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T-Interpolative Berinde Weak Contraction Mapping Theorem in Cone Metric Spaces with c-Distance

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Abstract

We prove a fixed point theorem for T-interpolative Berinde weak contractions in cone metric spaces with c-distance. A consequence of the main result is stated.

1 Introduction and Preliminaries

Definition 1.1. [1] Let E be a real Banach space and let θ denote the zero element in E. A cone P is a subset of E such that

(1) P is a nonempty, closed and $P \neq \{\theta\}$.

(2) If a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$.

(3) $x \in P$ and $-x \in P$ imply $x = \theta$.

Notation 1.2. [1] Regardless of $P \subset E$, the symbol $\preceq$ with reference to P is given by $x \preceq y$ if and only if $y - x \in P$. On the other hand, $\prec$ will represent $x \preceq y$ but $x \neq y$. Also, $x \ll y$ will represent $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

Definition 1.3. [1] If there exists a number K such that

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|$$

for all $x, y \in E$, then the cone P is called normal. The smallest natural number K satisfying the above inequality is called the normal constant of P.

Definition 1.4. [2] Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \mapsto E$ satisfies the following conditions:

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(1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$.

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$.

(3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 1.5. [2] Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$.

(2) For all $c \in E$ with $\theta \ll c$, if there is a natural number N such that $d(x_r, x_q) \ll c$ for all $r, q > N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) A cone metric space (X,d) is called complete if every Cauchy sequence in X is convergent.

Lemma 1.6. *[3]*

(1) *If E is a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

(2) *If $c \in int(P)$, $\theta \preceq a_n$, and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.*

Definition 1.7. [4] Let (X,d) be a cone metric space. A function $q : X \times X \mapsto E$ is called a c-distance on X, if the following conditions hold:

($q1$) $\theta \preceq q(x,y)$ for all $x, y \in X$.

($q2$) $q(x,y) \preceq q(x,z) + q(z,y)$ for all $x, y, z \in X$.

($q3$) For each $x \in X$ and $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x,y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$.

($q4$) For all $c \in E$ with $\theta \ll c$, one can find $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

Example 1.8. [5,6] Let $E = \mathbb{R}^2$ and $P = \{(x,y) \in E : x, y \geq 0\}$. Let $X = [0,1]$ and define a mapping $d : X \times X \mapsto E$ by $d(x,y) = (|x-y|, |x-y|)$ for all $x, y \in X$. Then (X,d) is a complete cone metric space. Define a mapping $q : X \times X \mapsto E$ by $q(x,y) = (y,y)$ for all $x, y \in X$. Then q is a c-distance on X.

Lemma 1.9. *[4] Let (X,d) be a cone metric space and q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then the following hold:*

(1) *If* $q(x_n, y) \preceq u_n$ *and* $q(x_n, z) \preceq u_n$, *then* $y = z$.

(2) *If* $q(x_n, y_n) \preceq u_n$ *and* $q(x_n, z) \preceq u_n$, *then* $\{y_n\}$ *converges to* z.

(3) *If* $q(x_n, x_m) \preceq u_n$ *for* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence in* X.

(4) *If* $q(y, x_n) \preceq u_n$, *then* $\{x_n\}$ *is a Cauchy sequence in* X.

Definition 1.10. [7] Let (X, d) be a cone metric space, P be a solid cone, and $T : X \mapsto X$. Then:

(a) T is a continuous operator if $\lim x_n = x^*$ implies that $\lim Tx_n = Tx^*$ for all sequences $\{x_n\}$ in X.

(b) T is a sequentially convergent mapping if for every sequence $\{x_n\}$, if $\{Tx_n\}$ is convergent, then $\{x_n\}$ is also convergent.

(c) T satisfies the subsequence condition if for every sequence $\{x_n\}$ for which $\{Tx_n\}$ is convergent, then $\{x_n\}$ has a convergent subsequence.

Definition 1.11. [1] Let X be a set and d be a cone metric on X, q be a c-distance on X, and $f, T : X \mapsto X$ be two mappings. A mapping f is said to be a T-Reich contraction if there exist constants $k, l, r \in [0, 1)$ with $k + l + r < 1$ such that

$$q(Tfx, Tfy) \preceq kq(Tx, Ty) + lq(Tx, Tfx) + rq(Ty, Tfy)$$

for all $x, y \in X$.

Theorem 1.12. *[1] Let* (X, d) *be a complete cone metric space,* P *be a solid cone, and* q *be a* c*-distance on* X*. In addition , let* $T : X \mapsto X$ *be a one-to-one, continuous function that is subsequentially convergent, and* $f : X \mapsto X$ *be a mapping satisfying the contractive condition*

$$q(Tfx, Tfy) \preceq kq(Tx, Ty) + lq(Tx, Tfx) + rq(Ty, Tfy)$$

for all $x, y \in X$, *where* $k, l, r \in [0, 1)$ *are constants such that* $k + l + r < 1$. *Then* f *has a unique fixed point* $x^* \in X$, *and for any* $x \in X$, *the iterative sequence* $\{fx_n\}$ *converges to the fixed point. If* $u = fu$, *then* $q(Tu, Tu) = \theta$.

Definition 1.13. [8] Let X be a set and r be a metric on X. A mapping $T : X \mapsto X$ is called an interpolative Berinde weak operator if it fulfills

$$r(Ta, Tb) \leq \lambda r(a, b)^{\alpha} r(a, Ta)^{1-\alpha},$$

where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, and for all $a, b \in X$, $a, b \notin Fix(T)$.

Alternatively, the interpolative Berinde weak operator is given as follows:

Definition 1.14. [8] Let X be a set and r be a metric on X. A mapping $T : X \mapsto X$ is called an interpolative Berinde weak operator if it fulfills

$$r(Ta, Tb) \leq \lambda r(a, b)^{\frac{1}{2}} r(a, Ta)^{\frac{1}{2}},$$

where $\lambda \in [0, 1)$ and for all $a, b \in X$, $a, b \notin Fix(T)$.

Motivation/Novelty: The novelty of the c-distance in fixed point theory lies in its ability to adapt classical metric space techniques to generalized cone metric spaces over Banach spaces. Unlike the traditional w-distance, the c-distance does not require the underlying topological space to rely on continuous contractive conditions, unlocking fixed point theorems for non-normal or non-solid cones. The c-distance is significant since it relaxes the continuity assumption, it is also independent of normal cones, and finally it generalizes vector-valued metrics. On the other hand , the novelty of the interpolative Berinde weak contraction lies in its ability to dynamically "squeeze" and adjust the contraction factor using exponents, bridging classical contractive principles and previously unresolvable nonlinear settings. Unlike traditional operators, it guarantees fixed points for broader, non-continuous mappings without requiring unique fixed points in every generalized space.

2 Main Result

Definition 2.1. Let X be a set and d be a cone metric on X, j be a c-distance on X, and $g, H : X \mapsto X$ be two mappings. A mapping g will be called a T-interpolative Berinde weak contraction mapping if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$j(Hga, Hgb) \preceq \lambda j(Ha, Hb)^{\alpha} j(Ha, Hga)^{1-\alpha}$$

for each $a, b \in X$, $a, b \notin Fix(g)$.

Remark 2.2. A c-distance $j : X \times X \mapsto E$ maps two points in a space X to an ordered topological vector space E (like a Banach algebra). Unlike a standard metric, a c-distance does not require symmetry and does not imply $x = y$ when $j(x, y) = \theta$ (zero vector). It must instead satisfy conditions bounding elements under specific limits, such as a generalized triangle inequality and sequential regularity. It is in this spirit that the powers and products of the cone-valued quantities appearing in the previous definition are defined.

Theorem 2.3. *Let the set X be complete and d be a cone metric on X, P be a solid cone, and j be a c-distance on X. In addition, let $H : X \mapsto X$ be a one-to-one, continuous function and subsequentially convergent and $g : X \mapsto X$ be a mapping satisfying the contractive condition*

$$j(Hga, Hgb) \preceq \lambda j(Ha, Hb)^{\alpha} j(Ha, Hga)^{1-\alpha}$$

for each $a, b \in X$, with $a, b \notin Fix(g)$, $\lambda \in [0, 1)$, and $\alpha \in (0, 1)$. Then the function g possesses exactly one fixed point $x^* \in X$, and for any $x \in X$, the sequence of iterates $\{gx_n\}$ converges to the fixed point. If $u = gu$, then $j(Hu, Hu) = \theta$.

Proof. Choose $x_0 \in X$. Set $x_1 = gx_0, x_2 = gx_1 = g^2x_0, \cdots, x_{n+1} = gx_n = g^{n+1}x_0$. Now, we have

$$\begin{aligned} j(Hx_n, Hx_{n+1}) &= j(Hgx_{n-1}, Hgx_n) \\ &\preceq \lambda j(Hx_{n-1}, Hx_n)^{\alpha} j(Hx_{n-1}, Hgx_{n-1})^{1-\alpha} \\ &= \lambda j(Hx_{n-1}, Hx_n)^{\alpha} j(Hx_{n-1}, Hx_n)^{1-\alpha} \\ &= \lambda j(Hx_{n-1}, Hx_n). \end{aligned}$$

So,

$$\begin{aligned} j(Hx_n, Hx_{n+1}) &\preceq \lambda j(Hx_{n-1}, Hx_n) \\ &\preceq \lambda^2 j(Hx_{n-2}, Hx_{n-1}) \\ &\vdots \\ &\preceq \lambda^n j(Hx_0, Hx_1), \end{aligned}$$

where $\lambda < 1$. Note that

$$j(Hgx_{n-1}, Hgx_n) = j(Hx_n, Hx_{n+1}) \preceq \lambda j(Hx_{n-1}, Hx_n). \tag{2.1}$$

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} j(Hx_n, Hx_m) &\preceq j(Hx_n, Hx_{n+1}) + j(Hx_{n+1}, Hx_{n+2}) + \cdots + j(Hx_{m-1}, Hx_m) \\ &\preceq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) j(Hx_0, Hx_1) \\ &\preceq \frac{\lambda^n}{1-\lambda} j(Hx_0, Hx_1) \to \theta \quad \text{as } n \to \infty. \end{aligned}$$

Thus, Lemma 1.9(3) shows that $\{Hx_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $Hx_n \to v$ as $n \to \infty$. Since H is subsequentially convergent, $\{x_n\}$ has a convergent subsequence. So, there are $x^* \in X$ and $\{x_{n_i}\}$ such that $x_{n_i} \to x^*$ as $i \to \infty$. Since H is continuous, we obtain that $\lim Hx_{n_i} = Hx^*$. Since the limit is unique, this implies that $Hx^* = v$. Now by Definition 1.7(q3) we have

$$j(Hx_n, Hx^*) \preceq \frac{\lambda^n}{1-\lambda} j(Hx_0, Hx_1). \tag{2.2}$$

On the other hand by using (2.1), we have

$$\begin{aligned} j(Hx_n, Hgx^*) &= j(Hgx_{n-1}, Hgx^*) \\ &\preceq \lambda j(Hx_{n-1}, Hx^*) \\ &\preceq \lambda \frac{\lambda^{n-1}}{1-\lambda} j(Hx_0, Hx_1) \\ &= \frac{\lambda^n}{1-\lambda} j(Hx_0, Hx_1). \end{aligned} \tag{2.3}$$

By Lemma 1.9(1), (2.2) and (2.3), we have $Hx^* = Hgx^*$. Since H is 1-1, $x^* = gx^*$. Thus, x^* is a fixed point of g. Suppose that $u = gu$, then

$$\begin{aligned} j(Hu, Hu) &= j(Hgu, Hgu) \\ &\preceq \lambda j(Hu, Hu)^{\alpha} j(Hu, Hgu)^{1-\alpha} \\ &= \lambda j(Hu, Hu)^{\alpha} j(Hu, Hu)^{1-\alpha} \\ &= \lambda j(Hu, Hu). \end{aligned}$$

Since $\lambda < 1$, Lemma 1.6(1) shows that $j(Hu, Hu) = \theta$. Finally, suppose there is another fixed point y^* of g, then we have

$$\begin{aligned} j(Hx^*, Hy^*) &= j(Hgx^*, Hgy^*) \\ &\preceq \lambda j(Hx^*, Hy^*)^{\alpha} j(Hx^*, Hgx^*)^{1-\alpha} \\ &= \lambda j(Hx^*, Hy^*)^{\alpha} j(Hx^*, Hy^*)^{1-\alpha} \\ &= \lambda j(Hx^*, Hy^*). \end{aligned}$$

Since $\lambda < 1$, Lemma 1.6(1) shows that $j(Hx^*, Hy^*) = \theta$. Also we have $j(Hx^*, Hx^*) = \theta$. Thus, Lemma 1.9(1) implies $Hx^* = Hy^*$. Since H is 1-1, $x^* = y^*$. Therefore the fixed point is unique. This completes the proof. □

If $Hx = x$ in the above theorem, then we get the following:

Corollary 2.4. *Let X be a set and d be a complete cone metric on X, P be a solid cone and j be a c-distance on X. Let $g : X \mapsto X$ be a mapping satisfying the contractive condition*

$$j(gx, gy) \preceq \lambda j(x, y)^{\alpha} j(x, gx)^{1-\alpha}$$

for all $x, y \in X$, $x, y \notin Fix(g)$, $\lambda \in [0, 1)$, and $\alpha \in (0, 1)$. Then g has a unique fixed point $x^* \in X$, and for any $x \in X$, the iterative sequence $\{gx_n\}$ converges to the fixed point. If $u = gu$, then $j(u, u) = \theta$.

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