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On Some New Inequalities for a Specific Class of Functions

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Abstract

This paper introduces a new class of functions motivated by a symmetry-type inequality satisfied by convex functions. By incorporating a real parameter, this class extends traditional convexity to accommodate more general functional behaviors while preserving useful structural properties. Several propositions are established, including new functional and integral inequalities inspired by classical results, such as the Hermite-Hadamard and Fejér integral inequalities. The paper concludes with perspectives for future research.

1 Introduction

Convex functions form a fundamental class in mathematics, providing a natural framework for problems where linearity is replaced by a notion of curvature that preserves inequality structures. The formal definition is given below.

Definition 1. *Let $a, b \in \mathbb{R}$ with $a < b$. We define the class of convex functions $\mathcal{E}(a, b)$ by*

$$\mathcal{E}(a,b) = \Big\{ f : [a,b] \to \mathbb{R};\ \ for\ any\ \ x, y \in [a,b]\ and\ \lambda \in [0,1], \\ f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \Big\}.$$

Every function in $\mathcal{E}(a, b)$ is called convex on $[a, b]$

Functions in this class possess numerous remarkable properties: they are continuous on their domain, differentiable almost everywhere, and any local minimum is also a global minimum. These characteristics make convexity central to the study of inequalities, variational methods, and optimization. Furthermore, classical results, such as the Hermite-Hadamard and Fejér integral inequalities rely fundamentally on convexity. Given the vast literature on the topic, we cite only a representative selection [1–17].

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One interesting and non-trivial property of a function $f \in \mathcal{E}(a,b)$ is that, for any $x \in [a,b]$, the following inequality holds:

$$f(a+b-x) \leq f(a) + f(b) - f(x). \tag{1}$$

See, for example, [17, Corollary 1.4.7, p. 31]. Motivated by this property, we introduce a broader class of functions that contains $\mathcal{E}(a,b)$ as a special case, yet allows for more general behaviors beyond convexity. The flexibility of this class comes from the introduction of a real parameter $\theta \geq -1$, which governs the strength of the inequality. A formal definition is given below.

Definition 2. *Let $a,b \in \mathbb{R}$ with $a < b$ and $\theta \geq -1$. We define the class of functions $\mathcal{F}(a,b;\theta)$ by $\mathcal{F}(a,b;\theta) = \left\{ f : [a,b] \to \mathbb{R}; \text{ for any } x \in [a,b],\ f(a+b-x) \leq f(a) + f(b) - \theta f(x) \right\}$.*

We now examine the richness of the class $\mathcal{F}(a,b;\theta)$ through general properties and concrete examples. First of all, based on Equation (1), we have

$$\mathcal{E}(a,b) \subseteq \mathcal{F}(a,b;1).$$

We now claim that the class $\mathcal{F}(a,b;\theta)$ is richer than the convex class $\mathcal{E}(a,b)$. To demonstrate this, for any $\epsilon \in \mathbb{R}$, we can consider the following trigonometric class of functions:

$$\mathcal{G}(\epsilon) = \left\{ f : [0,1] \to \mathbb{R};\quad f(x) = 1 + \epsilon \sin(2\pi x) \right\}.$$

Then, for any $f \in \mathcal{G}(\epsilon)$, we have $f(0) = f(1) = 1$, and $f''(x) = -4\pi^2 \epsilon \sin(2\pi x)$, which changes sign for $x \in [0,1]$ and $\epsilon \neq 0$. Therefore, in this case, f is not convex, implying that

$$\mathcal{G}(\epsilon) \not\subseteq \mathcal{E}(0,1).$$

On the other hand, we have

$$\begin{aligned} f(1-x) &= 1 + \epsilon \sin[2\pi(1-x)] = 1 - \epsilon \sin(2\pi x) = 2 - [1 + \epsilon \sin(2\pi x)] \\ &= f(0) + f(1) - f(x). \end{aligned}$$

Therefore, we have

$$\mathcal{G}(\epsilon) \subseteq \mathcal{F}(0,1;1).$$

The proposed class contains this non-standard trigonometric class, which is of interest for at least $\theta = 1$.

In order to motivate the role of the parameter θ, note that the negative constant function $f(x) = -1$ satisfies $f \in \mathcal{F}(a,b;2)$. The concave function $f(x) = \sqrt{x}$, $x \in [0,1]$, belongs to $\mathcal{F}(0,1;0)$ because $f(1-x) = \sqrt{1-x} \leq 1 = f(0) + f(1) - 0 \times f(x)$. As another example, let us consider the basic function $f(x) = x(1-x)$, $x \in [0,1]$. Then it is obviously concave because $f''(x) = -2 < 0$. Furthermore, we have

$$f(1-x) = x(1-x) = 0 + 0 + x(1-x) = f(0) + f(1) + f(x),$$

and so $f \in \mathcal{F}(0,1;-1)$. More technically, using the inequality, for any $x \in [0,1]$,

$$\arctan(1-x) \leq \frac{\pi}{4} + \frac{1}{2}\arctan(x),$$

we directly show that the concave function $f(x) = \arctan(x)$, $x \in [0,1]$, belongs to $\mathcal{F}(0,1;-1/2)$. The proposed class $\mathcal{F}(a,b;\theta)$ thus contains some concave functions for certain values of θ, which highlights its complex nature.

The aim of this paper is to establish new functional and integral inequalities assuming that the primary function f belongs to $\mathcal{F}(a,b;\theta)$. Several propositions are formulated and proved in detail, highlighting the mathematical significance and potential applications of this class of functions.

The rest of the paper is organized as follows: Section 2 presents direct inequalities for functions $f \in \mathcal{F}(a,b;\theta)$. Section 3 establishes integral inequalities inspired by the classical Hermite-Hadamard and Fejér integral inequalities. Finally, conclusions and perspectives for future work are provided in Section 4.

2 Direct Inequalities

The proposition below presents two fundamental inequalities satisfied by any function belonging to $\mathcal{F}(a,b;\theta)$. Both involve θ and the function values at the endpoints of the interval $[a,b]$, namely $f(a)$ and $f(b)$.

Proposition 2.1. *Let $a, b \in \mathbb{R}$ with $a < b$, $\theta \geq -1$, and $f \in \mathcal{F}(a,b;\theta)$. Then we have*

- $(1-\theta)f(a) \geq 0$,
- $(1-\theta)f(b) \geq 0$.

Proof. By the definition of $\mathcal{F}(a,b;\theta)$, selecting $x = a$, we get

$$f(b) = f(a+b-x) \leq f(a) + f(b) - \theta f(x) = f(a) + f(b) - \theta f(a) = f(b) + (1-\theta)f(a),$$

which can be arranged as

$$(1-\theta)f(a) \geq 0.$$

In a similar way, selecting $x = b$ gives

$$f(a) = f(a+b-x) \leq f(a) + f(b) - \theta f(x) = f(a) + f(b) - \theta f(b) = f(a) + (1-\theta)f(b),$$

which can be arranged as

$$(1-\theta)f(b) \geq 0.$$

This ends the proof of the proposition. □

Since θ and f can take various signs, this proposition offers initial insight into the complex interactions among θ, $f(a)$, and $f(b)$. Note that these inequalities become trivial in the standard convex case corresponding to $\theta = 1$.

The proposition below presents another direct inequality satisfied by functions in $\mathcal{F}(a, b; \theta)$, involving θ as well as the function values at the midpoint $(a + b)/2$ and the endpoints a and b.

Proposition 2.2. *Let $a, b \in \mathbb{R}$ with $a < b$, $\theta > -1$, and $f \in \mathcal{F}(a, b; \theta)$. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{1+\theta}[f(a) + f(b)].$$

Proof. By the definition of $\mathcal{F}(a, b; \theta)$, selecting $x = (a + b)/2$ gives

$$f\left(\frac{a+b}{2}\right) = f(a+b-x) \leq f(a) + f(b) - \theta f(x) = f(a) + f(b) - \theta f\left(\frac{a+b}{2}\right),$$

which can be arranged as

$$(1+\theta) f\left(\frac{a+b}{2}\right) \leq f(a) + f(b),$$

so that, since $\theta > -1$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{1+\theta}[f(a) + f(b)].$$

This achieves the proof of the proposition. $\square$

If f is convex, then $f \in \mathcal{F}(a, b; \theta)$ with $\theta = 1$, and Proposition 2.2 gives the classical convex inequality with the weight $\lambda = 1/2$, i.e.,

$$f\left(\frac{a+b}{2}\right) = f(\lambda a + (1-\lambda) b) \leq \lambda f(a) + (1-\lambda) f(b) = \frac{1}{2}[f(a) + f(b)].$$

The proposition below offers another point of view of $\mathcal{F}(a, b; \theta)$ by reversing the inequality defining it.

Proposition 2.3. *Let $a, b \in \mathbb{R}$ with $a < b$, $\theta > 0$, and $f \in \mathcal{F}(a, b; \theta)$. Then, for any $x \in [a, b]$, we have*

$$f(a+b-x) \leq \frac{1}{\theta}\left[f(a) + f(b) - f(x)\right].$$

If we take $\theta \in (-1, 0)$, this inequality is reversed.

Proof. By the definition of $\mathcal{F}(a, b; \theta)$, selecting $x = a + b - y$ with $y \in [a, b]$ gives

$$f(y) = f(a+b-x) \leq f(a) + f(b) - \theta f(a+b-y),$$

which can be arranged as

$$f(y) - f(a) - f(b) \leq -\theta f(a + b - y),$$

so that, since $\theta > 0$,

$$f(a + b - y) \leq \frac{1}{\theta} \left[f(a) + f(b) - f(y) \right].$$

Clearly, if $\theta \in (-1, 0)$, this last inequality is reversed. This concludes the proof of the proposition. □

These inequalities have numerous applications across mathematics. In the next section, we use them to establish several important integral inequalities.

3 Integral Inequalities

The first integral inequality involving $\mathcal{F}(a, b; \theta)$ is presented in the proposition below.

Proposition 3.1. *Let $a, b \in \mathbb{R}$ with $a < b$, $\theta > -1$, and $f \in \mathcal{F}(a, b; \theta)$ with f integrable. Then we have*

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \frac{1}{1+\theta}[f(a) + f(b)].$$

Proof. Let $\tau \in [0, 1)$ be an intermediate parameter (we can put $\tau = 1/2$). Then, by a standard linearization of the integral based on τ, a change of variables, i.e., $x = a + b - y$, and the definition of $\mathcal{F}(a, b; \theta)$, we get

$$\begin{aligned}
\int_a^b f(x)dx &= (1-\tau) \int_a^b f(x)dx + \tau \int_a^b f(x)dx \\
&= (1-\tau) \int_a^b f(a+b-y)dy + \tau \int_a^b f(x)dx \\
&\leq (1-\tau) \int_a^b [f(a) + f(b) - \theta f(y)]dy + \tau \int_a^b f(x)dx \\
&= (1-\tau)[f(a) + f(b)] \int_a^b dx - \theta(1-\tau) \int_a^b f(x)dx + \tau \int_a^b f(x)dx \\
&= (1-\tau)[f(a) + f(b)](b-a) + [\tau - \theta(1-\tau)] \int_a^b f(x)dx.
\end{aligned}$$

From this, we derive

$$(1+\theta)(1-\tau) \int_a^b f(x)dx \leq (1-\tau)[f(a) + f(b)](b-a),$$

so that, after simplification by $1 - \tau$,

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \frac{1}{1+\theta}[f(a) + f(b)].$$

This ends the proof of the proposition. □

If f is convex, then $f \in \mathcal{F}(a, b; \theta)$ with $\theta = 1$, and Proposition 3.1 gives the classical right-hand side of the Hermite-Hadamard integral inequality.

Proposition 3.1 can be extended with the use of an auxiliary symmetric function, as described below.

Proposition 3.2. *Let $a, b \in \mathbb{R}$ with $a < b$, $\theta > -1$, $f \in \mathcal{F}(a, b; \theta)$, and $g : [a, b] \to (0, +\infty)$ be a symmetric function to $(a + b)/2$, i.e., for any $x \in [a, b]$,*

$$g(a + b - x) = g(x), \tag{2}$$

with fg and g integrable. Then, we have

$$\int_a^b f(x)g(x)dx \leq \frac{1}{1+\theta}[f(a) + f(b)] \int_a^b g(x)dx.$$

Proof. Let $\tau \in [0, 1)$ be an intermediate parameter (we can put $\tau = 1/2$). Then, by a standard linearization of the integral based on τ, a change of variables, i.e., $x = a + b - y$, and the use of the definition of $\mathcal{F}(a, b; \theta)$, Equation (2) and the fact that g is non-negative, we get

$$\begin{aligned}
\int_a^b f(x)g(x)dx &= (1-\tau)\int_a^b f(x)g(x)dx + \tau \int_a^b f(x)g(x)dx \\
&= (1-\tau)\int_a^b f(a+b-y)g(a+b-y)dy + \tau \int_a^b f(x)g(x)dx \\
&= (1-\tau)\int_a^b f(a+b-y)g(y)dy + \tau \int_a^b f(x)g(x)dx \\
&\leq (1-\tau)\int_a^b [f(a) + f(b) - \theta f(y)]g(y)dy + \tau \int_a^b f(x)g(x)dx \\
&= (1-\tau)[f(a) + f(b)]\int_a^b g(x)dx - \theta(1-\tau)\int_a^b f(x)g(x)dx + \tau \int_a^b f(x)g(x)dx \\
&= (1-\tau)[f(a) + f(b)]\int_a^b g(x)dx + [\tau - \theta(1-\tau)]\int_a^b f(x)g(x)dx.
\end{aligned}$$

From this, we derive

$$(1+\theta)(1-\tau)\int_a^b f(x)g(x)dx \leq (1-\tau)[f(a) + f(b)]\int_a^b g(x)dx,$$

so that, after simplification by $1 - \tau$,

$$\int_a^b f(x)g(x)dx \leq \frac{1}{1+\theta}[f(a) + f(b)] \int_a^b g(x)dx.$$

This ends the proof of the proposition. □

Choosing $g = 1$, Proposition 3.2 reduces to Proposition 3.1.

If f is convex, then $f \in \mathcal{F}(a, b; \theta)$ with $\theta = 1$, and Proposition 3.2 gives the classical right-hand side of the Féjer integral inequality.

4 Conclusion

In this paper, we introduced and studied the class $\mathcal{F}(a, b; \theta)$, which generalizes convex functions via a symmetry-type inequality parameterized by $\theta \geq -1$. Several new functional and integral inequalities were established, extending classical results, such as the Hermite-Hadamard and Fejér integral inequalities. Future research directions include exploring applications of this class in optimization, refining the obtained bounds for specific subclasses, and investigating multidimensional or discrete analogues.

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