manifold	$B(W) =$
Extended generalized $C*-\phi-$recurrent	$\frac{1}{2n}[a_0+(2n+1)a_1+a_2+a_3+a_5+a_6].A(W)]$ $+\frac{1}{4n^2}[a_4+2na_7]\{dr(W)-r.A(W)\}$
Extended generalized $C-\phi-$recurrent	0
Extended generalized $L-\phi-$recurrent	$-\frac{1}{4n^2(2n-1)}[dr(W)-r.A(W)]$
Extended generalized $V-\phi-$recurrent	$\frac{1}{2n}.A(W)-\frac{1}{4n^2(2n+1)}[dr(W)-r.A(W)]$
Extended generalized $P*-\phi-$recurrent	$\{a+(2n-1)b\}\left[\left(\frac{r}{2n(2n+1)}-1\right)A(W)-\frac{1}{2n(2n+1)}dr(W)\right]$
Extended generalized $P-\phi-$recurrent	0
Extended generalized $M-\phi-$recurrent	$\frac{1}{4n}\left[1+\frac{1}{2n}\right]A(W)-\frac{1}{16n^3}[dr(W)-r.A(W)]$
Extended generalized $W_0-\phi-$recurrent	0
Extended generalized $W_0^*-\phi-$recurrent	$\frac{1}{n}.A(W)$
Extended generalized $W_1-\phi-$recurrent	$\frac{1}{n}.A(W)$
Extended generalized $W_1^*-\phi-$recurrent	0
Extended generalized $W_2-\phi-$recurrent	$\frac{1}{2n}\left[1+\frac{1}{2n}\right]A(W)-\frac{1}{8n^3}[dr(W)-r.A(W)]$
Extended generalized $W_3-\phi-$recurrent	$\frac{1}{2n}\left[1-\frac{1}{2n}\right]A(W)+\frac{1}{8n^3}[dr(W)-r.A(W)]$
Extended generalized $W_4-\phi-$recurrent	$\frac{1}{2n}.A(W)$
Extended generalized $W_5-\phi-$recurrent	$\frac{1}{2n}.A(W)$
Extended generalized $W_6-\phi-$recurrent	0
Extended generalized $W_7-\phi-$recurrent	$-\frac{1}{(4n)^2}A(W)+\frac{1}{8n^3}[dr(W)-r.A(W)]$
Extended generalized $W_8-\phi-$recurrent	0
Extended generalized $W_9-\phi-$recurrent	$\frac{1}{2n}[1+\frac{1}{2n}]A(W)-\frac{1}{8n^3}[dr(W)-r.A(W)]$

It is observed from the above table that, in a generalized $\mathcal{T}-\phi-$recurrent LP-Kenmotsu manifold if $\mathcal{T}$ is equal to $\mathcal{C},\mathcal{P},\mathcal{W}_0,\mathcal{W}_1^*,\mathcal{W}_6$ and $\mathcal{W}_8$, then the 1-form $\mathcal{B}$ vanishes (that is, $\mathcal{B}=0$). This is not possible. Hence we can state the following:

Theorem 4.4. *There exists no quasi-generalized $\{\mathcal{C},\mathcal{P},\mathcal{W}_0,\mathcal{W}_1^*,\mathcal{W}_6,\mathcal{W}_8\}-\phi-$recurrent para-Kenmotsu manifold.*

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