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Polynomial Reich Contractions of the Derivative Type in Metric Spaces with Applications

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Abstract

Olatinwo [4] introduced contractions of the derivative type, and used them to prove a variant of the Banach fixed point theorem and theorems for implicit contractions. In this paper, some results concerning Reich polynomial contractions of the derivative type are established in metric spaces, together with an illustrative example. Finally, we apply our result to the Fredholm integral equation.

1 Preliminaries

Theorem 1.1. *[1] Let (X, r) be a metric space, and suppose $T : X \mapsto X$ is a mapping satisfying the condition*

$$r(Tx, Ty) \leq kr(x, y)$$

for all $x, y \in X$ and $k \in [0, 1)$. If (X, r) is complete, then T has a unique fixed point.

Definition 1.2. [2] A self-mapping T on a metric space (X, r) is termed a polynomial contraction if there is a collection of functions $a_0, \ldots, a_k : X \times X \mapsto \mathbb{R}_+$ and $\lambda \in [0, 1)$ such that for some natural number k, the inequality

$$\sum_{i=0}^{k} a_i(Tx, Ty)r^i(Tx, Ty) \leq \lambda \sum_{i=0}^{k} a_i(x, y)r^i(x, y)$$

holds true.

Theorem 1.3. *[2] Consider a complete metric space (X, r) endowed with $T : X \mapsto X$, a polynomial contraction with respect to a collection of functions $a_0, \ldots, a_k : X \times X \mapsto \mathbb{R}^+$. Assuming further that the conditions below are in effect:*

(i) The mapping T is continuous.

(ii) There is an index $j \in \{1, 2, \ldots, k\}$ and $A_j > 0$ such that the corresponding coefficient function a_j satisfies the inequality $a_j(x, y) \geq A_j$ for all $x, y \in X$.

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Then T possesses exactly one fixed point $x^* \in X$. Moreover, for any starting value $x_0 \in X$ the Picard sequence $\{x_n\}$ converges to x^*.

Theorem 1.4. *[3] Let (X, r) be a complete metric space, and let $f : X \mapsto X$ be a Reich type single valued (a, b, c)-contraction, that is, there exist nonnegative numbers a, b, c with $a + b + c < 1$ such that*

$$r(f(x), f(y)) \leq ar(x, y) + br(x, f(x)) + cr(y, f(y))$$

for each $x, y \in X$. Then f has a unique fixed point.

Definition 1.5. [4] Let (X, r) be a metric space. X is said to be d-bounded if

$$\delta_d(X) = \sup\{r(x, y)|x, y \in X\} < \infty.$$

Definition 1.6. [4] A self-mapping T for $k \in [0, 1)$ defined on a metric space (X, r) such that

$$\left.\frac{d\varphi}{dt}\right|_{t=r(Tx,Ty)} \leq k \left.\frac{d\varphi}{dt}\right|_{t=r(x,y)}$$

for all $x, y \in X$ is called a contraction of the derivative type, where $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfies $\left.\frac{d\varphi}{dt}\right|_{t=\epsilon} > 0$, $\epsilon > 0$.

Theorem 1.7. *[4] Assume that T is a self-mapping on a complete metric space (X, d) such that T is a contraction of the derivative type. Suppose that X is d-bounded. Let $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be defined as in the previous definition. Then T admits a unique fixed point $z \in X$, such that $\lim_{n\to\infty} T^n(x) = z$ for each x.*

Motivation/Novelty: The motivation for introducing Reich polynomial contractions of the derivative type is to unify Reich contractions, polynomial contractions, and derivative-type contractions within a single framework, thereby introducing a novel contraction which is more advanced than Banach contractions. The novelty of this contraction lies in substituting classical linear contraction constants with flexible higher order polynomial expressions of distance. This allows for a robust algebraic framework that accommodates complex nonlinear behaviors, particularly in differential, integral, and fractional boundary value problems.

2 Main Result

Definition 2.1. A mapping $T : X \mapsto X$ on a metric space (X, r) will be called a polynomial Reich contraction of the derivative type if $0 < \lambda < \frac{1}{3}$, $k \in \mathbb{N}$, and there exists a collection of functions $a_0, a_1, \ldots, a_k : X \times X \mapsto \mathbb{R}^+$, satisfying the following:

$$\sum_{i=0}^{k} a_i(Tx, Ty) \left[\left.\frac{d\varphi}{dt}\right|_{t=r(Tx,Ty)}\right]^i \leq \lambda \Bigg\{ \sum_{i=0}^{k} a_i(x, y) \left[\left.\frac{d\varphi}{dt}\right|_{t=r(x,y)}\right]^i + \sum_{i=0}^{k} a_i(x, Tx) \left[\left.\frac{d\varphi}{dt}\right|_{t=r(x,Tx)}\right]^i + \sum_{i=0}^{k} a_i(y, Ty) \left[\left.\frac{d\varphi}{dt}\right|_{t=r(y,Ty)}\right]^i \Bigg\}$$

for all $x, y \in X$, where $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a function as defined in Definition 1.6.

Remark 2.2. The contraction factor $0 < \lambda < \frac{1}{3}$ quantifies how much the mapping shrinks the distance between any two points in the metric space.

Remark 2.3. The generalized Reich type condition introduced above encompasses various new classes of contractive mappings as particular instances.

(*i*) Setting $k = 1$, $a_0 \equiv 0$, and $a_1 \equiv 1$, yields the derivative type Reich contraction, which satisfies

$$\left.\frac{d\varphi}{dt}\right|_{t=r(Tx,Ty)} \leq \lambda\left[\left.\frac{d\varphi}{dt}\right|_{t=r(x,y)} + \left.\frac{d\varphi}{dt}\right|_{t=r(x,Tx)} + \left.\frac{d\varphi}{dt}\right|_{t=r(y,Ty)}\right]$$

for all $x, y \in X$.

(*ii*) More generally, setting $k = m$, with $a_j \equiv 0$, for $0 \leq j \leq m-1$ and $a_m \equiv 1$, yields a pure Reich contraction of the derivative type of m-power, defined by the condition

$$\left[\left.\frac{d\varphi}{dt}\right|_{t=r(Tx,Ty)}\right]^m \leq \lambda\left\{\left[\left.\frac{d\varphi}{dt}\right|_{t=r(x,y)}\right]^m + \left[\left.\frac{d\varphi}{dt}\right|_{t=r(x,Tx)}\right]^m + \left[\left.\frac{d\varphi}{dt}\right|_{t=r(y,Ty)}\right]^m\right\}$$

for all $x, y \in X$, and a certain $\lambda \in [0, \frac{1}{3})$.

(*iii*) Setting $k = 1$, $a_0 \equiv 0$, $a_1 \equiv 1$, and

$$\left.\frac{d\varphi}{dt}\right|_{t=r(Tx,Ty)} = \left.\frac{d\varphi}{dt}\right|_{t=r(x,y)} = \left.\frac{d\varphi}{dt}\right|_{t=r(x,Tx)} = \left.\frac{d\varphi}{dt}\right|_{t=r(y,Ty)} = t$$

yields the Reich contraction

$$r(Tx, Ty) \leq \lambda[r(x, y) + r(y, Ty) + r(x, Tx)]$$

for all $x, y \in X$ and $\lambda \in [0, \frac{1}{3})$.

(*iv*) More generally, setting $k = m$, with $a_j \equiv 0$, for $0 \leq j \leq m-1$ and $a_m \equiv 1$, and

$$\left.\frac{d\varphi}{dt}\right|_{t=r(Tx,Ty)} = \left.\frac{d\varphi}{dt}\right|_{t=r(x,Tx)} = \left.\frac{d\varphi}{dt}\right|_{t=r(y,Ty)} = \left.\frac{d\varphi}{dt}\right|_{t=r(x,y)} = t$$

yields a pure Reich contraction of m-power, defined by the condition

$$r^m(Tx, Ty) \leq \lambda[r^m(x, y) + r^m(x, Tx) + r^m(y, Ty)]$$

for all $x, y \in X$ and $\lambda \in [0, \frac{1}{3})$.

Theorem 2.4. *Consider a complete metric space (X, d) endowed with a mapping $T : X \mapsto X$ that satisfies a polynomial Reich contractive condition of the derivative type with respect to a collection of functions $a_1, \ldots, a_k : X \times X \mapsto \mathbb{R}^+$. Assume further that the following conditions hold:*

(*i*) *X is d-bounded.*

(*ii*) *The coefficient function* a_i *is symmetric in its arguments and continuous in its arguments for each* $i = 0, 1, \ldots, k$.

(*iii*) *There is an index* $j \in \{1, 2, \ldots, k\}$ *and* $A_j > 0$ *such that the corresponding coefficient function* a_j *satisfies the inequality* $a_j(x, y) \geq A_j$ *for all* $x, y \in X$.

Then T *possesses exactly one fixed point* $x^* \in X$. *Moreover, for any starting value* $x_0 \in X$, *the Picard sequence* $\{x_n\}$ *converges to* x^*.

Proof. Let $x_0 \in X$ be chosen arbitrarily. Define the sequence $\{x_n\}_{n=0}^{\infty}$ recursively by $x_{n+1} = Tx_n$ for all n and also define

$$J_n = \sum_{i=0}^{k} a_i(x_n, x_{n+1}) \left[\left. \frac{d\varphi}{dt} \right|_{t=r(x_n, x_{n+1})} \right]^i.$$

Letting $x = x_n$ and $y = x_{n+1}$ in Definition 2.1, we conclude that

$$J_{n+1} \leq \lambda [J_n + J_{n+1} + J_n]$$

which implies $J_{n+1} \leq \gamma J_n$, where $\gamma = \frac{2\lambda}{1-\lambda} < 1$, since $\lambda < \frac{1}{3}$. By induction we obtain

$$\begin{aligned} J_n &\leq \gamma^n J_0 \\ &\leq \gamma^n J_0^* \end{aligned}$$

where, $J_0^* = \sum_{i=0}^{k} a_i(x_0, x_1) \left[\left. \frac{d\varphi}{dt} \right|_{t=\delta_r(X)} \right]^i$, and $r(x_0, x_1) \leq \delta_r(X)$, and $\delta_r(X)$ is stated as in Definition 1.5. Now from (iii), there is an index j such that $a_j(x_n, x_{n+1}) \geq A_j > 0$. Then we have

$$A_j \left[\left. \frac{d\varphi}{dt} \right|_{t=r(x_n, x_{n+1})} \right]^j \leq \sum_{i=0}^{k} a_i(x_n, x_{n+1}) \left[\left. \frac{d\varphi}{dt} \right|_{t=r(x_n, x_{n+1})} \right]^i = J_n \leq \gamma^n J_0^*$$

which implies that

$$\left. \frac{d\varphi}{dt} \right|_{t=r(x_n, x_{n+1})} \leq \left(\frac{\gamma^n J_0^*}{A_j} \right)^{\frac{1}{j}}.$$

Applying the triangle inequality for $m > n \geq 0$, it follows that

$$\left. \frac{d\varphi}{dt} \right|_{t=r(x_n, x_m)} \leq \sum_{l=n}^{m-1} \left. \frac{d\varphi}{dt} \right|_{t=r(x_l, x_{l+1})} \leq \left(\frac{J_0^*}{A_j} \right)^{\frac{1}{j}} \sum_{l=n}^{\infty} \gamma^{\frac{l}{j}} = \left(\frac{J_0^*}{A_j} \right)^{\frac{1}{j}} \frac{\gamma^{\frac{n}{j}}}{1 - \gamma^{\frac{1}{j}}}.$$

Since $\gamma < 1$, the right hand side approaches to zero, thus, $\left. \frac{d\varphi}{dt} \right|_{t=r(x_n, x_m)} \to 0$. By the condition on φ, we have $r(x_n, x_m) \to 0$. This shows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, r), ensuring its convergence to a point $l^* \in X$. To confirm that l^* is a fixed point, we substitute $x = l^*$ and $y = x_n$ in Definition 2.1, then we have

$$J(Tl^*, x_{n+1}) \leq \lambda [J(l^*, Tl^*) + J(x_n, x_{n+1}) + J(l^*, x_n)],$$

where $J(u,v) = \sum_{i=0}^{k} a_i(u,v)\left[\frac{d\varphi}{dt}\bigg|_{t=r(u,v)}\right]^i$. Letting $n \to \infty$ in the above inequality and using the fact that $\frac{d\varphi}{dt}\bigg|_{t=0} = 0$ (by the condition on φ) we obtain, $J(Tl^*, l^*) \leq \lambda J(l^*, Tl^*)$. Owing to $\lambda < \frac{1}{3}$, and a_i is symmetric in its arguments, it follows that $Tl^* = l^*$. For uniqueness suppose that l^* and b^* are both fixed points of T, then using Definition 2.1 with $x = l^*$ and $y = b^*$, we obtain, $J(l^*, b^*) \leq \lambda[J(l^*, b^*) + J(l^*, l^*) + J(b^*, b^*)]$. The condition on φ gives $\frac{d\varphi}{dt}\bigg|_{t=0} = 0$, thus, $J(l^*, l^*) = J(b^*, b^*) = 0$. So we obtain $J(l^*, b^*) \leq \lambda J(l^*, b^*)$. Owing to $\lambda < \frac{1}{3}$ and (iii), leads to $l^* = b^*$, and this completes the proof. □

Example 2.5. We consider (X, r) a metric space with $X = [0,1]$ and the standard Euclidean metric $r(x,y) = |x - y|$. Define $T : X \mapsto X$ by

$$T(x) = \frac{x^2}{3}.$$

Also define $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ by $\varphi(x) = x^3$. T is not a classical Reich contraction. For instance, taking $x = 0$ and $y = 1$, we have $r(T0, T1) = \frac{1}{3}$, and $\lambda[r(0, T0) + r(1, T1) + r(0,1)] = \frac{5}{3}\lambda$. Thus we obtain

$$\frac{1}{3} \leq \frac{5}{3}\lambda$$

that is $\frac{1}{5} \leq \lambda$, which contradicts the fact that λ should lie in $[0, \frac{1}{3})$. We admit T is a derivative type polynomial Reich contraction with $k = 1$, $a_0(x,y) = 0$ and $a_1(x,y) = 1 + x^2 + y^2$. To verify the contractive condition we must establish that for some $\lambda \in [0, \frac{1}{3})$, the inequality below holds for all $x, y \in X$:

$$a_1(Tx,Ty)\frac{d\varphi}{dt}\bigg|_{t=r(Tx,Ty)} \leq \lambda\left[a_1(x,y)\frac{d\varphi}{dt}\bigg|_{t=r(x,y)} + a_1(x,Tx)\frac{d\varphi}{dt}\bigg|_{t=r(x,Tx)} + a_1(y,Ty)\frac{d\varphi}{dt}\bigg|_{t=r(y,Ty)}\right].$$

Figure 1 shows that the "BlueGreenYellow" surface (RHS of the contraction inequality) dominates the "LakeColors" surface (LHS of contraction inequality). This demonstrates that the inequality is satisfied for all $x, y \in [0,1]$ with $\lambda = \frac{1}{6}$. Moreover the function $a_1(x,y)$ satisfies the lower bound $a_1(x,y) \geq 1 = A_1 > 0$. Therefore, the requirements of the above theorem hold, ensuring that T has exactly one fixed point, which is $x = 0$.

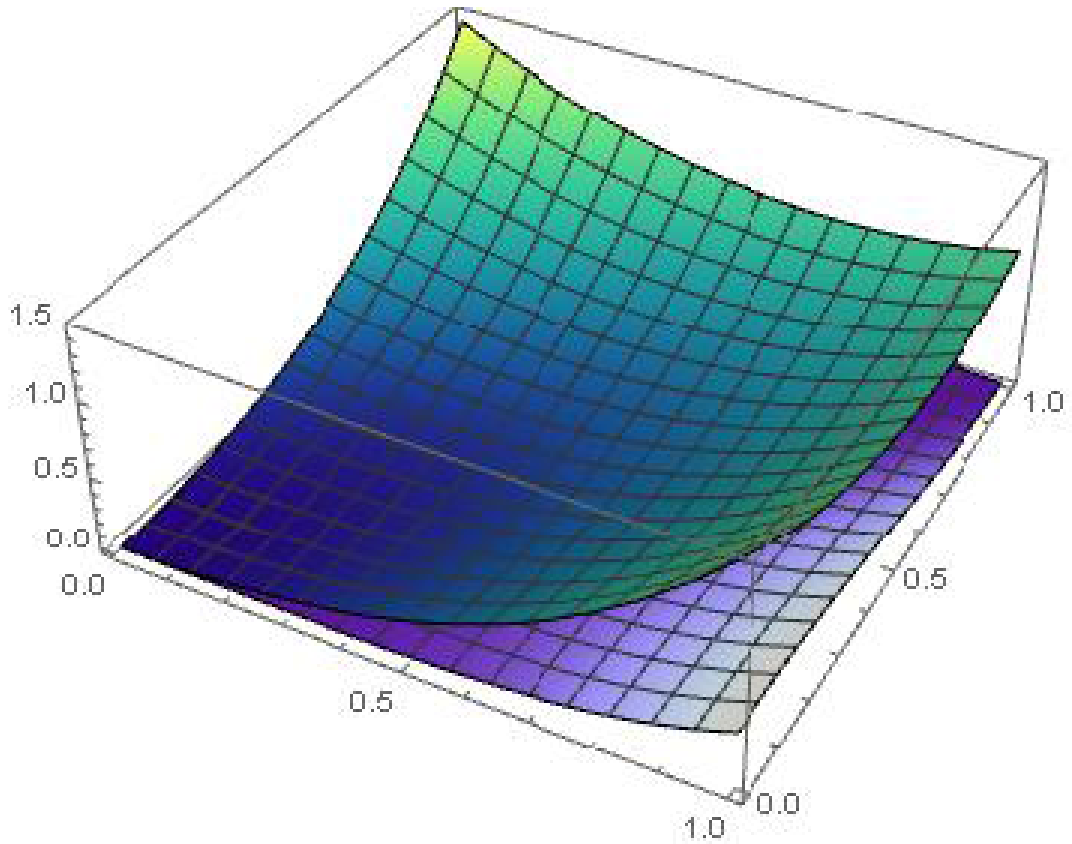

Figure 1: Comparison of both sides of the polynomial Reich contraction of the derivative type inequality in the above example over $[0, 1]$.

By Remark 2.3 and Theorem 2.4, we obtain the following corollaries

Corollary 2.6. *Assume that T is a self-mapping on a complete metric space (X, r) such that T is a Reich contraction of the derivative type. Suppose that X is d-bounded. Let $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be given in Definition 1.6. Then T admits a unique fixed point $x^* \in X$, such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$.*

Corollary 2.7. *Assume that T is a self-mapping on a complete metric space (X, r) such that T is a pure Reich contraction of the derivative type of m-power. Suppose that X is d-bounded. Let $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be given as in Definition 1.6. Then T admits a unique fixed point $x^* \in X$, such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$.*

Corollary 2.8. *Assume T is a self-mapping on a complete metric space (X, r) such that T is a Reich contraction. Then T admits a unique fixed point $x^* \in X$, such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$.*

Corollary 2.9. *Assume T is a self-mapping on a complete metric space (X, r) such that T is a pure Reich contraction of m-power. Then T admits a unique fixed point $x^* \in X$, such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$.*

3 Application

We apply our result to establish an existence theorem for a nonlinear Fredholm integral equation. Let $Y = C[0,1]$ denote the set of all real continuous functions on $[0,1]$ equipped with the metric $p(b,c) = |b-c| = \max_{t\in[0,1]} |b(t) - c(t)|$, for all $b, c \in C[0,1]$. Then (Y, p) is a complete metric space. Now we consider the non-linear Fredholm integral equation

$$b(t) = c(t) + \int_0^1 K(t, s, b(s))ds, \tag{3.1}$$

where $s, t \in [0,1]$. Assume that $K : [0,1] \times [0,1] \times Y :\mapsto \mathbb{R}$ and $c : [0,1] \mapsto \mathbb{R}$ are continuous, where $c(t)$ is a given function in Y.

Theorem 3.1. *Suppose (Y, p) is a metric space equipped with the metric $p(b,c) = |b-c| = \max_{t\in[0,1]} |b(t)-c(t)|$ for all $b, c \in Y$, and $F : Y \mapsto Y$ be an operator on Y defined by*

$$Fb(t) = c(t) + \int_0^1 K(t, s, b(s))ds. \tag{3.2}$$

If there exists $\lambda \in [0, \frac{1}{3})$ such that for all $b, c \in Y$, $s, t \in [0,1]$, satisfying the following inequality

$$\left.\frac{d\varphi}{dq}\right|_{q=|K(t,s,b(s))-K(t,s,c(s))|} \leq \lambda M(b(s), c(s))$$

where $M(b(s), c(s)) = \left.\frac{d\varphi}{dq}\right|_{q=|b(s)-c(s)|} + \left.\frac{d\varphi}{dq}\right|_{q=|b(s)-Fb(s)|} + \left.\frac{d\varphi}{dq}\right|_{q=|c(s)-Fc(s)|}$. Then the integral equation (3.2) has a unique solution in Y.

Proof. From (3.1) and (3.2), we obtain

$$\begin{aligned}
\left.\frac{d\varphi}{dq}\right|_{q=|Fb(t)-Fc(t)|} &= \left.\frac{d\varphi}{dq}\right|_{q=\left|\int_0^1 K(t,s,b(s))ds - \int_0^1 K(t,s,c(s))ds\right|} \\
&\leq \left.\frac{d\varphi}{dq}\right|_{q=\int_0^1 |K(t,s,b(s))-K(t,s,c(s))|ds} \\
&\leq \int_0^1 \left.\frac{d\varphi}{dq}\right|_{q=|K(t,s,b(s))-K(t,s,c(s))|} ds \\
&\leq \int_0^1 \lambda \left[\left.\frac{d\varphi}{dq}\right|_{q=|b(s)-c(s)|} + \left.\frac{d\varphi}{dq}\right|_{q=|b(s)-Fb(s)|} + \left.\frac{d\varphi}{dq}\right|_{q=|c(s)-Fc(s)|}\right] ds.
\end{aligned}$$

Taking the maximum on both sides for all $t \in [0,1]$, we conclude that

$$\left.\frac{d\varphi}{dq}\right|_{q=p(Fb,Fc)} \leq \lambda \left[\left.\frac{d\varphi}{dq}\right|_{q=p(b,c)} + \left.\frac{d\varphi}{dq}\right|_{q=p(b,Fb)} + \left.\frac{d\varphi}{dq}\right|_{q=p(c,Fc)}\right].$$

Now set $k = 1$, and define the coefficient functions $a_0, a_1 : Y \times Y \mapsto \mathbb{R}^+$ by $a_0(x,y) = 0$ and $a_1(x,y) = C$, where C is a positive real number. Then the above inequality becomes

$$a_1(Fb,Fc)\left.\frac{d\varphi}{dq}\right|_{q=p(Fb,Fc)} \leq \lambda \left[a_1(b,c)\left.\frac{d\varphi}{dq}\right|_{q=p(b,c)} + a_1(b,Fb)\left.\frac{d\varphi}{dq}\right|_{q=p(b,Fb)} + a_1(c,Fc)\left.\frac{d\varphi}{dq}\right|_{q=p(c,Fc)}\right]$$

which shows that F satisfies the condition of a Reich polynomial contraction of the derivative type with a bounded coefficient a_1 and $k = 1$. Since $Y = C[0, 1]$ is a complete metric space, we find that all the conditions of Theorem 2.4 are satisfied and hence the integral equation (3.2) has a unique solution in Y. □

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